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PSEUDOSPECTRA OF THE KRONECKER SUM OF EVEN ORDER TENSORS

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Abstract The evolution of theories associated with matrices into broader frameworks occurs consistently. Pseudospectra is a generalization to spectra, which can be used in perturbation analysis. This paper studies the S-pseudospectra of the Kronecker sum of even order tensors. The pseudospectral mapping theorem for the Kronecker sum of even order tensors is proved. Stability of generalized tensor Sylvester equation is analysed and the results are applied in the solvability and stability of a tensor dynamical system.

1 Introduction

Sylvester equation which is used in mathematics and in applied sciences [6], is a matrix equation of the form

$$AX + XB = C,$$

with A, B square matrices of size n, m respectively and X, C are $n \times m$ matrices. When A and $(-B)$ have no common eigenvalues, this equation has a unique solution for X.

Well-known notions for matrices are generalized to tensors that have application in mechanics, electrodynamics etc.; see [9, 18]. Tensors, denoted by Euler script letters $\mathcal{A} := (\mathcal{A}_{i_1 i_2 \cdots i_m})$ is a multi-array of entries where $\mathcal{A}_{i_1 i_2 \cdots i_m} \in \mathbb{F}$, $i_j = 1, 2, \ldots, n_j$ for $j = 1, 2, \ldots, m$ and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Here m is the order and $(n_1, n_2, \ldots, n_m)$ is the dimension of the tensor $\mathcal{A}$. If $n_1 = n_2 = \cdots = n_m = n$, then $\mathcal{A}$ is called an m^{th} order n-dimensional tensor. The set of all m^{th} order n-dimensional complex tensors is denoted by $CT_{m,n}$. If m is even, then $\mathcal{A}$ is an even order tensor. An $n \times n$ matrix is a 2-order n-dimensional tensor. The following operations and notions on tensors are used subsequently.

Definition 1.1. ([4]) Let $\mathcal{A}, \mathcal{B} \in CT_{2m,n}$. The *Einstein product* of $\mathcal{A}$ and $\mathcal{B}$ denoted by $\mathcal{A} \star \mathcal{B}$ is a tensor in $CT_{2m,n}$ defined by

$$(\mathcal{A} \star \mathcal{B})_{i_1 \cdots i_m j_1 \cdots j_m} = \sum_{k_1, \ldots, k_m = 1}^{n} \mathcal{A}_{i_1 \cdots i_m k_1 \cdots k_m} \mathcal{B}_{k_1 \cdots k_m j_1 \cdots j_m}.$$

$(CT_{2m,n}, \star)$ is a unital algebra; see [2]. The *zero tensor* and *identity tensor* in $CT_{2m,n}$ is denoted by $\mathcal{O}_{2m,n}$ and $\mathcal{I}_{2m,n}$ respectively, where $\mathcal{I}_{2m,n} = \delta_{i_1 j_1} \cdots \delta_{i_m j_m}$ ($\delta_{ij} = 1$ if $i = j, \delta_{ij} = 0$ if $i \neq j$). The *inverse* of $\mathcal{A}$ is denoted by $\mathcal{A}^{-1}$. The *transpose* of $\mathcal{A} = (\mathcal{A}_{i_1 \cdots i_m j_1 \cdots j_m})$ is defined by $\mathcal{A}^T = (\mathcal{A}_{j_1 \cdots j_m i_1 \cdots i_m})$, and the *conjugate transpose* by $\mathcal{A}^* = (\overline{\mathcal{A}}_{j_1 \cdots j_m i_1 \cdots i_m})$. It is true that

$(\mathcal{A}^*)^* = \mathcal{A}$ and $(\mathcal{A} \star \mathcal{B})^* = \mathcal{B}^* \star \mathcal{A}^*$. $\mathcal{A}$ is called *S-normal* if $\mathcal{A} \star \mathcal{A}^* = \mathcal{A}^* \star \mathcal{A}$, $\mathcal{A}$ is called *S-unitary* if $\mathcal{A} \star \mathcal{A}^* = \mathcal{I}_{2m,n} = \mathcal{A}^* \star \mathcal{A}$, and $\mathcal{A}$ is called *S-self adjoint* if $\mathcal{A} = \mathcal{A}^*$.

The elements in $\{\mathcal{A}_{i_1\cdots i_m i_1\cdots i_m} : 1 \le i_j \le n, 1 \le j \le m\}$ are called *S-diagonal* entries of $\mathcal{A}$. $\mathcal{A}$ is *S-diagonal* if $\mathcal{A}_{i_1\cdots i_m j_1\cdots j_m} = 0$ whenever $i_k \ne j_k$ for some $1 \le k \le m$. $\mathcal{A}$ is *S-upper triangular* if $\mathcal{A}_{i_1\cdots i_m j_1\cdots j_m} = 0$ whenever $i_k \ge j_k$ for all $1 \le k \le m$, and $i_p \ne j_p$ for some $1 \le p \le m$.

Remark 1.2. (i) (*Diagonalization of normal tensors*, [13]) Let $\mathcal{A} \in CT_{2m,n}$ be S-normal. Then $\mathcal{A} = \mathcal{U} \star \mathcal{D} \star \mathcal{U}^*$ for some $\mathcal{U}, \mathcal{D} \in CT_{2m,n}$ with $\mathcal{U}$ S-unitary and $\mathcal{D}$ S-diagonal with entries as $S\sigma(\mathcal{A})$.

(ii) (*Schur decomposition of tensors*, [13]) Let $\mathcal{A} \in CT_{2m,n}$. Then $\mathcal{A} = \mathcal{U} \star \mathcal{T} \star \mathcal{U}^*$ for some $\mathcal{U}, \mathcal{T} \in CT_{2m,n}$ with $\mathcal{U}$ S-unitary and $\mathcal{T}$ S-upper triangular with S-diagonal entries as $S\sigma(\mathcal{A})$.

(iii) (*Singular value decomposition of tensors*, [2, 20]) Let $\mathcal{A} \in CT_{2m,n}$. Then $\mathcal{A} = \mathcal{U} \star \mathcal{D} \star \mathcal{V}^*$ for some $\mathcal{U}, \mathcal{V}, \mathcal{D} \in CT_{2m,n}$ with $\mathcal{U}$, $\mathcal{V}$ S-unitary and $\mathcal{D}$ S-diagonal with entries as S-singular values.

In [10], norm of an even order tensor is defined as follows.
For $\mathcal{A} = (\mathcal{A}_{i_1\cdots i_m j_1\cdots j_m}) \in CT_{2m,n}$, the map $\Phi : CT_{m,n} \to CT_{m,n}$ defined by

$$(\Phi(\mathcal{X}))_{i_1\cdots i_m} = \sum_{j_1,\ldots,j_m=1}^{n} \mathcal{A}_{i_1\cdots i_m j_1\cdots j_m} \mathcal{X}_{j_1\cdots j_m}, \quad 1 \le i_k \le n,\ 1 \le k \le m,$$

is linear. Further define $\mathcal{A}\mathcal{X} = \Phi(\mathcal{X})$ and $\|\mathcal{A}\| = \|\Phi\|$ where

$$\|\Phi\| = \sup\{\|\Phi(\mathcal{X})\|_F : \mathcal{X} \in CT_{m,n}, \|\mathcal{X}\|_F \le 1\},$$

and the Frobenius norm as, $\|\mathcal{X}\|_F := \sqrt{\sum_{i_1,\ldots,i_m=1}^{n} |\mathcal{X}_{i_1\cdots i_m}|^2}$.

Here we define an operation that takes an even order tensor to a tensor with half of its order.

Definition 1.3. Let $\mathcal{X} \in CT_{2m,n}$. Then $ten_{1/2}(\mathcal{X}) \in CT_{m,n^2}$ is defined as

$$ten_{1/2}(\mathcal{X})_{[i_1 i_{m+1}][i_2 i_{m+2}]\ldots[i_m i_{2m}]} = \mathcal{X}_{i_1 i_2 \ldots i_{2m}}$$

Note that $\|ten_{1/2}(\mathcal{X})\|_F = \|\mathcal{X}\|_F$.

Matrix equations involving Kronecker product and Kronecker sum of matrices appears in mathematics and theoretical physics; see [6, 8]. As a generalization, tensor Kronecker product is defined in [1] to study a particular singular value decomposition (TKPSVD). The following is the definition for tensor Kronecker product.

Definition 1.4. ([1, 17]) Let $\mathcal{B} = (\mathcal{B}_{i_{m+1}\cdots i_{2m}})$ and $\mathcal{C} = (\mathcal{C}_{i_1\cdots i_m}) \in CT_{m,n}$. The *tensor Kronecker product* of $\mathcal{B}$ and $\mathcal{C}$ is denoted by $\mathcal{B} \otimes \mathcal{C}$, is an element CT_{m,n^2} where

$$\mathcal{B} \otimes \mathcal{C}_{[i_1 i_{m+1}][i_2 i_{m+2}]\cdots[i_m i_{2m}]} = \mathcal{B}_{i_{m+1}\cdots i_{2m}} \mathcal{C}_{i_1\cdots i_m}.$$

The following basic properties of the tensor Kronecker product are used subsequently.

Remark 1.5. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in CT_{m,n}$ and $\alpha \in \mathbb{C}$. Then

(i) $(\alpha\mathcal{A}) \otimes \mathcal{B} = \mathcal{A} \otimes (\alpha\mathcal{B}) = \alpha(\mathcal{A} \otimes \mathcal{B})$.

(ii) $(\mathcal{A} + \mathcal{B}) \otimes \mathcal{C} = (\mathcal{A} \otimes \mathcal{C}) + (\mathcal{B} \otimes \mathcal{C})$.

(iii) $\mathcal{A} \otimes (\mathcal{B} + \mathcal{C}) = (\mathcal{A} \otimes \mathcal{B}) + (\mathcal{A} \otimes \mathcal{C})$.

Proposition 1.6. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in CT_{2m,n}$. The following are true.*

(i) $(\mathcal{A} \otimes \mathcal{B}) \star (\mathcal{C} \otimes \mathcal{D}) = (\mathcal{A} \star \mathcal{C}) \otimes (\mathcal{B} \star \mathcal{D})$.

(ii) $(\mathcal{A}\otimes\mathcal{B})^* = \mathcal{A}^*\otimes\mathcal{B}^*$.

(iii) $(\mathcal{A}\otimes\mathcal{B})^{-1} = \mathcal{A}^{-1}\otimes\mathcal{B}^{-1}$ *for every* $\mathcal{A},\mathcal{B}$ *invertible.*

(iv) $\|\mathcal{A}\otimes\mathcal{B}\|_2 = \|\mathcal{A}\|_2\|\mathcal{B}\|_2$.

This paper introduces the Kronecker sum of even order tensors as follows.

Definition 1.7. Let $\mathcal{B} = (\mathcal{B}_{i_{2m+1}\cdots i_{4m}}), \mathcal{C} = (\mathcal{C}_{i_1\cdots i_{2m}}) \in CT_{2m,n}$. The *tensor Kronecker sum* of $\mathcal{B}$ and $\mathcal{C}$ is an element in CT_{2m,n^2} denoted as $\mathcal{B}\oplus\mathcal{C}$ and is defined as

$$\mathcal{B}\oplus\mathcal{C} = \mathcal{I}_{2m,n}\otimes\mathcal{B} + \mathcal{C}\otimes\mathcal{I}_{2m,n}.$$

Tensors have different types of eigenvalues. Based on a constrained variational approach, eigenvalues, eigenvectors, singular values, and singular vectors for tensors are proposed in [12]. Eigenvalues, H-eigenvalues, E-eigenvalues, and Z-eigenvalues of tensors are introduced in [18]. S-eigenvalues for even order tensors are discussed in [4, 18]. In [4], L.-B Cui et al. analyzed the square matrix unfolding of even-order tensors. They studied the S-spectrum to relate it to higher-order singular value decomposition (HOSVD) of even-order tensors.

Definition 1.8. ([4]) Let $\mathcal{A} = (\mathcal{A}_{i_1\cdots i_m j_1\cdots j_m}) \in CT_{2m,n}$. The set of all *S-eigenvalues* of $\mathcal{A}$ is denoted by $S\sigma(\mathcal{A})$ and defined by

$$S\sigma(\mathcal{A}) = \{z\in\mathbb{C} : \mathcal{A}\mathcal{X} = z\mathcal{X} \text{ for some non-zero } \mathcal{X}\in CT_{m,n}\}.$$

If $\mathcal{A}\mathcal{X} = z\mathcal{X}$ for some non-zero $\mathcal{X}$, then z is an *S-eigenvalue* of $\mathcal{A}$ and $\mathcal{X}$ is the corresponding *S-eigentensor*. The square root of the S-eigenvalues of $\mathcal{A}^*\star\mathcal{A}$ are *S-singular values* of $\mathcal{A}$, denoted by $s_1(\mathcal{A}) \le s_2(\mathcal{A}) \le \cdots \le s_{n^m}(\mathcal{A})$; see [4]. The maximum and minimum S-singular values of $\mathcal{A}$ are $\|\mathcal{A}\|$ and $\left\|\mathcal{A}^{-1}\right\|^{-1}$; see [16].

The pseudospectrum was introduced as a generalization of eigenvalues for the stability analysis of linear equations. The pseudospectra demonstrates a more practical approach than the spectra when analyzing the norm behavior of non-normal matrices.

Definition 1.9. Let $A \in CT_{2,n}$ and $\epsilon > 0$. The *ϵ-pseudospectrum* of A is defined by

$$\Lambda_\epsilon(A) = \sigma(A) \cup \left\{z\in\mathbb{C} : \left\|(A - zI)^{-1}\right\| \ge \epsilon^{-1}\right\}.$$

The concept of pseudospectra, initially rooted in matrices and bounded operators in Hilbert spaces, has been extended to include operators in Banach spaces and of unital Banach algebras; see [7, 11, 21]. In [3], Liqun Qi et al. defined the pseudospectra of a general tensor using unstructured perturbation bounds in the Frobenius norm. The following definition is a generalization of S-spectrum and hence called S-pseudospectrum. The definition for S-pseudospectra of even order tensors we are using here is entirely different from the one studied in [3]. In [10], S-pseudospectra of even order tensors and its Kronecker product are studied in detail. The properties of S-spectra and S-pseudospectra of Kronecker product used in this article are from [10].

Definition 1.10. Let $\mathcal{A} \in CT_{2m,n}$ and $\epsilon > 0$. The *S-pseudospectrum* of $\mathcal{A}$ for ϵ is denoted by $S\Lambda_\epsilon(\mathcal{A})$ and defined as

$$S\Lambda_\epsilon(\mathcal{A}) = \{z\in S\sigma(\mathcal{A}+\mathcal{E}) : \ \mathcal{E}\in CT_{2m,n}, \|\mathcal{E}\| \le \epsilon\}.$$

Remark 1.11. Let $\mathcal{A} \in CT_{2m,n}$ and $\epsilon > 0$. Then the following are equivalent

(i) $z \in S\Lambda_\epsilon(\mathcal{A})$.

(ii) $\|(\mathcal{A} - z\mathcal{I}_{2m,n})\mathcal{X}\|_F \le \epsilon\|\mathcal{X}\|_F$ for some non-zero $\mathcal{X}\in CT_{m,n}$.

(iii) $z \in S\sigma(\mathcal{A}) \cup \left\{z\in\mathbb{C} : \left\|(\mathcal{A} - z\mathcal{I}_{2m,n})^{-1}\right\| \ge \epsilon^{-1}\right\}$.

(iv) $S\nu_{\min}(\mathcal{A} - z\mathcal{I}_{2m,n}) \le \epsilon$.

For $z_0 \in \mathbb{C}$ and $r > 0$, define $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$. For $\Omega_1, \Omega_2 \subseteq \mathbb{C}$, the sum of two sets we consider here is the Minkowski sum denoted by $\Omega_1 + \Omega_2$ which is the set of sums of an element of Ω_1 and an element of Ω_2.

Proposition 1.12. *Let $\mathcal{A} \in CT_{2m,n}$ and $\epsilon > 0$. Then*

(i) $S\sigma(\mathcal{A}) + D(0, \epsilon) \subseteq S\Lambda_\epsilon(\mathcal{A})$.

(ii) *$\mathcal{A}$ is S-normal if and only if* $S\Lambda_\epsilon(\mathcal{A}) = S\sigma(\mathcal{A}) + D(0, \epsilon)$.

(iii) $S\Lambda_\epsilon(\mathcal{A} + c\mathcal{I}_{2m,n}) = c + S\Lambda_\epsilon(\mathcal{A})$ *for every* $c \in \mathbb{C}$.

(iv) $S\Lambda_\epsilon(c\mathcal{A}) = c\, S\Lambda_{\frac{\epsilon}{|c|}}(\mathcal{A})$ *for every nonzero* $c \in \mathbb{C}$.

We develop certain results on S-eigenvalues and S-pseudospectra of the Kronecker sum of even order tensors. For certain non-normal even order tensors $\mathcal{A}$ and $\mathcal{B}$, it is possible for the Kronecker sum $\mathcal{A} \oplus \mathcal{B}$ to be non-normal. Consequently, the norm behavior of non-normal tensors with larger dimensions which is a Kronecker sum, can be determined by examining the S-pseudospectra of the input tensors, which possess comparably smaller dimensions. This analysis becomes particularly relevant when establishing a relationship between the S-pseudospectra of the participating tensors and their Kronecker sum. Different engineering and applied sciences phenomena are modeled into multilinear systems and tensor equations. The relationship developed are used to analyze a tensor dynamical system through generalized tensor Sylvester equation.

We define a generalized tensor Sylvester equation as an equation of the form

$$f(\mathcal{A}) \star \mathcal{X} + \mathcal{X} \star g(\mathcal{B}) = \mathcal{C} \tag{1.1}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{C} \in CT_{2m,n}$ and f, g are analytic functions on some open sets containing $S\sigma(\mathcal{A}), S\sigma(\mathcal{B})$ respectively. If f and g are identity maps, then the generalized tensor Sylvester equation becomes $\mathcal{A} \star \mathcal{X} + \mathcal{X} \star \mathcal{B} = C$, the usual tensor Sylvester equation. Whenever $\mathcal{A}, \mathcal{B} \in CT_{2,n}$, the usual tensor Sylvester equation becomes the well-known matrix Sylvester equation.

The following is an outline of the article.

In section 2, we develop various properties on the S-pseudospectra of the Kronecker sum of even order tensors. In section 3, an analog of the spectral mapping theorem for the S-pseudospectrum of the Kronecker sum is proved. In section 4, the results are used to analyze the solvability and stability of the generalized tensor Sylvester equation. We use these theory and results to study the solvability and stability of a tensor dynamical system.

2 Pseudospectra for the Kronecker sum of even order tensors

A permutation tensor in $CT_{2m,n}$ is a tensor obtained from the identity tensor $\mathcal{I}_{2m,n} = \mathcal{I}_{i_1 \ldots i_m i_{m+1} \ldots i_{2m}}$ by permuting its first m indices over the n^m choices.

Remark 2.1. Let $\mathcal{A}, \mathcal{B} \in CT_{2m,n}$. Then

(i) $S\sigma(\mathcal{A} \oplus \mathcal{B}) = \{\lambda + \mu : \lambda \in S\sigma(\mathcal{A}), \mu \in S\sigma(\mathcal{B})\}$.

(ii) If $\mathcal{X}_1, \mathcal{X}_2 \in CT_{m,n}$ are the S-eigentensors of $\mathcal{A}$ and $\mathcal{B}$ corresponding to the S-eigenvalues λ and μ, then $\mathcal{X}_2 \otimes \mathcal{X}_1$ is an S-eigentensor of $\mathcal{A} \oplus \mathcal{B}$ corresponding to the S-eigenvalue $\lambda + \mu$.

Proposition 2.2. *Let $\mathcal{A}, \mathcal{B} \in CT_{2m,n}$ and $\epsilon > 0$. Then* $S\Lambda_\epsilon(\mathcal{A} \oplus \mathcal{B}) = S\Lambda_\epsilon(\mathcal{B} \oplus \mathcal{A})$.

Proof. Let $\mathcal{A}, \mathcal{B} \in CT_{2m,n}$, then there exists permutation tensor $\mathcal{P} \in CT_{2m,n^2}$ such that $\mathcal{P} \star (\mathcal{A} \otimes \mathcal{B}) \star \mathcal{P}^T = \mathcal{B} \otimes \mathcal{A}$. Thus

$$\begin{aligned} \mathcal{A} \oplus \mathcal{B} &= \mathcal{I}_{2m,n} \otimes \mathcal{A} + \mathcal{B} \otimes \mathcal{I}_{2m,n} \\ &= \mathcal{P} \star (\mathcal{A} \otimes \mathcal{I}_{2m,n}) \star \mathcal{P}^T + \mathcal{P} \star (\mathcal{I}_{2m,n} \otimes \mathcal{B}) \star \mathcal{P}^T \\ &= \mathcal{P} \star (\mathcal{B} \oplus \mathcal{A}) \star \mathcal{P}^T. \end{aligned}$$

Hence for $\epsilon > 0$,

$$S\Lambda_\epsilon(\mathcal{A} \oplus \mathcal{B}) = S\Lambda_\epsilon(\mathcal{P} \star (\mathcal{B} \oplus \mathcal{A}) \star \mathcal{P}^T) = S\Lambda_\epsilon(\mathcal{B} \oplus \mathcal{A}).$$

□

Theorem 2.3. *Let $\mathcal{A} \in CT_{2m,n}$ and $\epsilon > 0$. Then*

(i) $S\Lambda_\epsilon(d_1\mathcal{I}_{2m,n} \oplus d_2\mathcal{I}_{2m,n}) = D(d_1 + d_2, \epsilon)$ *for every* $d_1, d_2 \in \mathbb{C}$.

(ii) $S\Lambda_\epsilon(\mathcal{A} \oplus d\mathcal{I}_{2m,n}) = \{d\} + S\Lambda_\epsilon(\mathcal{A})$ *for every* $d \in \mathbb{C}$.

(iii) $S\Lambda_\epsilon(\mathcal{A} \oplus \mathcal{E}) = \bigcup_{i=1}^{n^m} (\{e_i\} + S\Lambda_\epsilon(\mathcal{A}))$ *for every S-diagonal tensor* $\mathcal{E} \in CT_{2m,n}$ *with diagonal entries* $\{e_1, \ldots, e_{n^m}\}$.

(iv) $S\Lambda_\epsilon(\mathcal{A} \oplus \mathcal{N}) = \bigcup_{i=1}^{n^m} (\{\lambda_i\} + S\Lambda_\epsilon(\mathcal{A}))$ *for every S-normal tensor* $\mathcal{N} \in CT_{2m,n}$ *with* $S\sigma(\mathcal{N}) = \{\lambda_1, \ldots, \lambda_{n^m}\}$.

Proof. (i) Let $d_1, d_2 \in \mathbb{C}$ and $\epsilon > 0$, then

$$\begin{aligned} S\Lambda_\epsilon(d_1\mathcal{I}_{2m,n} \oplus d_2\mathcal{I}_{2m,n}) &= S\Lambda_\epsilon(\mathcal{I}_{2m,n} \otimes d_1\mathcal{I}_{2m,n} + d_2\mathcal{I}_{2m,n} \otimes \mathcal{I}_{2m,n}) \\ &= S\Lambda_\epsilon((d_1 + d_2)\mathcal{I}_{2m,n^2}) \\ &= \{z \in \mathbb{C} : |z - (d_1 + d_2)| \le \epsilon\}. \end{aligned}$$

(ii) Let $\mathcal{A} \in CT_{2m,n}$, $d \in \mathbb{C}$ and $\epsilon > 0$. Then

$$\begin{aligned} \Lambda_\epsilon(\mathcal{A} \oplus d\mathcal{I}_{2m,n}) &= S\Lambda_\epsilon(\mathcal{I}_{2m,n} \otimes \mathcal{A} + d\mathcal{I}_{2m,n} \otimes \mathcal{I}_{2m,n}) \\ &= S\Lambda_\epsilon(\mathcal{I}_{2m,n} \otimes (\mathcal{A} + d\mathcal{I}_{2m,n})) = S\Lambda_\epsilon(\mathcal{A} + d\mathcal{I}_{2m,n}). \end{aligned}$$

(iii) Let $\mathcal{E}$ be S-diagonal with diagonal entries $\{e_1, \ldots, e_{n^m}\}$, then $\mathcal{A} \oplus \mathcal{E}$ becomes block S-diagonal. Thus

$$\begin{aligned} S\Lambda_\epsilon(\mathcal{A} \oplus \mathcal{E}) &= \{z \in \mathbb{C} : s_1(z\mathcal{I}_{2m,n^2} - (\mathcal{A} \oplus \mathcal{E})) \le \epsilon\} \\ &= \left\{z \in \mathbb{C} : \min\left(s_1((z - e_1)\mathcal{I}_{2m,n} - \mathcal{A}), \ldots, s_1((z - e_{n^m})\mathcal{I}_{2m,n} - \mathcal{A})\right) \le \epsilon\right\} \\ &= \bigcup_{i=1}^{n^m} S\Lambda_\epsilon(e_i\mathcal{I}_{2m,n} + \mathcal{A}) = \bigcup_{i=1}^{n^m} (\{e_i\} + S\Lambda_\epsilon(\mathcal{A})). \end{aligned}$$

(iv) Let $\mathcal{N}$ be S-normal, then $\mathcal{N} = \mathcal{U} \star \mathcal{E} \star \mathcal{U}^*$ for some $\mathcal{U}$ S-unitary and $\mathcal{E}$ S-diagonal with diagonal entries $\{\lambda_1, \ldots \lambda_{n^m}\}$. Thus

$$\begin{aligned} &S\Lambda_\epsilon(\mathcal{A} \oplus \mathcal{N}) = S\Lambda_\epsilon(\mathcal{A} \oplus \mathcal{U} \star \mathcal{E} \star \mathcal{U}^*) = \left\{z : \left\|\left(z\mathcal{I}_{2m,n^2} - (\mathcal{A} \oplus \mathcal{U} \star \mathcal{E} \star \mathcal{U}^*)\right)^{-1}\right\| \ge \epsilon^{-1}\right\} \\ &= \left\{z : \left\|\left((z\mathcal{I}_{2m,n} - \mathcal{U} \star \mathcal{E} \star \mathcal{U}^*) \otimes \mathcal{I}_{2m,n} - \mathcal{I}_{2m,n} \otimes \mathcal{A}\right)^{-1}\right\| \ge \epsilon^{-1}\right\} \\ &= \left\{z : \left\|\left(\mathcal{U} \star (z\mathcal{I}_{2m,n} - \mathcal{E}) \star \mathcal{U}^* \otimes \mathcal{I}_{2m,n} - \mathcal{U} \star \mathcal{I}_{2m,n} \star \mathcal{U}^* \otimes \mathcal{A}\right)^{-1}\right\| \ge \epsilon^{-1}\right\} \\ &= \left\{z : \left\|\left((\mathcal{U} \otimes \mathcal{I}_{2m,n}) \star \left(((z\mathcal{I}_{2m,n} - \mathcal{E}) \otimes \mathcal{I}_{2m,n}) - (\mathcal{I}_{2m,n} \otimes \mathcal{A})\right) \star (\mathcal{U} \otimes \mathcal{I}_{2m,n})^*\right)^{-1}\right\| \ge \epsilon^{-1}\right\} \\ &= \left\{z : \left\|\left((z\mathcal{I}_{2m,n} - \mathcal{E}) \otimes \mathcal{I}_{2m,n} - \mathcal{I}_{2m,n} \otimes \mathcal{A}\right)^{-1}\right\| \ge \epsilon^{-1}\right\} \\ &= \left\{z : \left\|\left(z\mathcal{I}_{2m,n} - (\mathcal{A} \oplus \mathcal{E})\right)^{-1}\right\| \ge \epsilon^{-1}\right\} \\ &= S\Lambda_\epsilon(\mathcal{A} \oplus \mathcal{E}) = \bigcup_{i=1}^{n^m} S\Lambda_\epsilon(\lambda_i\mathcal{I}_{2m,n} + \mathcal{A}) = \bigcup_{i=1}^{n^m} (\{\lambda_i\} + S\Lambda_\epsilon(\mathcal{A})). \end{aligned}$$

□

Lemma 2.4. *Let $\mathcal{A}, \mathcal{B} \in CT_{2m,n}$ and $\epsilon > 0$. Further let $\mathcal{A} = \mathcal{U} \star \mathcal{T}_\mathcal{A} \star \mathcal{U}^*$ and $\mathcal{B} = \mathcal{V} \star \mathcal{T}_\mathcal{B} \star \mathcal{V}^*$ be the Schur decomposition of $\mathcal{A}$ and $\mathcal{B}$. Then $S\Lambda_\epsilon(\mathcal{A} \oplus \mathcal{B}) = S\Lambda_\epsilon(\mathcal{T}_\mathcal{A} \oplus \mathcal{T}_\mathcal{B})$.*

Proof. Let $\mathcal{A}, \mathcal{B} \in CT_{2m,n}$ and $\epsilon > 0$. Then

$$\begin{aligned} S\Lambda_\epsilon(\mathcal{A} \oplus \mathcal{B}) &= S\Lambda_\epsilon(\mathcal{U} \star \mathcal{T}_\mathcal{A} \star \mathcal{U}^* \oplus \mathcal{V} \star \mathcal{T}_\mathcal{B} \star \mathcal{V}^*) \\ &= S\Lambda_\epsilon(\mathcal{I}_{2m,n} \otimes \mathcal{U} \star \mathcal{T}_\mathcal{A} \star \mathcal{U}^* + \mathcal{V} \star \mathcal{T}_\mathcal{B} \star \mathcal{V}^* \otimes \mathcal{I}_{2m,n}) \\ &= S\Lambda_\epsilon\left((\mathcal{V} \otimes \mathcal{U}) \star (\mathcal{I}_{2m,n} \otimes \mathcal{T}_\mathcal{A} + \mathcal{T}_\mathcal{B} \otimes \mathcal{I}_{2m,n}) \star (\mathcal{V} \otimes \mathcal{U})^*\right). \end{aligned}$$

Since $\mathcal{U}$ and $\mathcal{V}$ are S-unitary, so is $\mathcal{V} \otimes \mathcal{U}$ and hence $S\Lambda_\epsilon(\mathcal{A} \oplus \mathcal{B}) = S\Lambda_\epsilon(\mathcal{T}_\mathcal{A} \oplus \mathcal{T}_\mathcal{B})$. □

Theorem 2.5. *Let $\mathcal{A}, \mathcal{B} \in CT_{2m,n}$ and $\epsilon_1, \epsilon_2 > 0$. Then*

$$S\Lambda_{\epsilon_1}(\mathcal{A}) + S\Lambda_{\epsilon_2}(\mathcal{B}) \subseteq S\Lambda_{\epsilon_1+\epsilon_2}(\mathcal{A} \oplus \mathcal{B}).$$

Proof. Let $\lambda_1 \in S\Lambda_{\epsilon_1}(\mathcal{A})$ and $\lambda_2 \in S\Lambda_{\epsilon_2}(\mathcal{B})$ for some $\epsilon_1, \epsilon_2 > 0$. There exist $\mathcal{X}_1, \mathcal{X}_2 \in CT_{m,n}$, such that $\|(\lambda_1 \mathcal{I}_{2m,n} - \mathcal{A})\mathcal{X}_1\| \leq \epsilon_1 \|\mathcal{X}_1\|$ and $\|(\lambda_2 \mathcal{I}_{2m,n} - \mathcal{B})\mathcal{X}_2\| \leq \epsilon_2 \|\mathcal{X}_2\|$. Further,

$$\begin{aligned}
&\big\|\big((\lambda_1+\lambda_2)\mathcal{I}_{2m,n^2} - (\mathcal{A} \oplus \mathcal{B})\big)(\mathcal{X}_2 \otimes \mathcal{X}_1)\big\| = \big\|\big((\lambda_1 \mathcal{I}_{2m,n} - \mathcal{A}) \oplus (\lambda_2 \mathcal{I}_{2m,n} - \mathcal{B})\big)(\mathcal{X}_2 \otimes \mathcal{X}_1)\big\| \\
&= \big\|\big(\mathcal{I}_{2m,n} \otimes (\lambda_1 \mathcal{I}_{2m,n} - \mathcal{A})\big)(\mathcal{X}_2 \otimes \mathcal{X}_1) + \big((\lambda_2 \mathcal{I}_{2m,n} - \mathcal{B}) \otimes \mathcal{I}_{2m,n}\big)(\mathcal{X}_2 \otimes \mathcal{X}_1)\big\| \\
&= \|\mathcal{X}_2 \otimes (\lambda_1 \mathcal{I}_{2m,n} - \mathcal{A})\mathcal{X}_1 + (\lambda_2 \mathcal{I}_{2m,n} - \mathcal{B})\mathcal{X}_2 \otimes \mathcal{X}_1\| \\
&\leq \|\mathcal{X}_2 \otimes (\lambda_1 \mathcal{I}_{2m,n} - \mathcal{A})\mathcal{X}_1\| + \|(\lambda_2 \mathcal{I}_{2m,n} - \mathcal{B})\mathcal{X}_2 \otimes \mathcal{X}_1\| \\
&\leq (\epsilon_1 + \epsilon_2)\|\mathcal{X}_2 \otimes \mathcal{X}_1\|.
\end{aligned}$$

□

The following example shows that the inclusion in Theorem 2.5 may be proper.

Example 2.6. Consider $\mathcal{A} = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}$ and $\mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $CT_{2,2}$. Then

$$\mathcal{A} \oplus \mathcal{B} = \begin{bmatrix} 1 & k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{I}_{2,2} + \mathcal{A} & \\ & \mathcal{I}_{2,2} + \mathcal{A} \end{bmatrix}.$$

Let $\epsilon > 0$. From proposition 2.4 of [5],

$$S\Lambda_\epsilon(\mathcal{A}) + S\Lambda_\epsilon(\mathcal{B}) = D(1, \epsilon) + S\Lambda_\epsilon(\mathcal{A}) = D(1, \epsilon + \sqrt{\epsilon^2 + k\epsilon}),$$

and

$$S\Lambda_{2\epsilon}(\mathcal{A} \oplus \mathcal{B}) = S\Lambda_{2\epsilon}(I_{2,2} + \mathcal{A}) = \{1\} + S\Lambda_{2\epsilon}(\mathcal{A}) = D(1, \sqrt{4\epsilon^2 + 2k\epsilon}).$$

So the inclusion in Theorem 2.5 is proper in this case.

3 Pseudospectral mapping theorem for the Kronecker sum of even order tensors

The spectral mapping theorem is a fundamental result in operator theory; see [19]. An analog of the spectral mapping theorem for pseudospectrum of matrices is proved in [14, 15]. This section proves the spectral and pseudospectral mapping theorem for the Kronecker sum of even order tensors. The usual mapping theorems for tensors become a particular case of this result.

Theorem 3.1. *If $\mathcal{A} \in CT_{2m,n}$ and f be an analytic function on an open set containing $S\sigma(\mathcal{A})$, then $f(S\sigma(\mathcal{A})) = S\sigma(f(\mathcal{A}))$.*

Theorem 3.2. *Let $\mathcal{A}, \mathcal{B} \in CT_{2m,n}$, f be an analytic function defined on an open set containing $S\sigma(\mathcal{A})$, and g be an analytic function defined on an open set containing $S\sigma(\mathcal{B})$. Then*

$$S\sigma(f(\mathcal{A}) \oplus g(\mathcal{B})) = f(S\sigma(\mathcal{A})) + g(S\sigma(\mathcal{B})).$$

Proof. From Remark 2.1 and Theorem 3.1,

$$S\sigma(f(\mathcal{A}) \oplus g(\mathcal{B})) = S\sigma(f(\mathcal{A})) + S\sigma(g(\mathcal{B})) = f(S\sigma(\mathcal{A})) + g(S\sigma(\mathcal{B})).$$

□

The analog of spectral mapping theorem for pseudospectrum is not valid, i.e., there exists $\mathcal{A} \in CT_{2,n}$ and an analytic function f on an open set containing $S\sigma(\mathcal{A})$ such that $S\Lambda_\epsilon(f(\mathcal{A})) \neq f(S\Lambda_\epsilon(\mathcal{A}))$ for some $\epsilon > 0$; see [14]. Hence, the analog of the spectral mapping theorem for the S-pseudospectrum of Kronecker sum is also untrue.

Theorem 3.3. *Let $\mathcal{A}, \mathcal{B} \in CT_{2m,n}$, f an analytic function defined on an open set containing $S\sigma(\mathcal{A})$, and g an analytic function defined on an open set containing $S\sigma(\mathcal{B})$. For $\epsilon > 0$, define*

$$\phi(\epsilon) = \sup_{\substack{\lambda \in S\Lambda_\epsilon(\mathcal{A}) \\ \mu \in S\Lambda_\epsilon(\mathcal{B})}} \inf \{r > 0 : \ f(\lambda) + g(\mu) \in S\Lambda_r(f(\mathcal{A}) \oplus g(\mathcal{B}))\}.$$

Then ϕ is well defined, $\lim_{\epsilon \to 0} \phi(\epsilon) = 0$, and $f(S\Lambda_\epsilon(\mathcal{A})) + g(S\Lambda_\epsilon(\mathcal{B})) \subseteq S\Lambda_{\phi(\epsilon)}(f(\mathcal{A}) \oplus g(\mathcal{B}))$.

Proof. Let $\lambda \in S\Lambda_\epsilon(\mathcal{A})$ and $\mu \in S\Lambda_\epsilon(\mathcal{B})$ for some $\epsilon > 0$. Then there exists $\mathcal{E}, \mathcal{F} \in CT_{2m,n}$ with $\|\mathcal{E}\| \leq \epsilon$, $\|\mathcal{F}\| \leq \epsilon$ such that $\lambda \in S\sigma(\mathcal{A}+\mathcal{E})$ and $\mu \in S\sigma(\mathcal{B}+\mathcal{F})$. Thus $f(\lambda) \in f(S\sigma(\mathcal{A}+\mathcal{E})) = S\sigma(f(\mathcal{A}+\mathcal{E})) = S\sigma(f(\mathcal{A}) + \mathcal{E}_1)$ where $\mathcal{E}_1 = f(\mathcal{A}+\mathcal{E}) - f(\mathcal{A})$. Also $g(\mu) \in S\sigma(g(\mathcal{B}+\mathcal{F})) = S\sigma(g(\mathcal{B}) + \mathcal{F}_1)$ where $\mathcal{F}_1 = g(\mathcal{B}+\mathcal{F}) - g(\mathcal{B})$. Thus

$$\begin{aligned} f(\lambda) + g(\mu) &\in S\sigma(f(\mathcal{A}) + \mathcal{E}_1) + S\sigma(g(\mathcal{B}) + \mathcal{F}_1) = S\sigma((f(\mathcal{A}) + \mathcal{E}_1) \oplus (g(\mathcal{B}) + \mathcal{F}_1)) \\ &= S\sigma(I_{2m,n} \otimes (f(\mathcal{A}) + \mathcal{E}_1) + (g(\mathcal{B}) + \mathcal{F}_1) \otimes I_{2m,n}) \\ &= S\sigma((f(\mathcal{A}) \oplus g(\mathcal{B})) + (\mathcal{E}_1 \oplus \mathcal{F}_1)). \end{aligned}$$

Define $\mathcal{E}' := \mathcal{E}_1 \oplus \mathcal{F}_1$. Thus

$$f(\lambda) + g(\mu) \in S\sigma((f(\mathcal{A}) \oplus g(\mathcal{B})) + \mathcal{E}').$$

Suppose $\|\mathcal{E}'\| = r'$. Since r' is independent of λ and μ, $f(\lambda) + g(\mu) \in S\Lambda_{r'}(f(\mathcal{A}) \oplus g(\mathcal{B}))$. Since the infimum in the $\phi(\epsilon)$ definition is taken over a nonempty set, ϕ is well defined. Also as ϵ goes to zero,

$$\phi(\epsilon) = \sup_{\substack{\lambda \in S\Lambda_\epsilon(\mathcal{A}) \\ \mu \in S\Lambda_\epsilon(\mathcal{B})}} \inf \{r > 0 : \ f(\lambda) + g(\mu) \in S\Lambda_r(f(\mathcal{A}) \oplus g(\mathcal{B}))\}$$

tends to,

$$\sup_{\substack{\lambda \in S\sigma(\mathcal{A}) \\ \mu \in S\sigma(\mathcal{B})}} \inf \{r > 0 : \ f(\lambda) + g(\mu) \in S\Lambda_r(f(\mathcal{A}) \oplus g(\mathcal{B}))\}.$$

Thus $\lim_{\epsilon \to 0} \phi(\epsilon) = 0$.

From the definition of $\phi(\epsilon)$, $f(\lambda) + g(\mu) \in S\Lambda_{\phi(\epsilon)}(f(\mathcal{A}) \oplus g(\mathcal{B}))$ whenever $\lambda \in S\Lambda_\epsilon(\mathcal{A})$ and $\mu \in S\Lambda_\epsilon(\mathcal{B})$. Hence $f(S\Lambda_\epsilon(\mathcal{A})) + g(S\Lambda_\epsilon(\mathcal{B})) \subseteq S\Lambda_{\phi(\epsilon)}(f(\mathcal{A}) \oplus g(\mathcal{B}))$. □

Theorem 3.4. *Let $\mathcal{A}, \mathcal{B} \in CT_{2m,n}$, f an analytic function defined on an open set containing $S\sigma(\mathcal{A})$, and g an analytic function defined on an open set containing $S\sigma(\mathcal{B})$. For sufficiently small $\epsilon > 0$, define*

$$\psi(\epsilon) = \sup_{z \in S\Lambda_\epsilon(f(\mathcal{A}) \oplus g(\mathcal{B}))} \inf \{r > 0 : \ z \in f(S\Lambda_r(\mathcal{A})) + g(S\Lambda_r(\mathcal{B}))\}.$$

Then ψ is well defined, $\lim_{\epsilon \to 0} \psi(\epsilon) = 0$, and $S\Lambda_\epsilon(f(\mathcal{A}) \oplus g(\mathcal{B})) \subseteq f(S\Lambda_{\psi(\epsilon)}(\mathcal{A})) + g(S\Lambda_{\psi(\epsilon)}(\mathcal{B}))$.

Proof. Let $z \in S\Lambda_\epsilon(f(\mathcal{A}) \oplus g(\mathcal{B}))$ for some sufficiently small $\epsilon > 0$. Then z lies in the neighborhood of $S\sigma(f(\mathcal{A}) \oplus g(\mathcal{B}))$. Also

$$S\sigma(f(\mathcal{A}) \oplus g(\mathcal{B})) = S\sigma(f(\mathcal{A})) + S\sigma(g(\mathcal{B})) = f(S\sigma(\mathcal{A})) + g(S\sigma(\mathcal{B})) \subseteq f(\Omega_1) + g(\Omega_2),$$

where Ω_1 and Ω_2 are the open sets containing $S\sigma(\mathcal{A})$ and $S\sigma(\mathcal{B})$ such that f, g are analytic on Ω_1 and Ω_2 respectively. Thus $z = f(\lambda) + g(\mu)$ where $\lambda \in \Omega_1$ and $\mu \in \Omega_2$. Hence there exists $s > 0, t > 0$ such that

$$z \in f(S\Lambda_s(\mathcal{A})) + g(S\Lambda_t(\mathcal{B})).$$

Define $r' := \max\{s, t\}$, then $z \in f(S\Lambda_{r'}(\mathcal{A})) + g(S\Lambda_{r'}(\mathcal{B}))$. Hence, $\psi(\epsilon)$ is defined over a nonempty set, and ψ becomes well-defined. Also as ϵ goes to zero,

$$\psi(\epsilon) = \sup_{z \in S\Lambda_\epsilon(f(\mathcal{A}) \oplus g(\mathcal{B}))} \inf \{r > 0 : \ z \in f(S\Lambda_r(\mathcal{A})) + g(S\Lambda_r(\mathcal{B}))\}$$

tends to

$$\sup_{z\in f(S\sigma(\mathcal{A}))+g(S\sigma(\mathcal{B}))} \inf\{r>0:\ z\in f(S\Lambda_r(\mathcal{A}))+g(S\Lambda_r(\mathcal{B}))\}.$$

Thus $\lim_{\epsilon\to 0}\psi(\epsilon)=0$.

Further if $z\in S\Lambda_\epsilon(f(\mathcal{A})\oplus g(\mathcal{B}))$, then $z\in f(S\Lambda_{\psi(\epsilon)}(\mathcal{A}))+g(S\Lambda_{\psi(\epsilon)}(\mathcal{B}))$. Hence $S\Lambda_\epsilon(f(\mathcal{A})\oplus g(\mathcal{B}))\subseteq f(S\Lambda_{\psi(\epsilon)}(\mathcal{A}))+g(S\Lambda_{\psi(\epsilon)}(\mathcal{B}))$. □

Remark 3.5. (i) For $\mathcal{B}=\mathcal{I}_{2m,n}$ and $g(z)=z$, Theorem 3.3 and Theorem 3.4 gives the spectral mapping theorem for even order tensors.

(ii) From Theorem 3.3 and Theorem 3.4,

$$f(S\Lambda_\epsilon(\mathcal{A}))+g(S\Lambda_\epsilon(\mathcal{B}))\subseteq S\Lambda_{\phi(\epsilon)}(f(\mathcal{A})\oplus g(\mathcal{B}))\subseteq f(S\Lambda_{\psi(\phi(\epsilon))}(\mathcal{A}))+g(S\Lambda_{\psi(\phi(\epsilon))}(\mathcal{B})),$$

and

$$S\Lambda_\epsilon(f(\mathcal{A})\oplus g(\mathcal{B}))\subseteq f(S\Lambda_{\psi(\epsilon)}(\mathcal{A}))+g(S\Lambda_{\psi(\epsilon)}(\mathcal{B}))\subseteq S\Lambda_{\phi(\psi(\epsilon))}(f(\mathcal{A})\oplus g(\mathcal{B})).$$

(iii) Since $\phi(0)=0=\psi(0)$,

$$S\sigma(f(\mathcal{A})\oplus g(\mathcal{B}))=f(S\sigma(\mathcal{A}))+g(S\sigma(\mathcal{B})).$$

Thus, the spectral mapping theorem for the Kronecker sum of even order tensors is a particular case of the pseudospectral mapping theorem for the Kronecker sum of even order tensors.

(iv) The functions $\phi(\epsilon)$ and $\psi(\epsilon)$ are optimal because the set inclusions are sharp and any other smaller functions can't replace them.

(v) If $f(z)=g(z)=z$, then for $\epsilon>0$,

$$S\Lambda_\epsilon(\mathcal{A})+S\Lambda_\epsilon(\mathcal{B})\subseteq S\Lambda_{\phi(\epsilon)}(\mathcal{A}\oplus\mathcal{B}) \quad\text{and}\quad S\Lambda_\epsilon(\mathcal{A}\oplus\mathcal{B})\subseteq S\Lambda_{\psi(\epsilon)}(\mathcal{A})+S\Lambda_{\psi(\epsilon)}(\mathcal{B}).$$

(vi) If $g(z)=0$, then

$$\begin{aligned}\phi(\epsilon)&=\sup_{\substack{\lambda\in S\Lambda_\epsilon(\mathcal{A})\\ \mu\in S\Lambda_\epsilon(\mathcal{B})}}\inf\{r>0: f(\lambda)+0\in S\Lambda_r(f(\mathcal{A})\oplus\mathcal{O}_{2m,n})\}\\ &=\sup_{\lambda\in S\Lambda_\epsilon(\mathcal{A})}\inf\{r>0: f(\lambda)\in S\Lambda_r(\mathcal{I}_{2m,n}\otimes f(\mathcal{A}))\}\\ &=\sup_{\lambda\in S\Lambda_\epsilon(\mathcal{A})}\inf\{r>0: f(\lambda)\in S\Lambda_r(f(\mathcal{A}))\}.\end{aligned}$$

$$\begin{aligned}\psi(\epsilon)&=\sup_{w\in S\Lambda_\epsilon(f(\mathcal{A})\oplus\mathcal{O}_{2m,n})}\inf\{r>0:\ w\in f(S\Lambda_r(\mathcal{A}))+0\}\\ &=\sup_{w\in S\Lambda_\epsilon(\mathcal{I}_{2m,n}\otimes f(\mathcal{A}))}\inf\{r>0:\ w\in f(S\Lambda_r(\mathcal{A}))\}\\ &=\sup_{w\in S\Lambda_\epsilon(f(\mathcal{A}))}\inf\{r>0:\ w\in f(S\Lambda_r(\mathcal{A}))\}.\end{aligned}$$

Thus, for $\mathcal{A}\in CT_{2,n}$, the pseudospectral mapping theorem proved in [14] can be deducted from the pseudospectral analog of spectral mapping theorem for Kronecker sum of even order tensors.

(vii) If $f(z)=\alpha+\beta z$ and $g(z)=\gamma$ for some α,β and $\gamma\in\mathbb{C}$, then

$$\begin{aligned}\phi(\epsilon)&=\sup_{\substack{\lambda\in S\Lambda_\epsilon(\mathcal{A})\\ \mu\in S\Lambda_\epsilon(\mathcal{B})}}\inf\{r>0:(\alpha+\beta\lambda)+\gamma\in S\Lambda_r((\alpha\mathcal{I}_{2m,n}+\beta\mathcal{A})\oplus\gamma\mathcal{I}_{2m,n})\}\\ &=\sup_{\lambda\in S\Lambda_\epsilon(\mathcal{A})}\inf\{r>0:((\alpha+\gamma)+\beta\lambda)\in S\Lambda_r((\alpha+\gamma)\mathcal{I}_{2m,n}+\beta(\mathcal{I}_{2m,n}\otimes\mathcal{A})\}\\ &=\sup_{\lambda\in S\Lambda_\epsilon(\mathcal{A})}\inf\{r>0:\lambda\in S\Lambda_{\frac{r}{|\beta|}}(\mathcal{A})\}=\epsilon|\beta|.\end{aligned}$$

$$\psi(\epsilon) = \sup_{z\in S\Lambda_\epsilon((\alpha+\beta\mathcal{A})\oplus\gamma\mathcal{I}_{2m,n})} \inf\{r>0:\ z\in(\alpha+\beta S\Lambda_r(\mathcal{A}))+\gamma\}$$
$$= \sup_{z\in(\alpha+\gamma)+\beta S\Lambda_{\frac{\epsilon}{|\beta|}}(\mathcal{A})} \inf\{r>0:\ z\in(\alpha+\gamma)+\beta S\Lambda_r(\mathcal{A})\} = \frac{\epsilon}{|\beta|}.$$

From (ii) of Remark 3.5, $\{\alpha+\gamma\}+\beta S\Lambda_\epsilon(\mathcal{A}) = S\Lambda_{\epsilon|\beta|}((\alpha+\beta\mathcal{A})\oplus\gamma\mathcal{I}_{2m,n})$.

Corollary 3.6. *Suppose f and g are analytic functions defined on nonempty open sets $\mathcal{D}_1$ and $\mathcal{D}_2$ in the complex plane. If for each $\mathcal{A}\in CT_{2m,n}$ and $\mathcal{B}\in CT_{2m,n}$ satisfying $S\sigma(\mathcal{A})\subseteq\mathcal{D}_1$, $S\sigma(\mathcal{B})\subseteq\mathcal{D}_2$, there exists a non-negative real-valued function $\eta=\eta(\epsilon,\mathcal{A},\mathcal{B})$ such that*

$$f(S\Lambda_\epsilon(\mathcal{A}))+g(S\Lambda_\epsilon(\mathcal{B})) = S\Lambda_{\eta(\epsilon,\mathcal{A},\mathcal{B})}(f(\mathcal{A})\oplus g(\mathcal{B}))$$

for ϵ sufficiently small, then $f+g$ is an affine function.

Proof. Assume that $f(S\Lambda_\epsilon(\mathcal{A}))+g(S\Lambda_\epsilon(\mathcal{B})) = S\Lambda_{\eta(\epsilon,\mathcal{A},\mathcal{B})}(f(\mathcal{A})\oplus g(\mathcal{B}))$ for some $\epsilon,\eta>0$. In particular if $\mathcal{A}=\mathcal{B}=\alpha\mathcal{I}_{2m,n}$ where $\alpha\in\mathbb{C}$, then

$$f(S\Lambda_\epsilon(\alpha\mathcal{I}_{2m,n}))+g(S\Lambda_\epsilon(\alpha\mathcal{I}_{2m,n})) = S\Lambda_{\eta(\epsilon,\alpha)}(f(\alpha\mathcal{I}_{2m,n})\oplus g(\alpha\mathcal{I}_{2m,n})).$$

i.e., $f+g(D(\alpha,\epsilon)) = D(f+g(\alpha),\eta(\epsilon,\alpha))$. It follows from Theorem 2.2 of [14], $f+g$ is an affine function. □

Remark 3.7. Let $\mathcal{A},\mathcal{B}\in CT_{2m,n}$, $f(z)=\alpha_1+\beta z$, and $g(z)=\alpha_2+\beta z$ where $\alpha_1,\alpha_2,\beta\in\mathbb{C}$ and $\beta\neq 0$. Then for $\epsilon>0$,

$$\phi(\epsilon) = \sup_{\substack{u\in S\Lambda_\epsilon(\mathcal{A})\\ w\in S\Lambda_\epsilon(\mathcal{B})}} \inf\{r\geq 0: (\alpha_1+\beta u)+(\alpha_2+\beta w)\in S\Lambda_r((\alpha_1\mathcal{I}_{2m,n}+\beta\mathcal{A})\oplus(\alpha_2\mathcal{I}_{2m,n}+\beta\mathcal{B}))\}$$
$$= \sup_{\substack{u\in S\Lambda_\epsilon(\mathcal{A})\\ w\in S\Lambda_\epsilon(\mathcal{B})}} \inf\{r\geq 0: (\alpha_1+\alpha_2)+\beta(u+w)\in S\Lambda_r((\alpha_1+\alpha_2)\mathcal{I}_{2m,n^2}+\beta(\mathcal{A}\oplus\mathcal{B}))\}$$
$$= \sup_{\substack{u\in S\Lambda_\epsilon(\mathcal{A})\\ w\in S\Lambda_\epsilon(\mathcal{B})}} \inf\{r\geq 0: u+w\in S\Lambda_{\frac{r}{|\beta|}}(\mathcal{A}\oplus\mathcal{B})\}$$
$$\leq \sup_{v\in S\Lambda_{2\epsilon}(\mathcal{A}\oplus\mathcal{B})} \inf\{r\geq 0: v\in S\Lambda_{\frac{r}{|\beta|}}(\mathcal{A}\oplus\mathcal{B})\} := 2\epsilon|\beta|.$$

$$\psi(\epsilon) = \sup_{u\in S\Lambda_\epsilon((\alpha_1\mathcal{I}_{2m,n}+\beta\mathcal{A})\oplus(\alpha_2\mathcal{I}_{2m,n}+\beta\mathcal{B}))} \inf\{r>0: u\in(\alpha_1+\beta S\Lambda_r(\mathcal{A}))+(\alpha_2+\beta S\Lambda_r(\mathcal{B}))\}$$
$$= \sup_{u\in S\Lambda_\epsilon((\alpha_1+\alpha_2)\mathcal{I}_{2m,n^2}+(\beta\mathcal{A}\oplus\beta\mathcal{B}))} \inf\{r>0:\ u\in\alpha_1+\alpha_2+\beta(S\Lambda_r(\mathcal{A})+S\Lambda_r(\mathcal{B}))\}$$
$$= \sup_{u\in S\Lambda_{\frac{\epsilon}{|\beta|}}(\mathcal{A}\oplus\mathcal{B}))} \inf\{r>0:\ u\in S\Lambda_r(\mathcal{A})+S\Lambda_r(\mathcal{B})\}$$
$$\geq \sup_{u\in S\Lambda_{\frac{\epsilon}{|\beta|}}(\mathcal{A}\oplus\mathcal{B}))} \inf\{r>0:\ u\in S\Lambda_{2r}(\mathcal{A}\oplus\mathcal{B})\} := \frac{\epsilon}{2|\beta|}.$$

Example 3.8. Consider $\mathcal{A},\mathcal{B}\in CT_{2m,n}$ be S-normal tensors with S-eigenvalues $\{\lambda_1,\lambda_2,\ldots,\lambda_{n^m}\}$ and $\{\mu_1,\mu_2,\ldots,\mu_{n^m}\}$ respectively. For $f(z)=g(z)=z$ and $\epsilon>0$,

$$\phi(\epsilon) = \sup_{\substack{z\in\bigcup_{i=1}^{n^m} D(\lambda_i,\epsilon)\\ w\in\bigcup_{j=1}^{n^m} D(\mu_j,\epsilon)}} \inf\left\{r\geq 0:\ z+w\in\bigcup_{i,j} D(\lambda_i+\mu_j,r)\right\} = 2\epsilon.$$

Also

$$\psi(\epsilon) = \sup_{z\in\bigcup_{i,j} D(\lambda_i+\mu_j,\epsilon)} \inf\left\{r\geq 0:\ z\in\left(\bigcup_{i=1}^{n^m} D(\lambda_i,r)\right)+\left(\bigcup_{j=1}^{n^m} D(\mu_j,r)\right)\right\} = \frac{\epsilon}{2}.$$

From Theorem 3.3 and Theorem 3.4,

$$S\Lambda_\epsilon(\mathcal{A}) + S\Lambda_\epsilon(\mathcal{B}) \subseteq S\Lambda_{2\epsilon}(\mathcal{A} \oplus \mathcal{B}) \subseteq S\Lambda_\epsilon(\mathcal{A}) + S\Lambda_\epsilon(\mathcal{B}),$$

i.e., $S\Lambda_\epsilon(\mathcal{A}) + S\Lambda_\epsilon(\mathcal{B}) = S\Lambda_{2\epsilon}(\mathcal{A} \oplus \mathcal{B})$.

4 Analysis of a tensor dynamical system

Let $\mathcal{Y}(t) = (\mathcal{Y}_{i_1 i_2 \dots i_m}(t)), t \geq 0$ be an m^{th} order n-dimensional tensor valued function. The derivative of $\mathcal{Y}(t)$ with respect to t is defined as

$$\dot{\mathcal{Y}}(t) = (\mathcal{Y}'_{i_1 i_2 \dots i_m}(t)).$$

Consider the tensor dynamical system $\dot{\mathcal{Y}} = f(\mathcal{A})\mathcal{Y}$, which drives $\dot{\mathcal{Z}} = \mathcal{C}\mathcal{Y} - g(\mathcal{B})\mathcal{Z}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C} \in CT_{2m,n}$, f and g are analytic functions defined on some open sets containing $S\sigma(\mathcal{A})$, $S\sigma(\mathcal{B})$ respectively, and $\mathcal{Y}(t), \mathcal{Z}(t)$ are m^{th} order n-dimensional tensor valued functions. If 0 is not an S-eigenvalue of $g(\mathcal{B}) \oplus f(\mathcal{A}^T)$, then there is a constant tensor $\mathcal{T} \in CT_{2m,n}$ such that if $\mathcal{Z}(0) = \mathcal{T}\mathcal{Y}(0)$, then $\mathcal{Z}(t) = \mathcal{T}\mathcal{Y}(t)$ for all $t > 0$. Then

$$\mathcal{Z} = \mathcal{T}\mathcal{Y}, \;\; \mathcal{T}\dot{\mathcal{Y}} = \mathcal{T}(f(\mathcal{A})\mathcal{Y}), \;\text{ and }\; \mathcal{T}\dot{\mathcal{Y}} = \mathcal{C}\mathcal{Y} - g(\mathcal{B})(\mathcal{T}\mathcal{Y}).$$

Hence,

$$\mathcal{T}(f(\mathcal{A})\mathcal{Y}) + g(\mathcal{B})(\mathcal{T}\mathcal{Y}) = \mathcal{C}\mathcal{Y},$$

This results in the following generalized tensor Sylvester equation; see 1.1 in the introduction part.

$$\mathcal{T} \star f(\mathcal{A}) + g(\mathcal{B}) \star \mathcal{T} = \mathcal{C}. \tag{4.1}$$

So, by analysing the solvability and stability of the generalized tensor Sylvester equation, the stability of a tensor dynamical system can be observed.

We can express 4.1 as,

$$\sum_{k_1,\dots,k_m=1}^{n} \mathcal{T}_{i_1\cdots i_m k_1\cdots k_m} f(\mathcal{A})_{k_1\cdots k_m j_1\cdots j_m} + \sum_{k_1,\dots,k_m=1}^{n} g(\mathcal{B})_{i_1\cdots i_m k_1\cdots k_m} \mathcal{T}_{k_1\cdots k_m j_1\cdots j_m} = \mathcal{C}_{i_1\cdots i_m j_1\cdots j_m},$$

which in terms of Kronecker product becomes

$$(f(\mathcal{A}^T) \otimes \mathcal{I}_{2m,n} + \mathcal{I}_{2m,n} \otimes g(\mathcal{B}))ten_{1/2}(\mathcal{T}) = ten_{1/2}(\mathcal{C}).$$

From the definition of Kronecker sum, we get the following expression.

$$(g(\mathcal{B}) \oplus f(\mathcal{A}^T))ten_{1/2}(\mathcal{T}) = ten_{1/2}(\mathcal{C}).$$

This will provide several real-world examples from electrical circuits, brain networks, and ecology when we view it for $CT_{2,n}$. The prey-predator model is a useful example in ecology, where the prey population ($\mathcal{Y}$) evolves on its own while also influencing the predator population ($\mathcal{Z}$), as predators eat prey. Furthermore, based on both its own population dynamics and the effect of the prey population, the predator population is evolving on its own.

Remark 4.1. The generalized tensor Sylvester equation $\mathcal{T} \star f(\mathcal{A}) + g(\mathcal{B}) \star \mathcal{T} = \mathcal{C}$ has a solution if and only if $0 \notin S\sigma(g(\mathcal{B}) \oplus f(\mathcal{A}^T))$ or $S\sigma(f(\mathcal{A})) \bigcap S\sigma(-g(\mathcal{B})) = \emptyset$.

4.1 Stability analysis of generalized tensor Sylvester equation

Consider the generalized tensor Sylvester equation

$$f(\mathcal{A}) \star \mathcal{X} + \mathcal{X} \star g(\mathcal{B}) = \mathcal{C}.$$

which in terms of tensor Kronecker sum becomes,

$$(f(\mathcal{A}) \oplus g(B^T))ten_{1/2}(\mathcal{X}) = ten_{1/2}(\mathcal{C}). \tag{4.2}$$

Suppose $\mathcal{A}$ and $\mathcal{B}$ are perturbed by $\delta\mathcal{A}$ and $\delta\mathcal{B}$ respectively. Let the corresponding perturbations of $f(\mathcal{A})$ and $g(\mathcal{B})$ are denoted as $\delta f(\mathcal{A})$ and $\delta g(\mathcal{B})$. For a fixed $\mathcal{C}$, the solution $\mathcal{X}$ will also get perturbed and satisfy

$$(f(\mathcal{A}) + \delta f(\mathcal{A})) \star (\mathcal{X} + \delta\mathcal{X}) + (\mathcal{X} + \delta\mathcal{X}) \star (g(\mathcal{B}) + \delta g(\mathcal{B})) = \mathcal{C}.$$

Thus,

$$[(f(\mathcal{A}) + \delta f(\mathcal{A})) \oplus (g(\mathcal{B}^T) + \delta g(\mathcal{B})^T)]ten_{1/2}(\mathcal{X} + \delta\mathcal{X}) = ten_{1/2}(\mathcal{C}).$$

$$[f(\mathcal{A}) \oplus g(\mathcal{B}^T) + \delta f(\mathcal{A}) \oplus \delta g(\mathcal{B})^T]ten_{1/2}(\mathcal{X} + \delta\mathcal{X}) = ten_{1/2}(\mathcal{C}). \tag{4.3}$$

From (4.2) and (4.3),

$$(f(\mathcal{A}) \oplus g(\mathcal{B}^T))ten_{1/2}(\delta\mathcal{X}) = -(\delta f(\mathcal{A}) \oplus \delta g(\mathcal{B})^T)ten_{1/2}(\mathcal{X} + \delta\mathcal{X}).$$

If $0 \notin S\sigma(f(\mathcal{A}) \oplus g(\mathcal{B}^T))$, then

$$\|\delta\mathcal{X}\|_F \leq \|(f(\mathcal{A}) \oplus g(\mathcal{B}^T))^{-1}\|\|(\delta f(\mathcal{A}) \oplus \delta g(\mathcal{B})^T)\|\|(\mathcal{X} + \delta\mathcal{X})\|_F,$$

$$\frac{\|\delta\mathcal{X}\|_F}{\|(\mathcal{X} + \delta\mathcal{X})\|_F} \leq \|(f(\mathcal{A}) \oplus g(\mathcal{B}^T))^{-1}\|\|(\delta f(\mathcal{A}) \oplus \delta g(\mathcal{B})^T)\|.$$

Remark 4.2. If $\mathcal{A}$ and $\mathcal{B}$ are perturbed by $\delta\mathcal{A}$ and $\delta\mathcal{B}$ respectively, the generalized tensor Sylvester equation $f(\mathcal{A}) \star \mathcal{X} + \mathcal{X} \star g(\mathcal{B}) = \mathcal{C}$ has a stable solution if and only if $0 \notin S\Lambda_\epsilon(f(\mathcal{A}) \oplus g(\mathcal{B}^T))$ for some $\epsilon > 0$, and

$$\frac{\|\delta\mathcal{X}\|_F}{\|(\mathcal{X} + \delta\mathcal{X})\|_F} < \frac{1}{\epsilon}\|(\delta f(\mathcal{A}) \oplus \delta g(\mathcal{B})^T)\|. \tag{4.4}$$

Hence the tensor dynamical system we have taken has a stable solution if and only if $0 \notin S\Lambda_\epsilon(g(\mathcal{B}) \oplus f(\mathcal{A}^T))$ for some $\epsilon > 0$, and

$$\frac{\|\delta\mathcal{T}\|_F}{\|(\mathcal{T} + \delta\mathcal{T})\|_F} < \frac{1}{\epsilon}\|(\delta g(\mathcal{B}) \oplus \delta f(\mathcal{A})^T)\|. \tag{4.5}$$

Using the results related to Kronecker sum we obtained, existence of a stable solution to the system can be analysed in terms of pseudospectra of the input tensors. From (ii) of Remark 3.5, if $0 \notin f(S\Lambda_{\psi(\epsilon)}(\mathcal{A})) + g(S\Lambda_{\psi(\epsilon)}(\mathcal{B}))$, the system have a stable solution.

4.2 Numerical example

Let us analyse a simple example.

Consider a tensor dynamical system as seen above, with $\mathcal{A}_{1111} = 1, \mathcal{A}_{1122} = 1, \mathcal{A}_{1211} = -1, \mathcal{A}_{2121} = 2, \mathcal{A}_{2212} = 2, \mathcal{B}_{1112} = 1, \mathcal{B}_{1211} = 1, \mathcal{B}_{1221} = 2, \mathcal{B}_{2112} = 1, \mathcal{B}_{2122} = 1, \mathcal{B}_{2221} = 1, \mathcal{C}_{1112} = 1, \mathcal{C}_{1122} = 1, \mathcal{C}_{1211} = 1, \mathcal{C}_{2122} = -1, \mathcal{C}_{2212} = -1, \mathcal{C}_{2221} = 1$ and all other entries 0. Let $f(z)$ and $g(z) = z^2$. Consider the perturbations $\delta\mathcal{A} = r\mathcal{I}_{4,2}$ and $\delta\mathcal{B} = t\mathcal{I}_{4,2}$ for $\mathcal{A}$ and $\mathcal{B}$ respectively. Then the relative error to $\mathcal{T}$ and its upper bound obtained for various values of r and t can be observed in the following table.

Table 1

r	t	$\frac{\|\delta\mathcal{T}\|_F}{\|(\mathcal{T}+\delta\mathcal{T})\|_F}$	$\frac{1}{\epsilon}\|(\delta g(\mathcal{B}) \oplus \delta f(\mathcal{A})^T)\|$
0.1	0.1	0.01950	0.08201
0.01	0.025	0.02935	0.14748
0.025	0.01	0.03996	0.14266
0.01	0.05	0.04631	0.26251
0.05	0.025	0.08382	0.30765
0.025	0.05	0.06525	0.31494
0.05	0.05	0.09934	0.41400

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