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Automorphism group of the symmetric difference bipartite graph

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Abstract Let n be a fixed positive integer and denote $[n] = \{1, 2, \ldots, n\}$. For integers k and i satisfying $1 \leq k \leq n-1$ and $1 \leq i \leq \min\{n-k, k-1\}$, we introduced a family of simple bipartite graphs G, called *symmetric difference bipartite graphs*, denoted by $SDB(n, k, i)$. The vertex set of G is partitioned as $V = V_1 \cup V_2$, where V_1 consists of all k-element subsets of $[n]$, and V_2 comprises all subsets of $[n]$ of size $k-i$ or $k+i$. An edge connects a vertex in V_1 to a vertex in V_2 if and only if their symmetric difference is an i-element set. This paper investigates the structural symmetries of the graphs $SDB(n, k, i)$ by determining their automorphism groups. We show that if n is an even integer greater than 2 and $k = \frac{n}{2}$, then the automorphism group of $SDB(n, k, i)$ is isomorphic to the direct product $S_n \times \mathbb{Z}_2$, where S_n denotes the symmetric group on n elements and $\mathbb{Z}_2$ is the cyclic group of order 2. In all other cases, the automorphism group is isomorphic to S_n. Furthermore, we explore the automorphism groups of the common neighborhood graphs associated with $SDB(n, 1, 1)$, revealing additional structural properties.

1 Introduction

Let G be a simple connected undirected graph with vertex set $V(G)$ and edge set $E(G)$. A graph H is a subgraph of the graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Two graphs G_1 and G_2 are said to be isomorphic if there exists a one-to-one correspondence ϕ from $V(G_1)$ to $V(G_2)$ such that $uv \in E(G_1) \iff \phi(u)\phi(v) \in E(G_2)$. An isomorphism from a graph G to itself is an automorphism of G. The set of all automorphisms of G forms a group under the operation of composition, called automorphism group of G and is denoted as $Aut(G)$[13]. Moreover, $Aut(G)$ is a subgroup of the group of all permutations of $V(G)$, $Sym(V(G))$ and we use the notation $Aut(G) \leq Sym(V(G))$. Any automorphism is a homomorphism from a graph to itself that is also bijective. More generally, a homomorphism from a graph to itself is called an endomorphism. The automorphism groups of certain classes of graphs were found in [12]. Kneser graphs are employed in many different applications and have been the subject of much research due to their structural features. The ability to convert many set-related combinatorial problems into graph theory makes Kneser type graphs significant.

Let us consider some important Kneser graphs. The Kneser graph $K(n, k)$[5] is defined as a graph with k-subsets of a fixed set with n elements as vertices, and the adjacency exists between two disjoint k-subsets. In [7, 11], the authors discussed codes and designs in Johnson graph $J(n, k)$, which is the graph whose vertices are the k-element subsets of $[n]$ as well, where two vertices U and W are adjacent if $|U \cap W| = k-1$. The bipartite Kneser graph[9, 8], $H(n, k)$, for a positive integer $n \geq 2k+1$ has a vertex set consisting of all k-element and $n-k$ element subsets of $[n]$, and an edge between any two vertices U and W exists when $U \subset W$ *or* $W \subset U$. The set inclusion graph $G(n, k, l)$ where $1 \leq k < l \leq n-1$, $k + l \leq n$ is the graph with vertices as $k-$ element and $l-$ element subsets of $[n]$, adjacency exists if one of them is contained in another. The automorphism group of the bipartite Kneser graph was determined in [9], whereas the authors in [6] found the spectrum and automorphism group of the graph $G(n, k, l)$. Recently, there are certain works [15],[16] about the automorphism group of bipartite graphs.

One of the measures of the connectivity and cohesion of a graph is the size of the common neighbourhood of its vertices. For a vertex $u \in V(G)$, the open neighbourhood of u is the set of all vertices which are adjacent to u and is denoted by $N(u)$. The union, $N(u) \cup u$ is the closed neighbourhood of u. The common neighbourhood of any two distinct vertices u, v in $V(G)$, denoted as $N(u, v)$, is the set of all vertices different from u and v, that are adjacent to both u and v. The authors in [1] introduced congraph or common neighbourhood graph of a simple graph G. It is defined as a graph with the same vertex set as of G, and two vertices are adjacent if and only they have at least one common neighbour in the graph G.

Hypercube graph Q_n [2] is a graph with vertex set $\{0,1\}^n$ and if two $n-$tuples differ in precisely one coordinate, then they are adjacent. The graph Q_n is isomorphic to the Boolean lattice graph BL_n for $n \geq 1$ [9], whose vertex set consists of all subsets of $[n]$, and two vertices are adjacent if their symmetric difference contains exactly one element.

Motivated by hypercube graphs, the authors in [14] introduced a new family of graphs known as *symmetric difference bipartite graphs* $SDB(n, k, i)$, where $1 \leq k \leq n-1$ and $1 \leq i \leq \min\{n-k, k-1\}$, and determined certain graph-theoretic properties of these graphs.

In this paper, we determined the automorphism group of the symmetric difference bipartite graphs $SDB(n, k, i)$ as S_n if $k \neq \frac{n}{2}$, and $S_n \times \mathbb{Z}_2$ if $k = \frac{n}{2}$. We established certain properties of the congraph of $SDB(n, 1, 1)$. Also, the automorphism group of the congraph of the symmetric difference bipartite graph was obtained.

2 Preliminaries

In this section, we provide some definitions and theorems for proving our main results in forthcoming sections.
Let $[n] = \{1, 2, 3, \ldots, n\}$, where $n \in \mathbb{N}$.

Definition 2.1. The bipartite graph $B(n, k)$[10] for $n \geq 4$ is a subgraph of the boolean lattice graph BL_n where $V(B(n,k)) = \{U \subset [n] : |U| = k \text{ or } k+1\}$ and $E(B(n,k)) = \{UW : U, W \in V(B(n,k)),\ U \subset W \text{ or } W \subset U\}$.

The automorphism group of the graph $B(n, k)$ is proved as the following:

Theorem 2.2. *[10] Let n and k be integers with $1 \leq k < \frac{n}{2}$, and let $\Gamma = (V, E) = B(n, k)$, with partition $V = P_1 \cup P_2, P_1 \cap P_2 = \emptyset$, where $P_1 = \{U : U \subset [n], |v| = k\}$ and $P_2 = \{W : W \subset [n], |W| = k+1\}$. If $n \neq 2k+1$, then $Aut(\Gamma) \cong S_n$, and if $n = 2k+1$, then $Aut(\Gamma) \cong S_n \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the cyclic group of order 2.*

Theorem 2.3. *[12] The automorphism group of the complete graph on n vertices, $Aut(K_n)$, is isomorphic to the symmetric group S_n.*

Theorem 2.4. *[12] The automorphism group of the cycle graph on n vertices, $Aut(C_n) \cong D_n$, where D_n is the dihedral group of order $2n$.*

The automorphism groups of most Kneser-type graphs have been determined by studying the Johnson graph $J(n, k)$. Many authors have investigated this graph and derived its automorphism group using different methods.

Theorem 2.5. *[11] Let $\Gamma = J(n, k)$ with $n \geq 4$ and $2 \leq k \leq \frac{n}{2}$. If $n \neq 2k$, then $Aut(\Gamma) \cong S_n$. If $n = 2k$, then $Aut(\Gamma) \cong S_n \times \mathbb{Z}_2$.*

Now consider the newly defined family of graphs known as the symmetric difference bipartite graphs introduced in [14].

Definition 2.6. [14] Let n be a fixed positive integer and $[n] = \{1, 2, \ldots, n\}$. For any fixed positive integer $1 \leq k \leq n-1$, and for a fixed i, $1 \leq i \leq \min\{n-k, k-1\}$, symmetric difference bipartite graph is defined as a bipartite graph with vertex set $V = V_1 \cup V_2$, where $V_1 = \{U \subset [n] : |U| = k\}$ and $V_2 = \{W \subseteq [n] : |W| = k - i \text{ or } k + i\}$, a vertex in V_1 is adjacent to a vertex in V_2 if their symmetric difference is an i-element subset of $[n]$.
When $k = 1$, in V_2, only $k + 1$ element subsets are taken.

The graph $SDB(n, k, i)$ is connected; in particular, the graphs $SDB(4, 1, 1)$ and $SDB(4, 2, 1)$ are planar. Various properties, including connectivity and diameter, have been determined.

Proposition 2.7. *[14] For $n \neq 2$, the girth of $SDB(n, 1, 1)$ is 6.*

Lemma 2.8. *[14] For $SDB(n, 1, 1)$ graph, n vertices have degree $n-1$ and $\frac{n(n-1)}{2}$ vertices have degree 2.*

Lemma 2.9. *[6] Let G be a connected bipartite graph with bipartition $V = V_1 \cup V_2$. For each $\phi \in Aut(G)$, we have either $\phi(V_1) = V_1$ and $\phi(V_2) = V_2$, or $\phi(V_1) = V_2$ and $\phi(V_2) = V_1$.*

Definition 2.10. [1, 7] Let G be a simple graph. The congraph of G denoted by $con(G)$ is defined as a graph with the same vertex set as $V(G)$ and with the adjacency matrix

$$A = [a_{ij}] = \begin{cases} 1 & if \;\; |N(u_i, u_j)| \geq 1 \;\; and \;\; i \neq j, \\ 0 & otherwise. \end{cases}$$

where $N(u_i, u_j)$ represents the common neighbourhood of two vertices u_i and u_j in $V(G)$.

Theorem 2.11. *[1] The common neighbourhood graph, $con(G)$ is connected if and only if the parent graph G is connected and non-bipartite.*

3 Main Results

There are various properties for the graph $SDB(n, k, i)$. In this section, we are interested to study the automorphism group of these graphs.

Lemma 3.1. *The degree sequence of the graph $SDB(n, k, i), k \neq 1$ is as follows:*

(i) $\binom{n}{k}$ *vertices have degrees* $\binom{k}{k-i} + \binom{n-k}{i}$.

(ii) $\binom{n}{k-i}$ *vertices have degrees* $\binom{n-k+i}{i}$.

(iii) $\binom{n}{k+i}$ *vertices have* $\binom{k+i}{k}$ *degrees.*

Proof. From the Definition 2.6, the part V_1 has $\binom{n}{k}$ vertices and, V_2 has $\binom{n}{k-i}$ and $\binom{n}{k+i}$ vertices. By the adjacency relation, the result holds. □

Proposition 3.2. *The symmetric difference bipartite graph $SDB(n, k, i)$ is a tree if and only if $n = 2$.*

Proof. When $n = 2$, then $k = 1$ and $i = 1$. From the Definition 2.6, the graph has two vertex partitions V_1 and V_2 having cardinalities 2 and 1 respectively. The vertices in V_1 are pendant vertices and the graph has no cycle. By the Proposition 2.7, for $n \neq 2, SDB(n, 1, 1)$ contains a cycle of length 6 and for $n > 2, k \neq 1, SDB(n, k, i)$ contains a cycle of length 4. Thus $SDB(2, 1, 1)$ is a tree. □

Lemma 3.3. *Consider the graph $SDB(n, k, i)$ with $1 \leq k \leq n-1$ and $1 \leq i \leq \min\{n-k, k-1\}$. If ϕ is an automorphism of the graph $SDB(n, k, i)$ such that $\phi(U) = U$ for every $U \in V_1$ (or $\phi(U) = U$ for every $U \in V_2$), then ϕ is an identity automorphism of the graph $SDB(n, k, i)$.*

Proof. Since ϕ is an automorphism of the graph $SDB(n, k, i)$, it is an isomorphism and hence a permutation of the vertex set of $SDB(n, k, i)$.

Let W be an arbitrary vertex of V_2. Then, either $|W|$ is $k - i$ or $k + i$.
Case (i): If $|W| = k - i$,
then $N(W) = \{U \in V_1,\ U \supset W$ and $|U \triangle W| = i\}$. By the definition of the graph $SDB(n, k, i)$, there are $\binom{n-k+i}{i}$ vertices in V_1, say $U_1, U_2, \ldots, U_{\binom{n-k+i}{i}}$ with a unique common neighbour W. Since $\phi(U) = U$ for every $U \in V_1$, $\phi(U_j) = U_j$ for $j = 1, 2, \ldots, \binom{n-k+i}{i}$. As W is arbitrary, we must have $\phi(W) = W$.
Case (ii): If $|W| = k + i$,

then $N(W) = \{U \in V_1,\ U \subset W$ and $|U \triangle W| = i\}$ and $|N(W)| = \binom{k+i}{k}$. Set those vertices in V_1 as $U'_1, U'_2, \dots, U'_{\binom{k+i}{k}}$ and W is the unique common neighbour of them. In this case also, if ϕ fixes each vertex in V_1, then $\phi(U'_j) = U'_j$ for $j = 1, 2, \dots, \binom{k+i}{k}$. Since W is arbitrary, then $\phi(W) = W$.
Thus, in both cases $\phi(W) = W$ for every $W \in V_2$ so that ϕ is an identity automorphism of the graph $SDB(n, k, i)$. □

Lemma 3.4. *Consider the graph $SDB(n, k, i)$ as defined in Definition 2.6. For any two vertices $U, U' \in V_1$, and for each ℓ with* $\max\{2k - n, 0\} \le \ell \le k - 1$*,*

$$|U \cap U'| = \ell \quad \text{if and only if} \quad |N(U) \cap N(U')| = \begin{cases} 1 & \text{if } k = 1, \\ \binom{\ell}{k-i} + \binom{n-2k+\ell}{i-k+\ell} & \text{if } k \neq 1 \text{ and } \ell \geq k - i, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Let $U, U' \in V_1$ such that $|U \cap U'| = \ell$. We now consider three cases based on values of k and $\ell = |U \cap U'|$:

Case (i): $k = 1$.
By the definition, when $k = 1$, each vertex in V_1 is a singleton set from $[n]$, and vertices in V_2 are 2-element subsets of $[n]$. Two distinct singleton vertices $U = \{u\}$, $U' = \{u'\}$, have symmetric difference $U \triangle U' = \{u, u'\}$, a 2-element subset, so both are adjacent to the same vertex $W = \{u, u'\} \in V_2$. Thus,

$$|N(U) \cap N(U')| = 1.$$

Case (ii): $k \neq 1$ and $\ell \geq k - i$.
Let $|U| = |U'| = k$ and $|U \cap U'| = \ell$. Then $|U \cup U'| = 2k - \ell$, and their symmetric difference is:

$$|U \triangle U'| = |U \cup U'| - |U \cap U'| = (2k - \ell) - \ell = 2(k - \ell).$$

We now count the number of common neighbors $W \in V_2$ such that $|U \triangle W| = i$ and $|U' \triangle W| = i$. That is, we want sets W such that W is at symmetric difference i from both U and U'. Such sets exist in two categories:

- **Subsets of size** $k - i$ (i.e., $W \subset U \cap U'$): The symmetric difference between U and W is of size i if $W \subset U \cap U'$ and $|W| = k - i$. So there are $\binom{\ell}{k-i}$ such sets.
- **Supersets of size** $k + i$ (i.e., $W \supset U \cup U'$): The symmetric difference between U and W is of size i if $W \supset U \cup U'$ and $|W| = k + i$. Since $|U \cup U'| = 2k - \ell$, we need to choose $(k + i) - (2k - \ell) = i - k + \ell$ additional elements from the remaining $n - 2k + \ell$ elements. So there are $\binom{n-2k+\ell}{i-k+\ell}$ such sets.

Therefore,

$$|N(U) \cap N(U')| = \binom{\ell}{k-i} + \binom{n-2k+\ell}{i-k+\ell}.$$

Case (iii): $\ell < k - i$**.**
If the intersection size ℓ is too small, there are not enough shared elements to form a subset of size $k - i$, nor is the union small enough to be contained in a set of size $k + i$. Thus, there are no sets $W \in V_2$ that are simultaneously at symmetric difference i from both U and U'. Therefore,

$$|N(U) \cap N(U')| = 0.$$

Finally, note that for distinct values $\ell \neq m$, the expression

$$\binom{\ell}{k-i} + \binom{n-2k+\ell}{i-k+\ell}$$

is distinct from

$$\binom{m}{k-i} + \binom{n-2k+m}{i-k+m},$$

ensuring the function is injective in ℓ, completing the proof. □

Example 3.5. The automorphism group of the graph $SDB(2,1,1)$ is isomorphic to $\mathbb{Z}_2$.

Proof. Let $G = SDB(2,1,1)$.
Here $V_1 = \{a = \{1\}, b = \{2\}\}$ and $V_2 = \{c = \{1,2\}\}$. Then the automorphism group of G, $Aut(G) = \{(a),(a,b)\}$. It is a permutation group with generator $\{(a,b)\}$ and so isomorphic to $\mathbb{Z}_2$. □

Example 3.6. $Aut(SDB(3,1,1)) \cong D_6$, where D_6 is the dihedral group of order 12.

The graph $SDB(3,1,1)$ has 6 vertices and 6 edges. Each vertex has degree 2. Then there exists an isomorphism ϕ from $V(SDB(3,1,1))$ to $V(SDB(3,1,1))$ with the property that $\phi(V_1) = V_1$ and $\phi(V_2) = V_2$; or $\phi(V_1) = V_2$ and $\phi(V_2) = V_1$, where $V_1 = \{\{1\},\{2\},\{3\}\}$ and $V_2 = \{\{1,2\},\{1,3\},\{2,3\}\}$.

Let $1 = \{1\}, 2 = \{2\}, 3 = \{3\}, 4 = \{1,2\}, 5 = \{1,3\}, 6 = \{2,3\}$. Then $Aut(SDB(3,1,1))$ $= \{i, (2,3)(4,5), (1,2)(5,6), (1,2,3)(4,6,5), (1,3,2)(4,5,6), (1,3)(4,6)(1,4)(2,5)(3,6),$ $(1,5,3,6,2,4), (1,4,2,6,3,5), (1,6)(2,4)(3,5), (1,5)(2,6)(3,4), (1,6)(2,5)(3,4)\}$, where i is the identity permutation.
As it is a permutation group with generators $(2,3)(4,5), (1,2)(5,6), (1,4)(2,5)(3,6)$, $Aut(SDB(3,1,1))$ is isomorphic to the dihedral group D_6.

Example 3.7. The automorphism group of the graph $SDB(3,2,1)$, $Aut(SDB(3,2,1)) \cong D_3$, where D_3 is the dihedral group of order 6.

By the Lemma 3.1, the order and size of the graph $SDB(3,2,1)$ are 7 and 9 respectively. Out of these 7 vertices, four of them have degree 3 and other vertices have degree 2. Then $Aut(SDB(3,2,1)) = \{(1), (1,2)(5,6), (2,3)(4,5), (1,2,3)(4,6,5), (1,3,2)(4,5,6), (1,3)(4,6)\}$, where $1 = \{1,2\}, 2 = \{1,3\}, 3 = \{2,3\}, 4 = \{1\}, 5 = \{2\}, 6 = \{3\}, 7 = \{1,2,3\}$. That is, $Aut(SDB(3,2,1))$ is the permutation group acting on a set of cardinality 7 and has order 6, generated by $(2,3)(4,5), (1,2)(5,6)$. So $Aut(SDB(3,2,1)) \cong D_3$ and hence it is isomorphic to D_3.

Theorem 3.8. *Let $n \geq 4$, and $G = SDB(n,1,1)$. Then $Aut(G) \cong S_n$.*

Proof. When $n \geq 4, k = 1$ and $i = 1$, $SDB(n,1,1)$ is the same as the bipartite graph $B(n,k)$. Then the result follows from the Definition 2.1 and from the Theorem 2.2. □

Theorem 3.9. *Let $n \geq 4$. For $1 < k \leq n-1$, and $1 < i \leq \min\{n-k, k-1\}$, the automorphism group of the symmetric difference bipartite graph $SDB(n,k,i)$ is given by,*

$$Aut(SDB(n,k,i)) \cong \begin{cases} S_n \times \mathbb{Z}_2 & if\ n \text{ is even and } k = \frac{n}{2}, \\ S_n & otherwise. \end{cases}$$

Proof. Let n, k and i be fixed positive integers with $[n] = \{1,2,3,\ldots,n\}$, $1 \leq k \leq n-1$, $1 \leq i \leq \min\{n-k, k-1\}$. Let S_n be a symmetric group on $[n]$. Let $\gamma \in S_n$ be fixed. Then γ is both one-one and onto function from $[n]$ to itself. Let $V = V_1 \cup V_2$ be as in the Definition 2.6. Define $\phi_\gamma : V \to V$ by $\phi_\gamma(U) = \gamma(U)$ for $U \in V$, where $\gamma(U) = \{\gamma(u_s) : u_s \in U = \{u_1, u_2, \ldots, u_t\}, 1 \leq s \leq t, t \in \{k, k-i, k+i\}\}$. Then,
ϕ_γ is one-one. For any $U, U' \in V$,

$$\begin{aligned} \phi_\gamma(U) = \phi_\gamma(U') &\iff \gamma(U) = \gamma(U') \\ &\iff U = U' \text{ since } \gamma \text{ is one-one.} \end{aligned}$$

Also, for any vertex $W \in V$, there exists $U = \gamma^{-1}(W)$ in V such that $\phi_\gamma(U) = W$. Such a γ^{-1} exists since γ is onto. Hence ϕ_γ is onto. So, ϕ_γ is a permutation on V. Let $S'_n = \{\phi_\gamma : \gamma \in S_n\}$. Then, S'_n is a permutation group on the vertex set V of $SDB(n,k,i), k \neq \frac{n}{2}$, which is isomorphic to S_n. For any two vertices $U, W \in V$,

$$\begin{aligned} UW \in E(SDB(n,k,i)) &\iff U \subset W \text{ or } W \subset U \text{ and } |U \triangle W| = i \\ &\iff \phi_\gamma(U) = \gamma(U) \subset \gamma(W) = \phi_\gamma(W) \\ &\text{or } \phi_\gamma(W) = \gamma(W) \subset \gamma(U) = \phi_\gamma(U) \text{ and } |\gamma(U) \triangle \gamma(W)| = i \\ &\iff \phi_\gamma(U)\phi_\gamma(W) \in E(SDB(n,k,i)) \end{aligned}$$

Thus ϕ_γ is an automorphism of $SDB(n,k,i)$ and the mapping $\zeta : S_n \to Aut(SDB(n,k,i))$, defined by $\zeta(\gamma) = \phi_\gamma$ is one to one. Therefore, $S'_n \leq Aut(SDB(n,k,i))$.
When $k = \dfrac{n}{2}, 1 \leq i \leq \frac{n}{2} - 1$, define $\delta : V \to V$ by $\delta(U) = [n] - U,$ for $U \in V$.

$$\begin{aligned}\delta(U) = \delta(W) &\iff [n] - U = [n] - W\\ &\iff U = W\end{aligned}$$

For any $W \in V,$ there exists $U = [n] - W \in V$ such that $\delta(U) = W$. Then δ is onto and hence δ is a permutation. Also, $\delta^2(U) = \delta(\delta(U)) = \delta([n] - U) = [n] - ([n] - U) = U$. Therefore, δ is a permutation of order 2 on V and $\delta \notin S'_n$.
If $U, W \in V$, then

$$\begin{aligned}UW \in E(SDB(n,k,i)) &\iff U \subset W \text{ or } W \subset U \text{ and } |U \triangle W| = i\\ &\iff \delta(W) \subset \delta(U) \text{ or } \delta(U) \subset \delta(W) \text{ and } |\delta(U)\triangle\delta(W)| = i\\ &\iff \delta(U)\delta(W) \in E(SDB(n,k,i))\end{aligned}$$

Therefore, when $k = \frac{n}{2}$ and n is even , δ is an automorphism on V. The group $<\delta>\cong \mathbb{Z}_2$ and it is a subgroup of the group $Aut(SDB(n, \frac{n}{2}, i))$.
If $\delta \notin S'_n,$ for every $\gamma \in S_n$,
$\delta \circ \phi_\gamma(U) = \delta(\phi_\gamma(U)) = \delta(\gamma(U)) = [n] - \gamma(U)$
$\phi_\gamma \circ \delta(U) = \phi_\gamma(\delta(U)) = \phi_\gamma([n] - U) = \gamma([n] - U) = [n] - \gamma(U)$.
Then $S'_n \times <\delta> \leq Aut(SDB(n, \frac{n}{2}, i))$.
Let ϕ be an arbitrary automorphism of $SDB(n,k,i)$ and $\phi|_{V_1}$ be the restriction of ϕ to V_1. Unlike the set inclusion graph and the bipartite Kneser graph, the graph $SDB(n,k,i)$ has $|V_1| \neq |V_2|$. Therefore, any automorphism ϕ satisfies only the first case in Lemma 2.9, that is, $\phi(V_1) = V_1$ and $\phi(V_2) = V_2$. Then there arises two cases.
Let us rename the vertex partitions V_1 and V_2 as V_k and $V_{k-i} \cup V_{k+i}$, respectively.
Case (i): If $\phi(V_k) = V_k, \phi(V_{k-i}) = V_{k-i}$ and $\phi(V_{k+i}) = V_{k+i}$, then to show that $\phi|_{V_k}$ is an automorphism S'_n of the Johnson graph $J(n,k)$.

If $\phi(V_k) = V_k$, then for every vertex $U \in V_k$, $\phi(U)$ must belongs to V_k and so $\phi|_{V_k}$ is a permutation on V_k. The vertex set of $J(n,k)$ is the same as the partite set V_k of $SDB(n,k,i)$ and any two vertices U, U' in $J(n,k)$ are adjacent if and only if $|U \cap U'| = k - 1$. Then, S'_n is a subgroup of $Aut(J(n,k)), 1 \leq k \leq \frac{n}{2}$. $\phi|_{V_k}$ is an automorphism of the graph $J(n,k)$, if $U, U' \in V_k$ are such that $|U \cap U'| = k - 1 \iff |\phi|_{V_k}(U) \cap \phi|_{V_k}(U')| = k - 1$.
For, by Lemma 3.1,

$$\begin{aligned}|U \cap U'| = k - 1 &\iff |N(U) \cap N(U')| = \binom{k-1}{k-i} + \binom{n-k-1}{i-1}\\ &\iff |N(\phi|_{V_k}(U)) \cap N(\phi|_{V_k}(U'))| = \binom{k-1}{k-i} + \binom{n-k-1}{i-1}\\ &\iff |\phi|_{V_k}(U) \cap \phi|_{V_k}(U')| = k - 1.\end{aligned}$$

If $|\phi|_{V_k}(U) \cap \phi|_{V_k}(U')| < k - 1$, say $k - h$, then $h > 1$ and $|\phi|_{V_k}(U) \cup \phi|_{V_k}(U')| = |\phi|_{V_k}(U)| + |\phi|_{V_k}(U')| - |\phi|_{V_k}(U) \cap \phi|_{V_k}(U')| = k + h$.

To find the common neighbouring vertices of $\phi|_{V_k}(U)$ and $\phi|_{V_k}(U')$ in the graph $SDB(n,k,i)$, we must construct a $k-i$-element subset of $\phi|_{V_k}(U) \cup \phi|_{V_k}(U')$, and a $k+i$-element superset that contains $\phi|_{V_k}(U) \cup \phi|_{V_k}(U')$.

There are $\binom{k-h}{k-i} + \binom{n-k-h}{i-h}$ such neighbours. Thus, the permutation $\phi|_{V_k}$ is an automorphism of the Johnson graph and hence there exists a permutation γ in the symmetric group S_n such that $\phi|_{V_k} = \phi_\gamma$, where ϕ_γ is the permutation on V_k. Now extend ϕ_γ as a permutation on V with the same γ. Then $\phi_\gamma \in S'_n \leq Aut(SDB(n,k,i))$.
If $\phi \in Aut(SDB(n,k,i)),$ then $I = \phi_\gamma^{-1}\phi$ is the identity automorphism of the graph $SDB(n,k,i)$ and therefore $\phi = \phi_\gamma \in S'_n$, for some $\gamma \in S_n$.
Case (ii): Since ϕ preserves degree, it maps either the vertices of V_j onto itself, for $j \in \{k, k-i, k+i\}$ or the vertices of V_{k-i} onto the vertices of V_{k+i} and vice versa.

Also, $\phi \circ \delta(V_k) = \phi(\delta(V_k)) = \phi(V_k) = V_k$. Then by the first case $\phi\delta = \phi_\gamma$, for some $\gamma \in S_n$. As δ is of order 2, $\phi = \phi_\gamma\delta \in < S_n \times \mathbb{Z}_2 >$.
When $k \neq \frac{n}{2}$, by case(i), $Aut(SDB(n,k,i)) \leq S'_n \cong S_n$.
If $k = \frac{n}{2}$, then $|V_{k-i}| = |V_{k+i}|$, and from cases (i) and (ii), $Aut(SDB(n,k,i)) \leq S'_n \times \langle\delta\rangle \cong S_n \times \mathbb{Z}_2$

□

Composition factors provide a fundamental framework for understanding group structure. A composition series for a group G is a sequence of subgroups $\{e\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_k = G$ where G_i is a proper normal subgroup of G_{i+1} and $\dfrac{G_{i+1}}{G_i}$ is a simple group for $1 \leq i \leq k-1$. These simple groups are called the composition factors of the group G. To some extent, one can reconstruct a group using its composition factors, and numerous attributes of a group are dictated by the characteristics of its composition factors. A group is said to be simple if its only normal subgroups are the identity subgroup and the group itself. A subgroup H of G is normal in G, $H \lhd G$ if and only if $xHx^{-1} \subseteq H$ for all x in G.

Proposition 3.10. *The composition factors for the automorphism group G of the symmetric difference bipartite graphs $SDB(n,k,i)$, $n \geq 5$, $k \neq \frac{n}{2}$ are $\mathbb{Z}_2$ and A_n, where $\mathbb{Z}_2$ is a cyclic group of order 2, A_n is the alternating group of order $\frac{n!}{2}$. Hence, $Aut(SDB(n,k,i)) \cong S_n$.*

Proof. The result follows from the fact that the symmetric group $S_n, n > 4$ has the composition series $\{(1)\} \subset A_n \subset S_n$ as the factor groups $\dfrac{S_n}{A_n} \cong \mathbb{Z}_2$ and A_n are simple. □

Now consider the case when $n \neq 4, k = \frac{n}{2}$.

Proposition 3.11. *The composition series for the automorphism group G of the symmetric difference bipartite graph $SDB(n,\frac{n}{2},i), n \geq 5$ is $G_0 \lhd G_1 \lhd G_2 \lhd G$ where G_2 is a normal subgroup of G of order $n!$ such that $\frac{G}{G_2}$ is $\mathbb{Z}_2$, G_1 is a normal subgroup of G_2 of order 2 such that $\frac{G_2}{G_1}$ is the alternating group A_n and G_0 is a normal subgroup(trivial subgroup) of G_1 of order 1 such that $\frac{G_1}{G_0}$ is the cyclic group $\mathbb{Z}_2$.*

Proof. For $n \geq 5$ and $k = \frac{n}{2}$, $Aut(SDB(n,k,i)) \cong S_n \times \mathbb{Z}_2$. Therefore, its composition series is a chain of subgroups of the form $\{(1)\} \subset A_n \subset A_n \times \mathbb{Z}_2 \subset S_n \times \mathbb{Z}_2$. Corresponding factor groups are $\mathbb{Z}_2, A_n$ and $\mathbb{Z}_2$. Hence the result. □

Example 3.12. The Theorem 3.8 shows that the automorphism group of the graph $SDB(4,1,1)$ is isomorphic to the symmetric group S_4 on the set $\{1,2,3,4\}$. The normal subgroup of S_4 is the alternating group A_4, and A_4 has a normal subgroup of order 4, say
$H = \{(1),(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\}$. Then $H' = \{(1),(1,2)(3,4)\}$ is a normal subgroup of H. Thus the composition series for $Aut(SDB(4,1,1))$ is $\{(1)\} \subset H' \subset H \subset A_4 \subset S_4$ and hence the composition factors are the simple groups $\dfrac{S_4}{A_4} \cong \mathbb{Z}_2$, $\dfrac{A_4}{H} \cong \mathbb{Z}_3$, $\dfrac{H}{H'} \cong \mathbb{Z}_2$ and $\dfrac{H'}{\{(1)\}} \cong \mathbb{Z}_2$.

The following example illustrates the decomposition of the automorphism group of the graph $SDB(4,2,1)$.

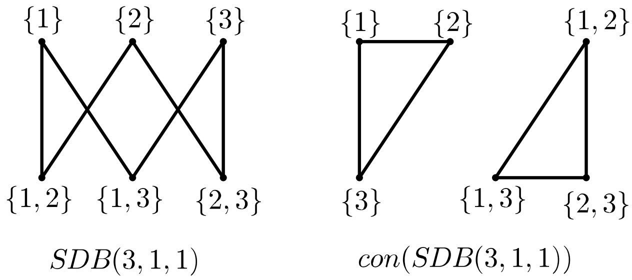
Example 3.13. With the help of the Theorem 3.9, the composition factors of the automorphism group of $SDB(4,2,1)$ are the cyclic groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2$.

Automorphism group of the congraph of the symmetric difference bipartite graph

Let us consider the common neighbourhood graph or the congraph of the symmetric difference bipartite graph $SDB(n,1,1)$ for $n > 2$. As the graph $SDB(n,1,1)$ is bipartite, its congraph is a disconnected graph with two components.

Example 3.14. Figure 1 below shows both the graph $SDB(3,1,1)$ and its congraph.

Figure 1. The graph of $SDB(3,1,1)$ and its congraph



Here we established certain properties of the congraph $con(SDB(n,1,1))$.

Theorem 3.15. *The graph $con(SDB(n,1,1))$ for $n > 2$, is a disconnected graph with two components in which one of them is a complete graph K_n, and the other is a regular graph of degree $2n-4$.*

Proof. From the Definition 2.6, the graph $SDB(n,1,1)$ is a connected bipartite graph with two disjoint vertex partitions V_1 and V_2, where $|V_1| = n$, $|V_2| = \binom{n}{2}$. Consider any two vertices $u, v \in V_1$. As $SDB(n,1,1)$ is connected, there exists a path $u = x_0x_1x_2\ldots x_{d-1}x_d = v$. Since $SDB(n,1,1)$ is bipartite, the length of the path d must be even. Thus in the graph $con(SDB(n,1,1))$, $u = x_0$ is adjacent to x_2, which is adjacent to x_4, which is adjacent to x_6 and hence x_{d-2} is adjacent to $x_d = v$. Thus $u = x_0x_2x_4\ldots x_{d-2}x_d = v$ is a path in $con(SDB(n,1,1))$. That is, there exists a path connecting any two vertices in V_1, and all the vertices in V_1 lie in the same component. Hence, the n vertices in V_1 of $SDB(n,1,1)$ lie entirely within one component, say $\mathfrak{C}_1$, of con$(SDB(n,1,1))$, such that each vertex u is adjacent to the remaining $n-1$ vertices. Therefore, this component is the complete graph K_n.

similarly, any two vertices in V_2 lie on the same component, say $\mathfrak{C}_2$, of $con(SDB(n,1,1))$. As $|V_2| = \binom{n}{2}$, maximum possible degree of each vertex in $\mathfrak{C}_2$ is $\binom{n}{2} - 1$. However, there are $\binom{n-2}{2}$ vertices that are non-adjacent to each vertex $u \in$ con$(SDB(n,1,1))$, since these $\binom{n-2}{2}$ vertices in the part V_2 have no common neighbours with u in $SDB(n,1,1)$. Thus the vertices in $\mathfrak{C}_2$ have degree $\left[\binom{n}{2} - 1\right] - \binom{n-2}{2} = 2n-4$.

Also, if $u \in V_1$ and $v \in V_2$, then $N(u) \in V_2$ and $N(v) \in V_1$. So they are not adjacent in $con(SDB(n,1,1))$. Hence the result. □

Corollary 3.16. *The order and size of the congraph, $con(SDB(n,1,1))$ where $n > 2$ are $\dfrac{n(n+1)}{2}$ and $\dfrac{n(n-1)^2}{2}$, respectively.*

Proof. Both the graphs $SDB(n,1,1)$ and $con(SDB(n,1,1))$ have the same vertex set. So the order of $con(SDB(n,1,1))$ is $\dfrac{n(n+1)}{2}$.

From the Theorem 3.15,

$$\begin{aligned} \text{size of con(SDB(n,1,1))} &= \text{size of the component } \mathfrak{C}_1 + \text{ size of the component } \mathfrak{C}_2 \\ &= \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{2} \\ &= \frac{n(n-1)^2}{2}. \end{aligned}$$

□

Let G_1 and G_2 be two vertex-disjoint graphs. The *join* $G_1 + G_2$ of the graphs G_1 and G_2 consists of the union $G_1 \cup G_2$ along with all edges joining every vertex of G_1 to every vertex of G_2.

The *congraph* con$(SDB(n,1,1))$, for $n \geq 3$, can be expressed as the join of one or more graphs. Based on this observation, we analyse the *automorphism group* of con$(SDB(n,1,1))$.

One of the important tools for characterizing the automorphism group of graphs is the *Wreath product* [4]. If G and H are permutation groups acting on sets Ω_1 and Ω_2, respectively, the wreath product, $G \wr H$ acts on $\Omega_1 \times \Omega_2$ in the following way:

- Each copy of G acts independently on elements of Ω_1.
- H permutes these copies of G according to its action on Ω_2.

The automorphism group of a disconnected graph can be described in terms of group products as follows:

Theorem 3.17. *[4] Let G be a graph whose connected components are n_1 copies of G_1, n_2 copies of $G_2, ..., n_r$ copies of G_r, where $G_1, G_2, \ldots, G_r$ are pairwise non-isomorphic connected graphs. Then,*

$$Aut(G) = (Aut(G_1) \wr S_{n_1}) \times (Aut(G_2) \wr S_{n_2}) \times \ldots \times (Aut(G_r) \wr S_{n_r}).$$

In the view of the above theorem, the automorphism groups of the congraphs of $SDB(n, 1, 1)$ are as follows:

Example 3.18. The Definitions 2.6 and 2.10 show that

$$V(con(SDB(3, 1, 1))) = V((SDB(3, 1, 1)) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$$

and the congraph is a disconnected graph consists of two copies of the complete graph K_3 as components. Then by Theorem 3.17, $Aut(con(SDB(3, 1, 1)) = Aut(K_3) \wr S_2 = S_3 \wr S_2$, which is a permutation group of order 72.

Remark 3.19. The congraph $con(SDB(3, 1, 1))$ is both vertex transitive and edge transitive. But not distance transitive. It is also Eulerian.

Proposition 3.20. *The automorphism group of the $con(SDB(4, 1, 1))$, $Aut(con(SDB(4, 1, 1))) \cong S_4 \times S_4 \times \mathbb{Z}_2$.*

Proof. The Figure 2 shows the graph of $con(SDB(4, 1, 1))$. The order and size of this graph are 10 and 18 respectively. As in the Theorem 3.15, it is a disconnected graph in which one component $\mathfrak{C}_1$ is a complete graph K_4 and the other component $\mathfrak{C}_2$ is a regular graph of degree 4. For $\mathfrak{C}_1$, $Aut(K_4) \cong S_4$. The component $\mathfrak{C}_2$ can be expressed as the join of two copies of K_1 with a cycle C_4 (in the Figure 2, the vertices k_1 and k_2 are joined with the 4−cycle in the component $\mathfrak{C}_2$ by dashed lines) and $Aut(\mathfrak{C}_2) \cong S_4 \times \mathbb{Z}_2$.
As $con(SDB(4, 1, 1)) = \mathfrak{C}_1 \cup \mathfrak{C}_2$, $Aut(con(SDB(4, 1, 1))) \cong S_4 \times S_4 \times \mathbb{Z}_2$.

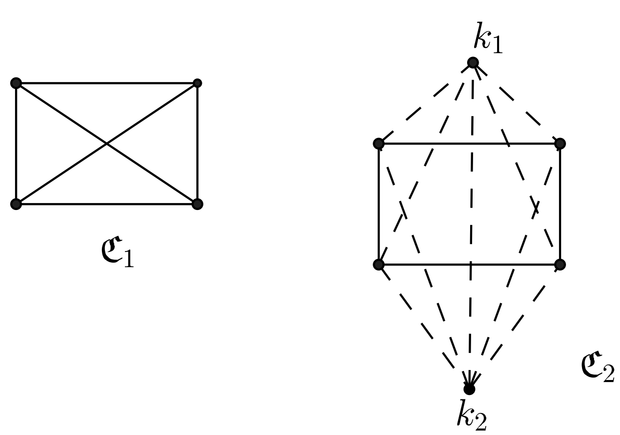


Figure 2. $con(SDB(4, 1, 1))$

□

Proposition 3.21. *For* $n \geq 5,\ Aut(con(SDB(n,1,1))) \cong S_n \times S_n$.

Proof. The automorphism group of each component of $con(SDB(n,1,1)), n \geq 5$ is isomorphic to the symmetric group S_n. Hence the result follows. □

Corollary 3.22. *For* $n > 4$*,* $Aut(con(SDB(n,1,1))) \cong Aut(\mathfrak{C}_1) \times Aut(\mathfrak{C}_2)$*, where* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *are connected components of* $con(SDB(n,1,1))$.

Proposition 3.23. *(i) The composition factors of* $Aut(con(SDB(3,1,1)))$ *are* $\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_3$.

(ii) The composition factors of $Aut(con(SDB(4,1,1)))$ *are* $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_2$.

(iii) For $n > 4$*, the composition factors of* $Aut(con(SDB(n,1,1)))$ *are* $\mathbb{Z}_2, A_n, \mathbb{Z}_2$*, and* A_n.

Proof. The result follows from the definition of $con(SDB(3,1,1))$ and the above propositions. □

4 Conclusion

In this paper, automorphism group of symmetric difference bipartite graphs $SDB(n,k,i)$ is discussed. If n is an even number greater than 2 and $k = \frac{n}{2}$, then the automorphism group of $SDB(n,k,i)$ is isomorphic to $S_n \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the cyclic group of order 2. Otherwise, the automorphism group of $SDB(n,k,i)$ is isomorphic to S_n. The automorphism group of the common neighbourhood graph of $SDB(n,k,i)$ for $k = i = 1$ were investigated. We determined the composition factors of the automorphism group of both $SDB(n,1,1)$ and $con(SDB(n,1,1))$ and verified the results using the computational tools like MAGMA and SAGE.

Conflict of Interest

The authors hereby declare that there is no potential conflict of interest.

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