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Frame Groupoids and Category of Representations of Lie groupoids

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Abstract A groupoid with smooth manifold structures on its object and morphism sets that makes various maps arise from the groupoid structure smooth is termed as a Lie groupoid. In this paper, we analyze the frame groupoids associated with vector bundles and discuss the category $Rep\,\mathcal{G}$ of all frame groupoid representations of a Lie groupoid $\mathcal{G}$, with the base preserving bundle morphisms that respects the representations as the morphisms . Further it is shown that this category is an additive monoidal category admitting kernel, cokernel and factorization of morphisms.

1 Introduction

The classical notion of representations of a Lie group can be generalized to the groupoid settings, namely the classical representation of a Lie groupoid. For a Lie groupoid $\mathcal{G} = G \rightrightarrows M$ and a smooth vector bundle (E, π, M), a representation of $\mathcal{G}$ over E is a Lie groupoid morphism from $\mathcal{G}$ to the frame groupoid Φ(E) (cf. [7]). All representations of a Lie groupoid G constitute the category $Re\,G$, and we explore its categorical properties in the following.

2 Preliminaries

In the following we assume familiarily with all basic concepts in category theory, groupoids and bundles. However, we recall briefly all essential definitions and results and for further details see cf.[7], cf.[8] and cf. [9].

A category comprises two classes $\mathcal{C}$ and $\nu\mathcal{C}$, referred to as the class of morphisms (arrows) and the class of objects (or vertices) along with a pair of mappings α (source map), β (target map) from $\mathcal{C}$ to $\nu\mathcal{C}$ and a map $\mathbf{1}$: $\nu\mathcal{C} \to \mathcal{C}$ with $A \mapsto 1_A$ called the object inclusion map(or unit map) together with a partial multiplication on the set $\mathcal{C} * \mathcal{C} = \{(h, g) \in \mathcal{C} \times \mathcal{C}\,|\alpha(h) = \beta(g)\}$ defined by $(h, g) \mapsto hg$ admitting the following conditions

(i) $\alpha(hg) = \alpha(g)$ & $\beta(hg) = \beta(h)\ \forall\ (h, g) \in \mathcal{C} * \mathcal{C}$

(ii) $j(hg) = (jh)g\,\forall\, j, h, g \in \mathcal{C}$ provided, $\alpha(j)= \beta(h)$ & $\alpha(h) = \beta(g)$

(iii) $\alpha(1_A)= \beta(1_A) = A \quad \forall\ A \in \nu\mathcal{C}$

(iv) $g1_{\alpha(g)} = g$ & $1_{\beta(g)}g = g \quad \forall\ g \in \mathcal{C}$.

We denote the category by $\mathcal{C} \rightrightarrows \nu\mathcal{C}$ or simply by $\mathcal{C}$. For A, $B \in \nu\mathcal{C}$ the set of all morphisms $g \in \mathcal{C}$ with $\alpha(g) = A$ and $\beta(g) = B$ is denoted by $\mathcal{C}(A, B)$. A morphism $m : A \to B$ in a category $\mathcal{C}$ is monic (also called *monomorphism*), if for any two parallel arrows $f_1, f_2 : D \to A$ in $\mathcal{C}$, the equality $m \circ f_1 = m \circ f_2 \Rightarrow f_1 = f_2$, that is, m is a monomorphism if it is left cancellable. Dually, a morphism $e : A \to B$ is epi (*epimorphism*) if it is right cancellable.

An object T is terminal in $\mathcal{C}$ if to each object A in $\mathcal{C}$ there is exactly one arrow $A \to T$ and an object S is initial in $\mathcal{C}$ if to each object A there is exactly one arrow $S \to A$ A zero object or null object Z in $\mathcal{C}$ is an object which is both initial and terminal

If $\mathcal{C}$ has a null object then for any two objects A and B of $\mathcal{C}$ there is a unique arrow $0_{A,B} : A \to 0 \to B$ called the *zero arrow* from A to B. (cf. [8]).

Definition 2.1. (cf. [8]) The *kernel* of a morphism $f : A \to B$ in a category $\mathcal{C}$ with zero objects is a pair (K, i) of an object K in $\mathcal{C}$ and a morphism $i : K \to A$ such that $f \circ i = 0$. Also if $i' : K' \to A$ is any other morphism with $f \circ i' = 0$ then there exist a unique arrow $h : K' \to K$ such that $i \circ h = i'$ (universal property).

Dually a pair (E, p) of an object E and a morphism $p : B \to E$ such that $p \circ f = 0$ and satisfying the universal property is called the *cokernel* of f in $\mathcal{C}$.

Definition 2.2. (cf. [6]) A *product* of two objects A and B in a category $\mathcal{C}$ is an object $A \Pi B$ together with two morphisms $p_1 : A \Pi B \to A$ and $p_2 : A \Pi B \to B$ that satisfies the universal property. ie, for any object C and any two morphisms $f_1 : C \to A, f_2 : C \to B$, there exist a unique morphism $h : C \to A \Pi B$ such that $p_i \circ h = f_i$ for $i = 1, 2$.

Similarly an object $A \amalg B$ together with morphisms $i_1 : A \to A \amalg B$ and $i_2 : B \to A \amalg B$ that satises the universal property is called the coproduct of A and B.

Definition 2.3. (cf. [8]) A category $\mathcal{C}$ is called *preadditive category* or *Ab-category* if each hom-set $\mathcal{C}(A, B)$ is an additive abelian group and composition is bilinear

$$ie.(g + g') \circ (f + f') = (g \circ f) + (g \circ f') + (g' \circ f) + (g' \circ f')$$

where $f, f' : A \to B$ and $g, g' : B \to C$. A preadditive category with a zero object in which every pair of objects admit a biproduct is called an additive category.

A category $\mathcal{P}$ is called a *preorder* if the hom-set $\mathcal{P}(p, p')$ contains atmost one morphism for any $p, p' \in \nu\mathcal{P}$. For any preorder P the relation $\subseteq$ defined by $p \subseteq p' \iff \mathcal{P}(p, p') \neq \phi$ is a quasi order on $\nu\mathcal{P}$. If $\subseteq$ is a partial order then P is said to be a *strict preorder* (see [9]).

Definition 2.4. (cf. [9]) Let $\mathcal{P}$ be a subcategory of $\mathcal{C}$. The pair ($\mathcal{C}$, $\mathcal{P}$) is called *category with subobjects* if the following conditions hold:

(i) $\mathcal{P}$ is a strict preorder with $\nu\mathcal{C} = \nu\mathcal{P}$.

(ii) Every $f \in \mathcal{P}$ is a monomorphism.

(iii) If $f, g \in \mathcal{P}$ and $f = g \circ h$ for some $h \in \mathcal{C}$ then $h \in \mathcal{P}$.

For each $C, D \in \nu\mathcal{C}$, the unique morphism in $\mathcal{P}$ from $C \to D$ is denoted by j_C^D and is called *inclusion*.Here the object C is referred to as a *subobject* of D.

Recall that a topological manifold M of dimension n is a Hausdorff, second countable, topological space in which each point has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. A coordinate chart on M is a pair (U, φ) where U is an open subset of M and φ is a homeomorphsm from U onto an open subset of $\mathbb{R}^n$. The collection of charts whose domains cover M is called an atlas. We say two charts (U, φ) and (V, ψ) are smoothly compatible if either $U \cap V = \phi$ or the transition map $\varphi \circ \psi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a smooth map in the sense of ordinary calculus. If any two charts in an atlas are smoothly compatible with each other, then the atlas is smooth. A smooth atlas is maximal if it is not properly contained in any larger smooth atlas and any such maximal smooth atlas is said to be a smooth structure on M, and a manifold M together with a smooth structure is called a smooth manifold (see cf. [5]).

Definition 2.5. (cf.[5]) Let M, N be two smooth manifolds. A map F: $M \to N$ is called a smooth map, if for every $p \in M$, there is a smooth chart (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and the composite map $\psi \circ F \circ \phi^{-1}$ is a smooth map from $\varphi(U)$ to $\psi(V)$.

A bijective smooth map F: $M \to N$ with a smooth inverse F^{-1}: $N \to M$ is called a diffeomorphism.

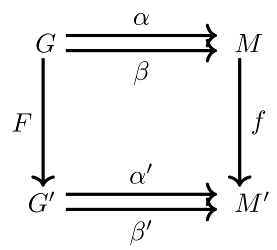
Definition 2.6. Let M be a smooth manifold and p be a point of M. The derivation at p is linear map $v : C^{\infty}(M) \to \mathbb{R}$ satisfying $v(fg) = f(p)vg + g(p)vf$ for all f,$g \in C^{\infty}(M)$. Set of all derivations of $C^{\infty}(M)$ at p is a vector space T_pM called the tangent space to M at p.

Definition 2.7. Let M and N be two smooth manifolds, F: $M \to N$ be a smooth map, for each $p \in M$ the diffential of F at p is a linear map $dF_p : T_pM \to T_{F(p)}N$ defined as, for $v \in T_pM$, $dF_p(v)$ is the derivation of $F(p)$ that acts on $f \in C^{\infty}(N)$ by the rule $dF_p(v)(f) = v(f \circ F)$, where $C^{\infty}(N)$ denotes the space of all smooth maps from M to N. The smooth map F is called a submersion if it's differential at each point is surjective.

2.1 Groupoid as a category.

A groupoid is a category $\mathbf{G} \rightrightarrows \mathbf{M}$ such that every arrow has an inverse. ie., for each $g \in \mathbf{G}$ ther is $g^{-1} \in \mathbf{G}$ such that $\alpha(g^{-1})$= $\beta(g)$, $\beta(g^{-1})$= $\alpha(g)$ and $g^{-1}g = 1_{\alpha(g)}$, $gg^{-1} = 1_{\beta(g)}$.

Definition 2.8. (cf. [7]) Let $\mathbf{G}$ and $\mathbf{G}'$ be groupoids on $\mathbf{M}$ and $\mathbf{M}'$ respectively. A morphism between them is a pair (F, f) of maps $F : \mathbf{G} \to \mathbf{G}'$, $f : \mathbf{M} \to \mathbf{M}'$ such that the diagram below commutes



and $F(hg) = F(h)F(g)$ for all $(h, g) \in \mathbf{G} * \mathbf{G}$, we say F is a morphism over f. If $\mathbf{M} = \mathbf{M}'$ and $f = id_{\mathbf{M}}$ then F is termed as a morphism over $\mathbf{M}$ or F is a base-preserving morphism.

Definition 2.9. (cf. [7]) A Lie groupoid is a groupoid $\mathbf{G} \rightrightarrows \mathbf{M}$ together with smooth structures on $\mathbf{G}$ and $\mathbf{M}$ such that the maps α, $\beta : \mathbf{G} \to \mathbf{M}$ are surjective submertions, the object inclusion map $x \mapsto 1_x$, $\mathbf{M} \to \mathbf{G}$ and the partial multiplication $\mathbf{G} * \mathbf{G} \to \mathbf{G}$ are smooth.

Example 2.10. (i) Any manifold $\mathbf{M}$ can be regarded as a Lie groupoid $\mathbf{M} \rightrightarrows \mathbf{M}$ with α = β = $id_{\mathbf{M}}$ and every morphism is an identity morphism. A groupoid in which every morphism is an identity morphism will be called a base groupoid.

(ii) Let $\mathbf{G} \times \mathbf{M} \to \mathbf{M}$ be a smooth action of a Lie group $\mathbf{G}$ on a manifold $\mathbf{M}$. Assign the product manifold $\mathbf{G} \times \mathbf{M}$ the structure of a Lie groupoid on $\mathbf{M}$ in the following way

- α be the projection into the second factor of $\mathbf{G} \times \mathbf{M}$, β be the group action itself.
- The object inclusion map is $x \mapsto 1_x$ = $(1, x)$.
- partial multiplication is $(g_2, y)(g_1, x) = (g_2 g_1, x)$ which is defines iff $y = g_1 x$.
- The inverse of (g, x) is (g^{-1}, gx).

then this structure $\mathbf{G} \times \mathbf{M}$ denoted by $\mathbf{G} \triangleleft \mathbf{M}$ is called the action groupoid $\mathbf{G} \times \mathbf{M} \to \mathbf{M}$.

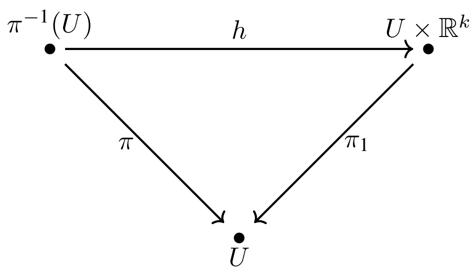
Let $\mathbf{G}$ and $\mathbf{G}'$ be two Lie groupoids with base $\mathbf{M}$ and $\mathbf{M}'$ respectively. A Lie groupoid morphism is the groupoid morphism (F, f) with both F and f are smooth and (F, f) is an isomorphism if both F and f are diffeomorphisms. If $f = id_M$, we call it a base preserving morphism between the Lie groupoids $\mathbf{G}$ and $\mathbf{G}'$.

3 Bundles and Representations of Lie Groupoid

A smooth real vector bundle E of rank k over a smooth manifold M is a triple (E, π, M), where E and M are smooth manifolds called total space and base space respectively and $\pi : E \to$ M is a surjective submersion, called bundle projection such that the following conditions hold;

(i) for each $p \in M$ the fiber $E_p = \pi^{-1}(p)$ of E over p is a real k dimensional vector space.

(ii) for every point $p \in M$ there exist a neighborhood U of p in M and a diffeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that the following diagram commutes. (cf. [5])



Example 3.1. If M is a smooth manifold of rank k, $TM = \sqcup T_pM$ be the disjoint union of the tangent spaces at all points of M and $\pi : TM \to M$, is the map defined by $\pi(p, v)) = p$ where $(p, v) \in T_pM$, then $\pi : TM \to M$ defines a smooth vector bundle of rank k.

If π: $E \to M$ and π': $E' \to M'$ are vector bundles, a smooth map $F : E \to E'$ is called a bundle homomorphism if there exists a map f: $M \to M'$ satisfying $\pi' \circ F= f \circ \pi$, with the property that for each $p \in M$; the restricted map $F|E_p : E_p \to E'_{f(p)}$ is linear .

3.1 Frame groupoid of bundles

Let (E, p, M) be a smooth vector bundle over a manifold M of dimension n. Consider the set $\Phi(E) = \{E_x \xrightarrow{\psi} E_y : x, y \in M \text{ and } \psi \text{ linear isomorphism}\}$. Define

$$\alpha : \Phi(E) \to M \text{ by } (E_x \xrightarrow{\psi} E_y) \mapsto x$$

$$\beta : \Phi(E) \to M \text{ by } (E_x \xrightarrow{\psi} E_y) \mapsto y$$

$$1 : M \to \Phi(E) \text{ by } x \mapsto (E_x \xrightarrow{id} E_x)$$

and a partial multiplication $m : \Phi(E) \times_M \Phi(E)$ by

$$m(E_x \xrightarrow{\psi} E_y, E_z \xrightarrow{\eta} E_x) \mapsto (E_z \xrightarrow{\psi \circ \eta} E_y).$$

Define $i : \Phi(E) \to \Phi(E)$ by $(E_x \xrightarrow{\psi} E_y) \mapsto (E_y \xrightarrow{\psi^{-1}} E_x)$, then $\Phi(E) \rightrightarrows M$ is a Lie groupoid called the frame groupoid of the vector bundle $E \to M$; (see [7]).
Next we discuss the representations (also termed classical representations) of a Lie groupoid $\mathcal{G} = G \rightrightarrows M$ and it is shown that the collection of all these representations of $\mathcal{G}$ is a category, which we denote by $Rep\,(\mathcal{G})$ (see. cf. [1]).

Definition 3.2. A representation of a Lie groupoid $\mathcal{G}$=$G \rightrightarrows M$ is the pair (E, ρ), where E is a smooth vector bundle over M and ρ is a base-preserving Lie groupoid morphism from $\mathcal{G} \to \Phi(E)$; (cf. [7])

Example 3.3. Consider the action $\star$ of the Lie group G = $\mathbb{R}$ on the smooth manifold $M = \mathbb{S}^1$ defined by $\star(r, z) = e^{2\pi ir} z$ and the corresponding action groupoid $\mathbb{R} \times \mathbb{S}^1 \rightrightarrows \mathbb{S}^1$ with source map the projection into the second component and target map the Lie group action. Now consider the tangent bundle T$\mathbb{S}^1 \to \mathbb{S}^1$ and define a map $f : \mathbb{R} \times \mathbb{S}^1 \to \Phi(T\mathbb{S}^1)$ by

$$f(r, z) = h_{z, e^{2\pi ir} z}$$

where $h_{z, e^{2\pi ir} z}$ is the isomorphism of the tangent space $T_z\, \mathbb{S}^1$ to $T_{ze^{2\pi ir}}\mathbb{S}^1$ given by

$$h_{z, e^{2\pi ir} z}(z, v) = (ze^{2\pi ir}, v).$$

Diagramatically,

$$\begin{array}{ccc} \mathbb{R}\times\mathbb{S}^1 & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & \mathbb{S}^1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id} \\ \Phi(\mathrm{T}\mathbb{S}^1) & \overset{\alpha'}{\underset{\beta'}{\rightrightarrows}} & \mathbb{S}^1 \end{array}$$

where f is a a smooth mapping satisfying $id \circ \alpha = \alpha' \circ f$ and $id \circ \beta = \beta' \circ f$. Also

$$\begin{aligned} f((r_1, z_1)(r_2, z_2)) &= f(r_1 + r_2, z_2) \\ &= h_{z_2, e^{2\pi i(r_1+r_2)} z_2} \\ &= h_{z_2, e^{2\pi i r_1} e^{2\pi i r_2} z_2} \\ &= h_{z_2, e^{2\pi i r_1} z_1} \\ &= h_{z_1, e^{2\pi i r_1} z_1} h_{z_2, e^{2\pi i r_2} z_2} \\ &= f((r_1, z_1)) f((r_2, z_2)) \end{aligned}$$

whenever the composition $(r_1, z_1)(r_2, z_2)$ is defined. Therefore f is a Lie groupoid morphism and hence defines a representation of the action groupoid $\mathbb{R}\times\ \mathbb{S}^1 \rightrightarrows \mathbb{S}^1$.

Let (V, ρ) be a representation of a Lie groupoid $\mathcal{G}$. A subrepresentation of $\mathcal{G}$ is a subbundle V' of V stable under the groupoid action, i.e, for each arrow $g{:}x \to y$ in $\mathcal{G}$ $\rho_g|_{V'_x}$ is an isomorphism from V'_x to V'_y.

3.2 Category of representations of Lie groupoids

Consider the class of all representations of a Lie groupoid $\mathcal{G}$ of the form (E, ρ), where E is any smooth vector bundle (of finite rank) over M and ρ is a Lie groupoid morphism from $\mathcal{G} \to \Phi(E)$ (see Definition 3.2). For two such representations (E, ρ) and (F, δ), a morphism between them is defined as a vector bundle morphisms $\mathfrak{a} : E \to F$ such that the diagram below commutes for all $g \in \mathcal{G}(x, x'), x, x' \in M$.

$$\begin{array}{ccc} E_x & \xrightarrow{\mathfrak{a}_x} & F_x \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow{\scriptstyle \delta(g)} \\ E_{x'} & \xrightarrow{\mathfrak{a}_{x'}} & F_{x'} \end{array}$$

this is a category, and denote it by $Rep\,(\mathcal{G})$. For two morphisms $\mathfrak{a} : (E, \rho) \to (F, \delta)$ and $\mathfrak{b} : (F, \delta) \to (K, \sigma)$ with vector bundles (E, π, M), (F, π', M) and (K, π'', M), the composition $\mathfrak{b}{\circ}\mathfrak{a}$ defined in the usual sense. i.e we have the following diagrams commute:

$$\begin{array}{ccccc} E & \xrightarrow{\mathfrak{a}} & F & \xrightarrow{\mathfrak{b}} & K \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi''} \\ M & \xrightarrow[id]{} & M & \xrightarrow[id]{} & M \end{array} \qquad \begin{array}{ccccc} E_x & \xrightarrow{\mathfrak{a}_x} & F_x & \xrightarrow{\mathfrak{b}_x} & K_x \\ {\scriptstyle \rho_g}\downarrow & & {\scriptstyle \delta_g}\downarrow & & \downarrow{\scriptstyle \sigma_g} \\ E_{x'} & \xrightarrow[\mathfrak{a}_{x'}]{} & F_{x'} & \xrightarrow[\mathfrak{b}_{x'}]{} & K_{x'} \end{array}$$

Since both $\mathfrak{a}$ and $\mathfrak{b}$ are smooth maps $\mathfrak{b} \circ \mathfrak{a}$ is smooth and from the above diagram it follows that,

$$\pi'' \circ (\mathfrak{b} \circ \mathfrak{a}) = \pi' \circ \mathfrak{a} = \pi$$

and for each $g : x \to x'$,

$$\begin{aligned}\sigma_g \circ (\mathfrak{b} \circ \mathfrak{a})_x &= \sigma_g \circ \mathfrak{b}_x \circ \mathfrak{a}_x \\ &= \mathfrak{b}_{x'} \circ \delta_g \circ \mathfrak{a}_x \\ &= \mathfrak{b}_{x'} \circ \mathfrak{a}_{x'} \circ \rho_g \\ &= (\mathfrak{b} \circ \mathfrak{b})_{x'} \circ \rho_g.\end{aligned}$$

For $x \in M$ and $\alpha \in E_x$, $\mathfrak{b} \circ \mathfrak{a}(\alpha) = \mathfrak{b}_x \circ \mathfrak{a}_x(\alpha)$. ie $(\mathfrak{b} \circ \mathfrak{a})|_{E_x} = \mathfrak{b}_x \circ \mathfrak{a}_x$, which is linear and so $\mathfrak{b} \circ \mathfrak{a}$ is again a morphism in this category. It is easy to see that the composition is associative and for each object (E, ρ) the identity bundle morphism i acts as the identity morphism.

If we replace the object set M of a frame groupoid $\Phi(E)$ by the set of fibers $\{E_x : x \in M\}$ of E, we obtain a subcategory of the category of vector spaces $\mathbf{Vect}$. Then ρ induces a functor $\rho' : \mathcal{G} \to \mathbf{Vect}$ whose arrow function is same as that of ρ and the object function maps each object x to the corresponding fiber E_x of E.

Example 3.4. Consider the smooth manifold $\mathbb{R}$, let $\mathcal{G} = \mathbb{R} \rightrightarrows \mathbb{R}$ be the base groupoid. Then any vector bundle E over $\mathbb{R}$ with trivial morphism $\mathcal{G} \to \Phi(E)$ is a representation of $\mathcal{G}$. So the category $Rep(\mathcal{G})$ coincides with the category $\mathbf{Vect}_{\mathbb{R}}$ of all smooth vector bundles over $\mathbb{R}$. Morphism between two such representations is the vector bundle morphism between them.

Theorem 3.5. *Let $\mathcal{G} = G \rightrightarrows M$ be a Lie groupoid. The category $Rep\,\mathcal{G}$ of all representations of $\mathcal{G}$ is an additive category.*

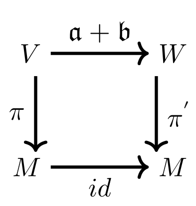
Proof. Let (V, ρ) and $(W, \delta) \in \nu$ Rep($\mathcal{G}$) and $\mathfrak{a}, \mathfrak{b} \in$ Hom((V, ρ),(W, δ)). Define $\mathfrak{a}+\mathfrak{b}$: $V \to W$ by

$$\begin{aligned}(\mathfrak{a} + \mathfrak{b})(\alpha) &= \mathfrak{a}(\alpha) + \mathfrak{b}(\alpha) \\ &= \mathfrak{a}_x(\alpha) + \mathfrak{b}_x(\alpha)\end{aligned}$$

provided $\alpha \in V_x$. Since $\mathfrak{a}$ and $\mathfrak{b}$ are smooth maps, $(\mathfrak{a} + \mathfrak{b})$ is also smooth and for each $\alpha \in V_x$,

$$\begin{aligned}\pi' \circ (\mathfrak{a} + \mathfrak{b})(\alpha) &= \pi'(\mathfrak{a}_x(\alpha) + \mathfrak{b}_x(\alpha)) \\ &= x \\ &= id \circ \pi(\alpha).\end{aligned}$$

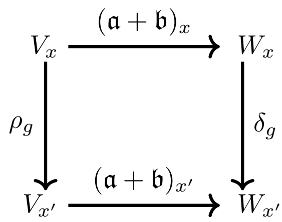
i.e, the following diagram commutes.



Also the restriction $(\mathfrak{a}+\mathfrak{b})|_{V_x} : V_x \to W_x$ is the linear map $\mathfrak{a}_x + \mathfrak{b}_x$ and $\mathfrak{a} + \mathfrak{b}$ is a smooth bundle morphism from $V \to W$. For each arrow g: $x \to x'$,

$$\begin{aligned}\delta_g \circ (\mathfrak{a} + \mathfrak{b})_x(\alpha) &= \delta_g(\mathfrak{a}_x(\alpha) + \mathfrak{b}_x(\alpha)) \\ &= \delta_g(\mathfrak{a}_x(\alpha)) + \delta_g(\mathfrak{b}_x(\alpha)) \\ &= \mathfrak{a}_{x'}\rho_g(\alpha) + \mathfrak{b}_{x'}\rho_g(\alpha) \\ &= (\mathfrak{a}_{x'} + \mathfrak{b}_{x'})(\rho_g(\alpha)) \\ &= (\mathfrak{a} + \mathfrak{b})_{x'} \circ \rho_g(\alpha)\end{aligned}$$

hence the diagram below commutes



and $\mathfrak{a}+\mathfrak{b} \in Hom((V,\rho),(W,\delta))$. If $\mathfrak{a},\mathfrak{b},\mathfrak{c} \in$ Hom((V, ρ),(W, δ)), then

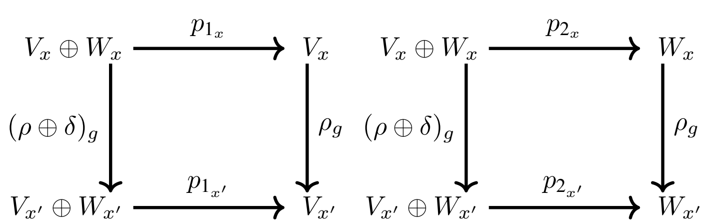
$$\begin{aligned}((\mathfrak{a}+\mathfrak{b})+\mathfrak{c})(\alpha) &= ((\mathfrak{a}_x+\mathfrak{b}_x)+\mathfrak{c}_x)(\alpha)\\ &= (\mathfrak{a}_x+(\mathfrak{b}_x+\mathfrak{c}_x))(\alpha)\\ &= (\mathfrak{a}+(\mathfrak{b}+\mathfrak{c}))(\alpha)\end{aligned}$$

and the zero bundle morphism 0: $V \to W$ defined by 0(α) = 0_{W_x} whenever $\alpha \in V_x$, acts as the identity element for addition. For $\mathfrak{a} \in$ Hom((V, ρ),(W, δ)), the map $-\mathfrak{a} : V \to W$ defined by $-\mathfrak{a}(\alpha) = -\mathfrak{a}_x(\alpha)$ where $\alpha \in V_x$, is the inverse for $\mathfrak{a}$. Clearly, for $\mathfrak{a}$, $\mathfrak{b} \in$ Hom((V, ρ),(W, δ)), we have $(\mathfrak{a}+\mathfrak{b})(\alpha) = (\mathfrak{b}+\mathfrak{a})(\alpha)$ provided $\alpha \in V_x$ and so Hom((V, ρ), (W, δ)) is an additive abelian group and the composition is bilinear. Therefore Rep($\mathcal{G}$) is a preadditive category.

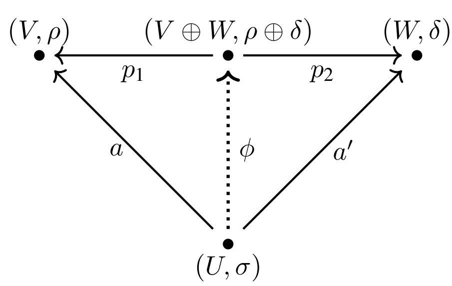
For (V, ρ), (W, δ) $\in \nu Rep(\mathcal{G})$, let $V \oplus W$ be the direct sum of the vector bundles V and W, which is again a smooth vector bundle, where fibre at each point $x \in M$ is given by the vector space direct sum $V_x \oplus W_x$. Now define $\rho \oplus \delta$ from the Lie groupoid $\mathcal{G}$ to the frame groupoid $\Phi(V\oplus W)$ of the direct sum bundle $V\oplus W$ such that each arrow g: $x \to y$ is mapped to an isomorphism ($\rho \oplus \delta$)(g) from $V_x \oplus W_x$ to $V_y \oplus W_y$ given by

$$(\rho \oplus \delta(g))(e, f) = (\rho_g(e), \delta_g(f)).$$

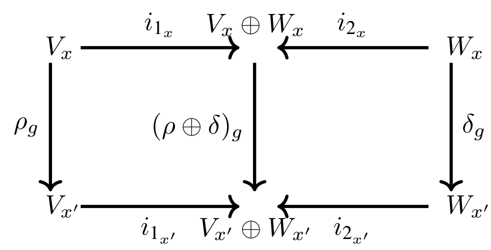
Then $\rho \oplus \delta$ is a Lie groupoid morphism and $(V \oplus W, \rho \oplus \delta)$ is an object in the category Rep($\mathcal{G}$). The projection maps $p_1 : (V \oplus W, \rho \oplus \delta) \to (V,\rho)$ and $p_2 : (V \oplus W, \rho \oplus \delta) \to (W,\delta)$ given by $p_1(v,w) = v$ and $p_2(v,w) = w$ respectively are smooth bundle morphisms such that the following diagrams commute.



hence p_1 and p_2 are morphisms in the category Rep($\mathcal{G}$). If (U, σ) is an object with two arrows a, a' as shown in the diagram below then there exist a morphism ϕ : (U, σ) $\to$ ($V \oplus W$, $\rho \oplus \delta$) in the category Rep($\mathcal{G}$), defined by, $\phi(u)$ = $(a(u), a'(u))$ that makes the diagrams commutes



therefore $((V \oplus W, \rho \oplus \delta), p_1, p_2)$ defines the product of (V,ρ) and (W,δ) in Rep($\mathcal{G}$). Also the inclusion maps $i_1 : (V,\rho) \to (V \oplus W, \rho \oplus \delta)$ and $i_2 : (W,\delta) \to (V \oplus W, \rho \oplus \delta)$, given by $i_1(\alpha)$ = $(\alpha, 0_{W_x})$ and $i_2(\beta)$ = ($0_{V_x}, \beta$), where $\alpha \in V_x$ and $\beta \in W_x$ respectively are smooth bundle morphisms that makes the following diagrams also commutes



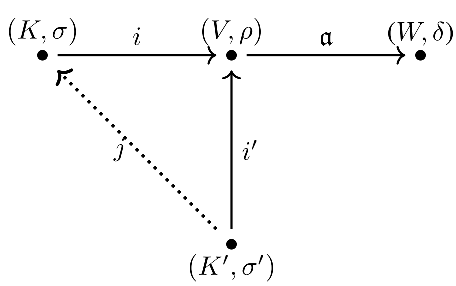
Hence $(V \oplus W, \rho \oplus \delta)$ together the inclusions i_1 and i_2 defines the coproduct of the objects (V, ρ) and (W, δ). For the zero object in this category, consider the smooth vector bundle $i : M \to M$ (of rank zero) and a map $\rho_0 : \mathcal{G} \to \phi(M)$ defined by $\rho_0(g) = 0_{x,y}$ where $g \in \mathcal{G}(x, y)$, and $0_{x,y}$ is the zero map $M_x \to M_y$. Clearly $(M, \rho_0) \in \nu$ Rep($\mathcal{G}$) and for any object (V, ρ) in this category there is a unique morphism $(M, \rho_0) \to (V, \rho)$ namely the zero bundle morphism that is trivial in each fiber. So the object (M, ρ_0) is an initial object in this category. Similarly for each object (V, ρ), there is a unique morphism $(V, \rho) \to (M, \rho_0)$ which is trivial in each fiber and so (M, ρ_0) is also a terminal. Therefore (M, ρ_0) defines the zero object in Rep($\mathcal{G}$) and the category Rep($\mathcal{G}$) is additive. □

Next we recall the following proposition which will be used in the sequel.

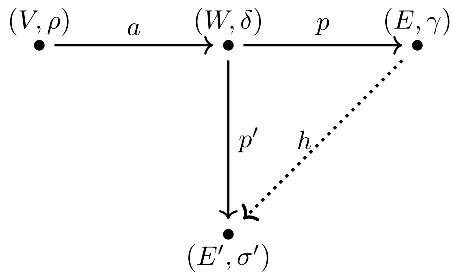
Proposition 3.6. *(cf. Theorem* 10.34 *in [5]) Let E and E' be smooth vector bundles over a smooth manifold M and let $F : E \to E'$ be a smooth bundle homomorphism over M. Then KerF= $\bigcup_{p \in M} Ker(F|_{E_p})$ and ImF= $\bigcup_{p \in M} Im(F|_{E_p})$ are smooth subbundles of E and E' respectively, if and only if F has constant rank.*

Proposition 3.7. *In the category Rep($\mathcal{G}$) of all classical representations of $\mathcal{G}$, any morphism with constant rank admits kernel and cokernel.*

Proof. Let $\mathfrak{a} : (V, \rho) \to (W, \delta)$ be a morphism in the category Rep($\mathcal{G}$) with constant rank and let K = Ker $\mathfrak{a}$ be the kernel of the bundle morphism $\mathfrak{a}$, where Ker $\mathfrak{a}$ = $\cup_{x \in M}$ Ker$\mathfrak{a}_x$, $\mathfrak{a}_x$= $\mathfrak{a}|_{V_x}$, is a smooth subbundle of V. Define a map σ from $\mathcal{G}$ to the frame groupoid $\phi(K)$ corresponding to K, by $\sigma(g)$=$\rho(g)|_{K_x}$, where x=$\alpha(g)$ and K_x= Ker($\mathfrak{a}_x$). Then σ is a representation of $\mathcal{G}$ and hence (K, σ) is an object in the category Rep($\mathcal{G}$). Define i:(K,σ) $\to$ (V, ρ) by $i(k)$= k, then (K, σ) together with this morphism defines kernel of $\mathfrak{a}$. The universal property follows, for if (K', σ') is any other object with a morphism i': (K',σ') $\to (V, \rho)$ satisfying $a \circ i'$= 0, then there is a unique morphism j: (K',σ') $\to (K, \sigma)$ which is given by $j(k)$= $i'(k)$, such that $i \circ j = i'$.



Similarly, $Im\,\mathfrak{a} = \cup_{x \in M} Im\,\mathfrak{a}_x$, is a smooth subbundle of W and E= W/Im $\mathfrak{a}$, is the quotient bundle. Then for each $x \in M$ we have the fiber E_x = W_x/Im $\mathfrak{a}_x$. Define a map $\gamma : \mathcal{G} \to \phi(E)$ as follows: for each arrow g: $x \to y$ in $\mathcal{G}$, $\gamma(g) : W_x/Im\mathfrak{a}_x \to W_y/Im\mathfrak{a}_y$ is given by $\gamma(g)(w + Im\,\mathfrak{a}_x) = \delta(g)(w) + Im\,\mathfrak{a}_y$. Then γ is a representation of $\mathcal{G}$ and (E, γ) is an object in the category $Rep(\mathcal{G})$. Let p: (W,δ) $\to (E, \gamma)$ be the quotient map, which is the usual projection in each fiber. Then the object (E, γ) together with the morphism p defines the cokernel of a. If (E', γ') is any object with a morphism p': (W,δ) $\to (E', \gamma')$, then there is a unique morphism h: (E,γ) $\to (E', \gamma')$ that makes the following diagram commute.



□

Recall that a category $\mathcal{C}$ equipped with a monoidal structure is called a monoidal category (c.f [8]). Next theorem shows that the category of all representations of a Lie groupoid is a monoidal category.

Theorem 3.8. *The category $Rep(\mathcal{G})$ of all representations of $\mathcal{G}$ is a monoidal category.*

Proof. Define the monoidal product $\otimes : Rep(\mathcal{G}) \times Rep(\mathcal{G}) \to Rep(\mathcal{G})$ as the direct sum functor whose object function

$$\otimes(((V,\rho),(W,\delta))) = (V \oplus W, \rho \oplus \delta)$$

and arrow function

$$(a,b) : ((V,\rho),(W,\delta)) \to ((V',\rho'),(W',\delta')) \text{ to } a \oplus b : (V \oplus W, \rho \oplus \delta) \to (V' \oplus W', \rho' \oplus \delta')$$

and is defined in each fiber by $a \oplus b\ ((v,w)) = (a(v), b(w))$. The monoidal unit is the zero object (M,ρ_0) in $Rep(\mathcal{G})$ and the associator α is defined with components

$$\alpha_{(U,\sigma),(V,\rho),(W,\delta)} : (U \oplus (V \oplus W), \sigma \oplus (\rho \oplus \delta)) \to (U \oplus V) \oplus W, (\sigma \oplus \rho) \oplus \delta)$$

given by $\alpha_{(U,\sigma),(V,\rho),(W,\delta)}(u \oplus (v \oplus w)) = (u \oplus v) \oplus w$. The right unitor $\lambda_{(U,\sigma)}$: $(M \oplus U, \rho_0 \oplus \sigma) \to (U,\sigma)$ that maps $(m \oplus u)$ to u and the left unitor $\varrho_{(U,\sigma)}$: $(U \oplus M, \sigma \oplus \rho_0) \to (U,\sigma)$ is given by $\varrho_{(U,\sigma)}\ (u \oplus m) = u$. □

Lemma 3.9. *The category $Rep(\mathcal{G})$, of all representations of $\mathcal{G}$ is a category with subobjects.*

Proof. For objects (V,ρ) and (W,δ) in the category $Rep(\mathcal{G})$, define $\subseteq$ by $(V,\rho) \subseteq (W,\delta)$ if and only if V is a sub representation of W. Then $\subseteq$ is a partial order in $\nu Rep(\mathcal{G})$ and for two objects $(V,\rho) \subseteq (W,\delta)$ the morphism $j^{(W,\delta)}_{(V,\rho)} : (V,\rho) \to (W,\delta)$ given by $j^{(W,\delta)}_{(V,\rho)}(v)$=$v$ for all $v \in V$ is the inclusion. Let $\mathcal{P}$ be the category with $\nu(\mathcal{P})$= νRep($\mathcal{G}$) and morphisms are precisely the monomorphisms of the form $j^{(W,\delta)}_{(V,\rho)}$. Then the subcategory $\mathcal{P}$ of Rep($\mathcal{G}$) is a strict preorder and for any $j^{(W,\delta)}_{(V,\rho)}, j^{(W,\delta)}_{(U,\sigma)} \in \mathcal{P}$ with some h $\in$ Rep($\mathcal{G}$) such that $j^{(W,\delta)}_{(V,\rho)}$= $j^{(W,\delta)}_{(U,\sigma)}h$, we have $j^{(W,\delta)}_{(V,\rho)}(v)$= $j^{(W,\delta)}_{(U,\sigma)}(h(v))$= $h(v)$. Hence $h \in \mathcal{P}$ and the category (Rep($\mathcal{G}$), $\mathcal{P}$) is a category with subobjects. □

Theorem 3.10. *In the category Rep($\mathcal{G}$) any morphism with constant rank has a normal factorisation.*

Proof. Let $\mathfrak{a} : (V,\rho) \to (W,\delta)$ be a morphism with constant rank. Then K= Ker($\mathfrak{a}$) and I = Im($\mathfrak{a}$) are subbundles of V. For each $x \in M$, $K_x^\perp \subset V_x$ is

$$K_x^\perp = \{v \in V_x : \forall k \in K_x, \langle v,k\rangle = 0\}$$

where $\langle .,. \rangle$ is the product given by the Riemannian metric. The spaces $K_x^\perp$ are the fibers of a vector bundle $K^\perp$ over M, which is a subbundle of V. Also V= $K \oplus K^\perp$ and $K^\perp$ gives sub representation of (V,ρ), namely $(K^\perp,\rho')$. Then the projection $q : (V,\rho) \to (K^\perp,\rho')$, $u = \mathfrak{a}|_{(K^\perp,\rho')}$ and j= $j^{(W,\delta)}_{(I,\delta')}$ gives a factorisation $\mathfrak{a} = quj$. □

Since every subbundle has an orthogonal complement, every inclusion splits in this category and for any sub representation (V', ρ') of (V, ρ) there is a sub representation $(V'^{\perp}, \rho'')$ such that (V, ρ)=$(V', \rho')$$\oplus$$(V'^{\perp}, \rho'')$. Hence the projection p:$(V, \rho) \to (V', \rho')$ is a right inverse for the inclusion $j((V', \rho'), (V, \rho))$. Consider a sub category $\mathfrak{R}(\mathcal{G})$ of Rep($\mathcal{G}$) obtained by restrictiong the morphism sets to be the set of all constant rank morphisms the category $\mathfrak{R}(\mathcal{G})$ has the natural choice of subobjects, every morphism has a normal factorisation and so it is seen that the category $\mathfrak{R}(\mathcal{G})$ is regular.

4 Concluding remarks

In this paper we discuss the fame groupoid of a bundle and the representaions of Lie groupoid. Further we consider the category $Rep\,(G)$ of all representations of a finite dimensional Lie groupoid and discused certain categorical properties. However there are several interesting questions such as how far the category $Rep\,(G)$ characterize the groupoids, that has to be considered in future research.

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