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Axiomatic characterization of the interval function of line graphs and its related graphs

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Abstract: In graph theory, line graphs play a critical role in understanding relationships between edges of a graph, where the edges of an original graph G become the vertices of its corresponding line graph $L(G)$. The transit function is a set function defined on the Cartesian product $V \times V$ of a non-empty set V to the power set of V satisfying the extensive, symmetric, and idempotent axioms. The interval function I_G of a connected graph G with vertex set V is a well studied transit function on V and is an important concept in metric graph theory.

In this paper, we propose certain independent first-order betweenness axioms on the interval function I_G of a graph G and prove that I_G satisfies these axioms if and only if the graph G is a line graph. Further, we extend this result to related classes of line graphs, including the line graphs of triangle-free graphs, line graphs of multigraphs without triangles, line graphs of linear hypergraphs of rank 3, line graphs of bipartite graphs, and line graphs of bipartite multigraphs.

Keywords: line graphs, forbidden subgraphs, transit function, interval function.

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1 Introduction and Preliminaries

Transit functions were introduced by Mulder [9] to generalize the three classical notions in mathematics, namely, convexity, interval, and betweenness in an axiomatic approach.

A *transit function* on V is a function $R : V \times V \to 2V$ that satisfies the following three axioms:

$(t1)$ $u \in R(u, v)$, for all $u, v \in V$.

$(t2)$ $R(u, v) = R(v, u)$, for all $u, v \in V$.

$(t3)$ $R(u, u) = \{u\}$, for all $u \in V$.

If V is the vertex set of a graph G, then the transit function defined on V is the transit function on G. Given a transit function R on V, one can define the underlying graph G_R of a transit function R on V as the graph with vertex set V, where two distinct vertices u and v are joined by an edge if and only if $R(u, v) = \{u, v\}$. The interval function I_G of a connected graph G is an important notion in metric graph theory and an essential concept in the study of the metric properties of graphs. It is defined as a function that maps for every pair of vertices u, v in G, the set

$$I_G(u, v) = \{w \mid w \text{ lies on a shortest } u, v\text{-path}\}.$$

When no confusion arises for the graph G, we write I instead of I_G. The first systematic study of the interval function is due to Mulder in [8], where the term interval function was coined. The following betweenness axioms were considered by Mulder in [8].

$(b1)$: If $x \in R(u, v)$, $x \neq v$ then $v \notin R(u, x)$.

$(b2)$: If $x \in R(u, v)$ then $R(u, x) \subseteq R(u, v)$.

$(b3)$: If $x \in R(u, v)$ and $y \in R(u, x)$ then $x \in R(y, v)$.

$(b4)$: If $x \in R(u, v)$ then $R(u, x) \cap R(x, v) = \{x\}$.

Nebeský addressed an interesting problem on the interval function I of a connected graph $G = (V, E)$ during the 1990s as follows. "Is it possible to give a characterization of the interval function I_G of a connected graph G by a set of simple axioms (first-order axioms) defined on an arbitrary transit function R on V?" Nebeský [12] proved that there exists such a characterization for the interval function $I(u, v)$ in terms of a set of first-order axioms on a transit function R. The axiomatic studies in metric graph theory captured attention due to the various characterizations of the interval function I by Nebeský [12, 13, 14], Nebeský and Mulder [11].

We now state the Mulder-Nebeský theorem stated in [11] characterizing the interval function of an arbitrary connected graph using axioms on an arbitrary transit function. In addition to the transit axioms $(t1), (t2)$ and the betweenness axioms $(b2), (b3), (b4)$, the following axioms $(s1)$ and $(s2)$ are required for the characterization.

$(s1)$ If $R(u, \bar{u}) = \{u, \bar{u}\}, R(v, \bar{v}) = \{v, \bar{v}\}, u \in R(\bar{u}, \bar{v})$ and $\bar{u}, \bar{v} \in R(u, v)$ then $v \in R(\bar{u}, \bar{v})$.

$(s2)$ If $R(u, \bar{u}) = \{u, \bar{u}\}, R(v, \bar{v}) = \{v, \bar{v}\}, \bar{u} \in R(u, v)$, $v \notin R(\bar{u}, \bar{v}), \bar{v} \notin R(u, v)$ then $\bar{u} \in R(u, \bar{v})$.

Theorem 1.1. *[11] Let $R : V \times V \longrightarrow 2^V$ be a function on V, satisfying the axioms $(t1), (t2)$, $(b2), (b3), (b4)$ with the underlying graph G_R and let I be the interval function of G_R. The following statements are equivalent.*

(a) $R = I$.

(b) *R satisfies axioms $(s1)$ and $(s2)$.*

The axiomatic characterization of the interval function of special class of graphs also became an interesting problem. For example, axiomatic characterization of the interval function of trees [16, 5], median graphs [17, 10, 8], geodetic graphs [14], block graphs [2], weakly modular graphs and their principal subclasses and superclasses, partial cubes and their principal subclasses and superclasses [3].

In [4] it is proved that if R satisfies the axioms $(b1)$ and $(b2)$, then the underlying graph G_R of R is connected and both the axioms $(b1)$ and $(b2)$ are necessary for the connectivity of G_R. It is clear that the axioms $(t1)$ and $(b2)$ imply the axiom $(t3)$. Hence, any function R: $V \times V \to 2^V$ satisfying the five classical axioms is a transit function. Furthermore, it is easy to see that we have the following implications. The axioms $(t1)$, $(t2)$ and $(b4)$ imply $(b1)$, see [2]. Axioms $(t1)$, $(t2)$, $(t3)$ and $(b3)$ imply axiom $(b4)$, see [11]. Hence if R is a transit function that satisfies the axiom $(b3)$ then R satisfies the axioms $(b4)$ and $(b1)$.

In this paper, we extend the characterization of the interval function to line graphs and some of its related graph classes. Harary and Norman formally coined the term line graph [15] in 1960, but, in 1932 Whitney [18] and in 1943 Krausz [7] used the construction of line graphs in their papers without mentioning the term line graph.

In the rest of this section, we fix some of the graph-theoretical notation and terminology used in this paper. In this paper, we consider only finite simple and undirected graphs, where an edge $e = uv$ joining the vertex u and the vertex v can be represented as the unordered set $\{u, v\}$. Let G be a graph and H a subgraph of G, H is called an *induced* subgraph if u, v are vertices in H such that uv is an edge in G, then uv must also be an edge in H. A path in G that is induced as a subgraph is an *induced path*. A graph G is said to be *H-free*, if G has no induced subgraph isomorphic to H. Let $G_1, G_2, \ldots, G_k$ be graphs. For a graph G, we say that G is *$(G_1, G_2, \ldots, G_k)$-free* if G has no induced subgraph isomorphic to G_i, $i \in \{1, \ldots, k\}$. Also, a graph G is said to have a *forbidden induced subgraph characterization* if G possess a characterization in terms of all its forbidden induced subgraphs. The *complement graph* of G is denoted as usual by $\overline{G}$ or co-G. The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2 = (V1 \cup V2, E1 \cup E2)$. A bipartite graph is a graph whose vertex set can be divided into two sets, such that each edge of the graph consists of a vertex from one set and a vertex from the other set. A *triangle* is a set of three pairwise adjacent vertices. A graph is triangle-free if it has no triangles. Clearly a bipartite graph is a triangle free graph. Given a family $\mathcal{F}$ of subsets of a set F, then the *intersection graph of $\mathcal{F}$* is the graph with vertex set $\mathcal{F}$ and two vertices (that is, subsets, say F_1 and F_2 of $\mathcal{F}$) form an edge of the intersection graph if $F_1 \cap F_2 \neq \emptyset$.

Formally, the *line graph* $L(G)$ of a graph G with edge set $E(G)$ is defined as the intersection graph of the edge set $E(G)$. That is, $L(G)$ has $E(G)$ as the vertex set and two vertices e_1 and e_2 are adjacent in $L(G)$, if and only if $e_1 \cap e_2 \neq \emptyset$. In this paper, by G a line graph, we mean that G is the line graph of some graph. An important fact is that line graph possess a forbidden induced subgraphs characterization. A forbidden induced subgraphs characterization of line graphs was provided by Beineke in [6]. That is, a graph G is a line graph if and only if G is $(K_5 - e, W_5, \bar{A}, \overline{C_4 \cup 2K_1}, \overline{P_2 \cup P_3}, \bar{R}$, claw, co-twin-$C_5$, co-twin-house)-free. These nine forbidden subgraphs of a line graph are depicted in Figure 2.

In this paper, along with the line graphs of arbitrary connected graphs, we also consider the line graphs of bipartite multigraphs and line graphs of multigraphs without triangles. A *multigraph* is a graph that allows multiple edges or parallel edges (that is, two vertices may be joined by more than one edge).

A *hypergraph* consists of a set of vertices V and a set of *hyperedges*, each of which is a subset of the vertex set V. Hypergraphs are characterized by their *rank*, which is the maximum size of any hyperedge. A hypergraph is of rank 3, if the largest hyperedge contains at most 3 vertices. So, each hyperedge in a rank 3 hypergraph will contain either $1, 2$, or 3 vertices. A hypergraph is *linear* if any two distinct hyperedges intersect in at most one vertex. In other words, no two hyperedges can share more than one vertex. We observe that linear hypergraph is an immediate generalization of an ordinary undirected graph.

From the definitions of multigraphs and hypergraphs, it follows that the line graphs of multigraphs and linear hypergraphs are simple undirected graphs. The line graphs that we consider in this paper are always assumed to be connected. We organize the paper as follows.

In Section 2, we provide an axiomatic characterization of the interval function of line graphs by defining a set of axioms on its interval function I. In Section 3, we extend the characterization to certain related classes of line graphs, namely line graphs of triangle-free graphs, line graphs of multigraphs without triangles, line graphs of linear hypergraphs of rank 3, line graphs of bipartite graphs, and line graphs of bipartite multigraphs. The relation between these graph classes are given in Figure 1.

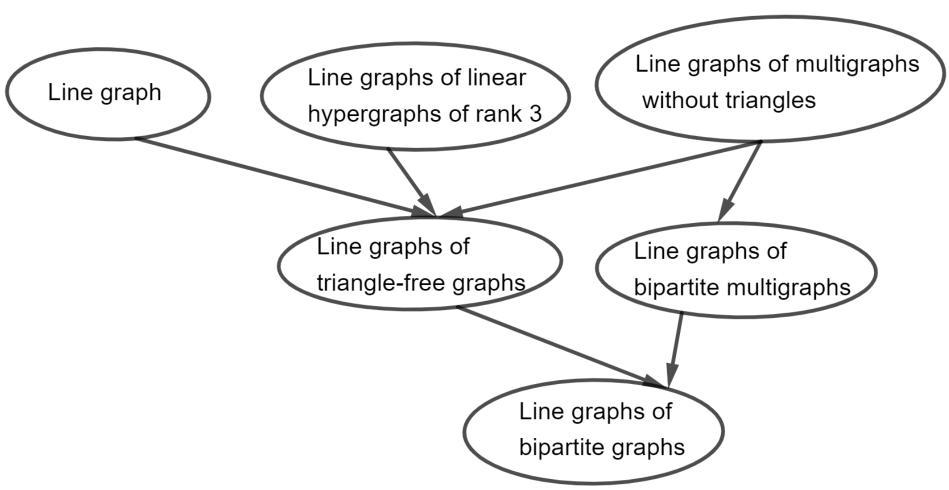

Figure 1. Relation between graph classes

2 Interval function of line graphs

In this section, we introduce three axioms for the characterization of interval function of the line graphs, denoted as (ln1), (ln2), and (ln3). The nine forbidden subgraphs of line graphs are the claw, co-twin-house, co-twin-C_5, $\overline{R}$, $\overline{C_4 \cup 2K_1}$, $\overline{P_2 \cup P_3}$, $K_5 - e$, $\overline{A}$, and W_5. Axiom (ln1) characterizes graphs that does not contain any of co-twin-house, co-twin-C_5, $\overline{R}$, $\overline{C_4 \cup 2K_1}$, $\overline{P_2 \cup P_3}$, and $K_5 - e$ as induced subgraphs, which form a subclass of line graphs. Axiom (ln2) characterizes W_5-free graphs, while axiom (ln3) characterizes $\overline{A}$-free graphs.

In [1], axiom (cw) is introduced for the characterization of claw-free graphs. The axiom and its corresponding characterization are presented below.

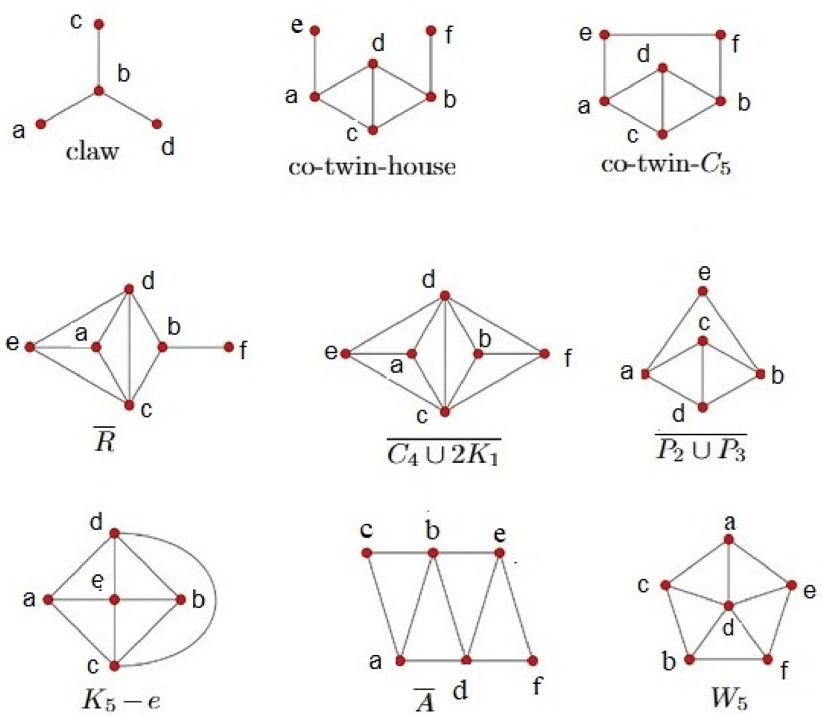

Figure 2. Forbidden subgraphs of line graphs

(cw) : For distinct vertices $a, b, c, d \in V$, $b \in R(a, c)$ and $b \in R(a, d)$ then $b \notin R(c, d)$.

Proposition 2.1. *[1] The interval function I of a connected graph G satisfies the axiom (cw) if and only if G is a claw-free graph.*

The axioms (ln1), (ln2), and (ln3) are defined as follows.

$(ln1)$: For distinct vertices $a, b, c, d, e, f \in V$, if $R(a, c) = \{a, c\}$, $R(b, c) = \{b, c\}$, $R(a, e) = \{a, e\}$, $R(b, f) = \{b, f\}$, $c, d \in R(a, b)$, $R(c, e) \setminus \{c\} = R(d, e) \setminus \{d\}$, and ($e = f$ or $R(e, b) \neq \{e, b\}$) then $R(c, d) \neq \{c, d\}$.

$(ln2)$: For distinct vertices $a, b, c, d, e, f \in V$, if $R(a, c) = \{a, c\}$, $R(b, c) = \{b, c\}$, $R(d, e) = \{d, e\}$, $R(d, f) = \{d, f\}$, $c, d \in R(a, b)$, $d, f \in R(b, e)$, $b, d \in (c, f)$, $a, d \in R(c, e)$ then $R(a, f) = \{a, f\}$.

$(ln3)$: For any distinct vertices $a, b, c, d, e, f \in V$, if $R(a, c) = \{a, c\}, R(b, c) = \{b, c\}, R(b, e) = \{b, e\}, R(b, f) = \{b, f\}$, $c, d \in R(a, b)$, $b, f \in R(d, e)$, $b, d \in R(c, f)$ and $b, c, d, f \in R(a, e)$, then $R(a, f) = \{a, f\}$ or $R(c, e) = \{c, e\}$.

Proposition 2.2. *The interval function I of a connected graph G satisfies the axiom $(ln1)$ if and only if G is a (co-twin-house, co-twin-C_5, $\overline{R}$, $\overline{C_4 \cup 2K_1}$, $\overline{P_2 \cup P_3}$, $K_5 - e$)-free graph.*

Proof. Let I be the interval function of a connected graph G. Assume that G contains at least one of the co-twin-house, co-twin-C_5, $\overline{R}$, $\overline{C_4 \cup 2K_1}$, $\overline{P_2 \cup P_3}$, $K_5 - e$ as an induced subgraph with vertices as labeled in Figure 2. Then $I(a, c) = \{a, c\}, I(b, c) = \{b, c\}, I(a, e) = \{a, e\}, I(b, f) = \{b, f\}$, $c, d \in I(a, b), I(c, e) \setminus \{c\} = I(d, e) \setminus \{d\}$, and $I(c, f) \setminus \{c\} = I(d, f) \setminus \{d\}$, ($e = f$ or

$I(e,b) \neq \{e,b\}$) but $I(c,d) = \{c,d\}$. That is, if G contain any one of the mentioned graphs as an induced subgraph, then I does not satisfy axiom $(ln1)$.

Conversely, assume that I does not satisfy axiom $(ln1)$ on G. That is $I(a,c) = \{a,c\}, I(b,c) = \{b,c\}, I(a,e) = \{a,e\}, I(b,f) = \{b,f\}$, $c,d \in I(a,b), R(c,e) \setminus \{c\} = I(d,e) \setminus \{d\}$, and $I(c,f) \setminus \{c\} = I(d,f) \setminus \{d\}$, ($e = f$ or $I(e,b) \neq \{e,b\}$) but $I(c,d) = \{c,d\}$. The other possible edges among the vertices a,b,c,d,e,f are ce, de, cf, df, af, eb and ef. Note that if $ce \in E(G)$ then $de \in E(G)$, since $R(c,e)\setminus\{c\} = I(d,e)\setminus\{d\}$ and if $cf \in E(G)$ then $df \in E(G)$, since $R(c,f) \setminus \{c\} = I(d,f) \setminus \{d\}$. Also, the assumption ($e = f$ or $I(e,b) \neq \{e,b\}$) means that either $e = f$ or $I(e,b) \neq \{e,b\}$ then by symmetry (ie; by interchanging the edge ea and fb), the assumption ($e = f$ or $I(e,b) \neq \{e,b\}$) means that either $e = f$ or $I(f,a) \neq \{f,a\}$ and $I(e,b) \neq \{e,b\}$. So we have the following cases;

- If non of ce, de, cf, df, af, eb or ef are edges, then the vertices a,b,c,d,e,f induces co-twin-house.
- If $ef \in E(G)$ and ce, de, cf, df, af, eb are not edges, then the vertices a,b,c,d,e,f induces co-twin-C_5.
- The vertices a,b,c,d,e,f induces $\overline{R}$ if both $ce \in E(G)$ and $de \in E(G)$ and cf, df, af, eb and ef are not edges.
- The vertices a,b,c,d,e,f induces $\overline{C_4 \cup 2K_1}$ if both ce, de, cf, df are edges and af, eb and ef are not edges.
- The vertices a,b,c,d,e,f induces $\overline{P_2 \cup P_3}$ if $e = f$ and non of ce and de are not edges.
- The vertices a,b,c,d,e,f induces K_5^- if $e = f$ and all the ce, de and eb are edges.

That is if I does not satisfy axiom $(ln1)$ on G, then G contains any one of (co-twin-house, co-twin-C_5, $\overline{R}$, $\overline{C_4 \cup 2K_1}$, $\overline{P_2 \cup P_3}$, K_5^- as an induced subgraph. □

Proposition 2.3. *The interval function I of a connected graph G satisfies the axiom $(ln2)$ if and only if G is W_5-free graph.*

Proof. Let I be the interval function of a connected graph G. If G contains a W_5 as an induced subgraph with vertices as given in Figure 2, then I does not satisfy the axiom (ln2), since $I(a,c) = \{a,c\}, I(b,c) = \{b,c\}, I(d,e) = \{d,e\}$, $I(d,f) = \{d,f\}$ $c,d \in I(a,b), d,f \in I(b,e), b,d \in I(c,f)$ and $a,d \in I(c,e)$ but $I(f,a) \neq \{f,a\}$.

Conversely, We have to prove that, if G is a graph free of W_5, then I satisfies the axiom $(ln2)$. Suppose that I does not satisfy $(ln2)$. This means that there are different vertices a,b,c,d,e, and f such that, $I(a,c) = \{a,c\}, I(b,c) = \{b,c\}$ $I(d,e) = \{d,e\}, I(d,f) = \{d,f\}, c,d \in I(a,b), d,f \in I(b,e), b,d \in I(c,f), a,d \in I(c,e)$ and $I(f,a) \neq \{f,a\}$. Since, $ac, bc \in E(G)$, and $c,d \in I(a,b)$ means that $d(a,b) = 2$ and both ad and db are edges. Next $bd, de \in E(G)$ and $d,f \in I(b,e)$ implies that $d(b,e) = 2$ hence both ef and bf are edges. Similarly, $cb, bf \in E(G)$, further provided that $b,d \in I(c,f)$, then $d(c,f) = 2$, therefore it is evident that, $cd, df \in E(G)$. Finally, we have $I(f,a) \neq \{f,a\}$. From these arguments, we can conclude that the vertices a,b,c,d,e,f induces a W_5, a contradiction. □

Proposition 2.4. *The interval function I of a connected graph G satisfies the axiom $(ln3)$ if and only if G is a $\overline{A}$-free graph.*

Proof. Let I be the interval function of a connected graph G. Assume that G contains a $\overline{A}$ as an induced subgraph with vertices as labeled in Figure 2. Then $R(a,c) = \{a,c\}, I(b,c) = \{b,c\}, I(b,e) = \{b,e\}, I(b,f) = \{b,f\}, c,d \in I(a,b)$, $b,f \in I(d,e)$, $b,d \in I(c,f)$, $b,c,d,f \in I(a,e)$, and $I(a,f) \neq \{a,f\}$, $I(c,e) \neq \{c,e\}$. That is if G contains $\overline{A}$ as an induced subgraph, then I does not satisfy axiom (ln3).

Conversely, We need to prove that, if G is $\overline{A}$-free, then I satisfies axiom $(ln3)$. Suppose that I does not satisfy $(ln3)$. That is there exist distinct vertices a,b,c,d,e, and f such that, $I(a,c) = \{a,c\}, I(b,c) = \{b,c\}, I(b,e) = \{b,e\}, I(b,f) = \{b,f\}, c,d \in I(a,b)$, $b,f \in I(d,e)$, $b,d \in I(c,f)$, $b,c,d,f \in I(a,e)$, and $I(a,f) \neq \{a,f\}$ and $I(c,e) \neq \{c,e\}$. We have $I(a,c) = \{a,c\}, I(b,c) = \{b,c\}$ and $c,d \in I(a,b)$ implies that $d(a,b) = 2$ and both ad and db are edges. Also $c,d \in I(a,b)$ and $b,f \in I(d,e)$ implies that cd and ef are edges. Now, $b,c,d,f \in I(a,e)$

implies that $ae \notin E(G)$. Since I does not satisfy axiom $(ln3)$, both ce and af are not edges. Thus the vertices a, b, c, d, e, f induced $\overline{A}$, a contradiction. □

The following theorem of characterizing interval function of line graphs can be readily deduced from Propositions 2.1, 2.2, 2.3, and 2.4.

Theorem 2.5. *The interval function I of a connected graph G satisfies the axioms (cw), (ln1), (ln2), and (ln3) if and only if G is a line graph.*

Now the characterization of line graphs using arbitrary transit function easily follow from Theorems 1.1 and 2.5

Theorem 2.6. *Let R be a transit function defined on the vertex set V of a graph G satisfying the axioms (b2), (b3), (s1), (s2), (cw), (ln1), (ln2), and (ln3) if and only if $R = I_{G_R}$ and G_R is a line graph.*

The axioms $(b2), (b3), (s1), (s2), (cw), (ln1), (ln2)$, and $(ln3)$ are independent, which is illustrated by the following examples.

Example 2.7. $(b3), (s1), (s2), (ln1), (ln2), (ln3), (cw) \not\Rightarrow (b2)$.
Let $V = \{a, b, c, d\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}$, $R(a, c) = \{a, b, c\}$, $R(a, d) = \{a, c, d\}$, $R(b, c) = \{b, c\}$, $R(b, d) = \{b, d\}$, $R(c, d) = \{c, d\}$,$R(x, x) = \{x\}$ and $R(x, y) = R(y, x)$ for other $x, y \in V$. We can see that R satisfies $(b3), (cw), (s1), (s2)$, $(ln1)$, $(ln2), (ln3)$. But $c \in R(a, d), b \in R(a, c)$, and $b \notin R(a, d)$. Therefore R does not satisfy the $(b2)$ axiom.

Example 2.8. $(b2), (s1), (s2), (ln1), (ln2), (ln3), (cw) \not\Rightarrow (b3)$.
Let $V = \{a, b, c, d, e\}$ and define a transit function R on V as follows: $R(a, b) = \{a, b\}$, $R(a, c) = \{a, c\}, R(a, d) = \{a, b, c, d\}, R(a, e) = V, R(b, c) = \{b, c\}, R(b, d) = \{b, d\}, R(b, e) = \{b, e\}, R(c, d) = \{c, d\}$, $R(c, e) = \{b, c, d, e\}$, $R(d, e) = \{d, e\}$, $R(x, x) = \{x\}$ and $R(x, y) = R(y, x)$ for other $x, y \in V$. It is easy to see that R satisfies $(b2), (s1), (cw)$, (s2), (ln1), (ln2), (ln3) on V. But $d \in R(a, e), b \in R(a, d)$, and $d \notin R(b, e)$. Therefore R does not satisfy the $(b3)$ axiom.

Example 2.9. $(b2), (b3), (s2), (ln1), (ln2), (ln3), (cw) \not\Rightarrow (s1)$.
Let $V = \{u, \overline{v}, v, w, \overline{u}\}$ and let R be a function on V. Define as follows; $R(v, \overline{v}) = \{v, \overline{v}\}$, $R(u, \overline{u}) = \{u, \overline{u}\}$, $R(u, v) = V$, $R(u, w) = \{u, w, \overline{u}\}$, $R(\overline{v}, w) = \{\overline{v}, w, v\}$, $R(\overline{u}, \overline{v}) = \{\overline{u}, \overline{v}, u\}$, $R(v, \overline{u}) = \{v, \overline{u}, w\}$, $R(w, \overline{u}) = \{w, \overline{u}\}$, and for any other distinct pair $x, y \in V, R(x, y) = \{x, y\}$, and for any $x \in V$, $R(x, x) = \{x\}$. It is easy to see that R satisfes axioms $(b2), (b3), (s2)$, $(ln1), (ln2), (ln3), (cw)$, but does not satisfy axiom $(s1)$ since,u and $\overline{u}$ adjacent in R and v and $\overline{v}$ are adjascent in R and $u \in R(\overline{u}, \overline{v})$, and $\overline{u}, \overline{v} \in R(u, v)$ but $v \notin R(\overline{u}, \overline{v})$.

Example 2.10. $(b2), (b3), (s1), (ln1), (ln2), (ln3), (cw) \not\Rightarrow (s2)$.
Let $V = \{u, \overline{u}, w, v, \overline{v}, z\}$ and let $R : V \times V \longrightarrow 2^V$ be a function on V define as follows, $R(u, w) = \{u, \overline{u}, w\}, R(u, v) = \{u, \overline{u}, w, v\}, R(u, \overline{v}) = \{u, \overline{v}, z\}, R(\overline{u}, v) = \{\overline{u}, w, v\}, R(\overline{u}, \overline{v}) = \{\overline{u}, \overline{v}, z, u\}$, $R(\overline{u}, z) = \{\overline{u}, w, z\}$, $R(w, \overline{v}) = \{w, \overline{v}, v\}$, $R(w, z) = \{w, z, v\}, R(v, z) = \{v, z, \overline{v}\}$ and for any other distinct pair $x, y \in V, R(x, y) = \{x, y\}$ and $x \in V, R(x, x) = \{x\}$. It is easy to see R satisfes axioms (b2), (b3), (s1), (ln1), (ln2), (ln3), (cw), but does not satisfy axiom (s2) since, $\overline{u} \in R(u, v)$, $\overline{v} \notin R(u, v)$ and $v \notin R(\overline{u}, \overline{v})$, but $\overline{u} \notin R(u, \overline{v})$.

Example 2.11. $(b2), (b3), (s1), (s2), (ln1), (ln2), (ln3) \not\Rightarrow (cw)$.
Let G be a claw graph and $V = V(G)$. Define a transit function R such that $R = I$ on $V(G)$. Then, it is easy to see that R satisfies axioms (b2),(b3),(s1),(s2),(ln1), (ln2), (ln3) on $V(G)$ but does not satisfy axiom (cw).

Example 2.12. (b2), (b3), (s1), (s2), (ln2), (ln3), (cw) $\not\Rightarrow$ (ln1).
Let G be a $K_5 - e$ and $V = V(G)$. Define a transit function R such that $R = I$ on $V(G)$. Then, it is easy to see that R satisfies axioms (b2),(b3),(s1),(s2),(cw),(ln2), (ln3) on $V(G)$ but does not satisfy axiom (ln1).

Example 2.13. $(b2), (b3), (s1), (s2), (ln1), (ln3), (cw) \not\Rightarrow (ln2)$.
Let G be a W_5 graph and $V = V(G)$. Define a transit function R such that $R = I$ on $V(G)$. Then, it is easy to see that R satisfies axioms (b2),(b3),(s1),(s2),(cw),(ln1), (ln3) on $V(G)$ but does not satisfy axiom (ln2).

Example 2.14. $(b2), (b3), (s1), (s2), (ln1), (ln2), (cw) \not\Rightarrow (ln3)$.
Let G be a $\bar{A}$ and $V = V(G)$. Define a transit function R such that $R = I$ on $V(G)$. Then, it is easy to see that R satisfies axioms (b2),(b3),(s1),(s2),(cw),(ln1), (ln2) on $V(G)$ but does not satisfy axiom (ln3).

3 Related graph classes of line graphs

In this section, we characterize interval function of some related graph classes of line graphs which include line graphs of triangle-free graphs (linear domino), line graphs of multigraphs without triangles, line graphs of linear hyper graphs of rank 3, line graphs of bipartite graphs, and line graphs of bipartite multigraphs.

3.1 Line graphs of triangle-free graphs (linear domino)

Line graphs of triangle-free graphs, also known as linear dominoes, form a subclass of line graphs. In [19], it is clear that the line graphs of triangle-free graphs posses a forbidden subgraph characterization and is give below.

Theorem 3.1. *[19] A graph G is a line graph of a triangle-free graph if and only if it is (claw, diamond)-free graph.*

The claw and diamond graphs are given in Figure 3. We introduce axiom (dm) to characterize the interval function of line graphs of triangle-free graphs.

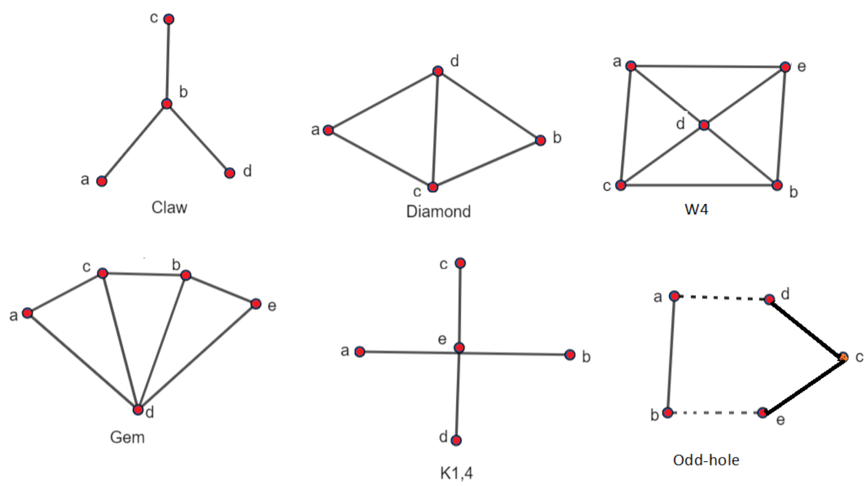

Figure 3. Forbidden subgraphs of related graph classes of line graphs

(dm): For any distinct vertices $a, b, c, d \in V$, if $R(a, d) = \{a, d\}, R(b, d) = \{b, d\}$ and $c, d \in R(a, b)$, then $a, b \in R(c, d)$.

Proposition 3.2. *The interval function I of a connected graph G satisfies the axiom (dm) if and only if G is a diamond-free graph.*

Proof. Let I be the interval function of a connected graph G. Assume that G contains a diamond as an induced subgraph with vertices as labeled in Figure 3, then $I(a, d) = \{a, d\}, I(b, d) = \{b, d\}$ and $c, d \in I(a, b)$ but $a, b \notin I(c, d)$. That is, if G contains a diamond as an induced subgraph, then I does not satisfy axiom (dm).

Conversely, assume that I does not satisfy axiom (dm) on graph G. That is $I(a,d) = \{a,d\}, I(b,d) = \{b,d\}$ and $c,d \in I(a,b)$ and $a,b \notin I(c,d)$. By the assumption $I(a,d) = \{a,d\}$ and $I(d,b) = \{d,b\}$, we have $d(a,b) = 2$. Then $c,d \in I(a,b)$ implies that ac, ad, bd and cb are edges. The assumption $a,b \notin R(c,d)$ implies that $d(c,d) = 1$ and hence cd is an edge. So the vertices a,b,c,d induces a diamond. □

The following theorem can be stated based on Propositions 2.1 and 3.2 .

Theorem 3.3. *The interval function I of a connected graph G satisfies the axioms (dm) and (cw) if and only if G is a line graph of a triangle-free graph.*

The characterization of line graphs of triangle-free graphs with respect to arbitrary transit functions follows directly from Theorems 1.1 and 3.3.

Theorem 3.4. *Let R be a transit function defined on a graph G satisfies the axioms (b2), (b3), (s1), (s2), (cw), and (dm) if and only if $R = I_{G_R}$ and G_R is a line graphs of triangle-free graph.*

3.2 Line graph of a bipartite graph

Line graphs of bipartite graphs are subclass of line graphs of triangle-free graphs. From [19], we have the forbidden subgraph characterization of line graphs of bipartite graphs as follows.

Theorem 3.5. *[19] A graph G is a line graph of a bipartite graph if and only if G is (claw, diamond, odd-hole)-free graph.*

In this section, we define axiom (oh) and prove that I of a graph G satisfies axiom (oh) if and only if G is an (odd-hole)-free graph and that leads to the characterization of line graphs of bipartite graphs. The claw, diamond, and odd-hole graphs are given in Figure 3

(oh): For different vertices $a,b,c,d,e \in V$, $R(a,b) = \{a,b\}$, $R(c,d) = \{c,d\}$, $R(c,e) = \{c,e\}$ $R(a,c) \cap R(b,c) = \{c\}$, $d \in R(a,c)$, $e \in R(b,c)$, then $d \in R(a,e)$ or $e \in R(b,d)$

Proposition 3.6. *The interval function I of a connected graph G satisfies the axiom (oh) if and only if G is a (odd-hole)-free graph.*

Proof. Let I be the interval function of a connected graph G. Assume that G contains a odd-hole as an induced subgraph with vertices as labeled in figure 3. Then $I(a,b) = \{a,b\}$, $I(a,c) \cap I(b,c) = \{c\}$, $d \in I(a,c)$, $e \in I(b,c)$, $I(c,d) = \{c,d\}$, $I(c,e) = \{c,e\}$ but $d \notin I(a,e)$ and $e \notin I(b,d)$. That is, if G contains an odd-hole graph as an induced subgraph, then I does not satisfy axiom (oh).

Conversely, suppose that G is a odd-hole-free graph, to prove that interval function I satisfies axiom (oh). If possible assume that I does not satisfies axiom (oh). That is $I(a,b) = \{a,b\}$, $I(a,c) \cap I(b,c) = \{c\}$, $d \in I(a,c)$, $e \in I(b,c)$, $I(c,d) = \{c,d\}$, $I(c,e) = \{c,e\}$, and $d \notin I(a,e)$ and $e \notin I(b,d)$. Since $d \in I(a,c)$, $e \in I(b,c)$, assume that P be the a,c-shortest path containing d and Q be the b,c-shortest path containing e. The assumption $R(a,b) = \{a,b\}$ and $I(a,b) \cap I(b,c) = \{c\}$ implies that $a \notin I(b,c)$ and $b \notin I(a,c)$. Hence the edge ab and the paths P and Q will forms an odd cycle of length at least five. Also the path P and Q has same length, otherwise either $a \in I(b,c)$ or $b \in I(a,c)$. Let P be $P : a = x_0x_1 \dots x_n = c$ and Q be $Q : b = y_0y_1 \dots y_n = c$. Since $d \notin I(a,e)$, $e \notin I(b,d)$, we have $de \notin E(G)$ and since $I(a,b) \cap I(b,c) = \{c\}$, we get $x_iy_j \notin E(G)$, for $i \neq j$, $i,j \in \{0,1,\dots,n-1\}$. That is only possible chords are x_iy_i, $i,j \in \{0,1,\dots,n-2\}$. Let a' and b' be the first vertices close to c such that $a'b' \in E(G)$. Then the edge ab and the c,a'-subpaths P and c,b'-subpaths Q will induces an odd cycle, a contradiction.

□

In light of Propositions 2.1, 3.2, and 3.6, the following theorem can be stated.

Theorem 3.7. *The interval function I of a connected graph G satisfies the axioms (dm), (cw), and (oh) if and only if G is a line graph of a bipartite graph.*

The characterization using arbitrary arbitrary transit function follows from Theorems 1.1 and 3.7

Theorem 3.8. *Let R be a transit function defined on a graph G satisfies the axioms (b2), (b3), (s1), (s2), (cw), (dm), and (oh) if and only if $R = I_{G_R}$ and G_R is a line graphs of a bipartite graph.*

3.3 Line graphs of bipartite multigraphs

Line graphs of bipartite multigraphs form a superclass of line graphs of bipartite graphs. Consider the following forbidden subgraph characterization of line graphs of bipartite multigraphs.

Theorem 3.9. *[19] A graph G is a line graph of a bipartite multigraph if and only if it is (W_4, claw, gem, odd-hole)-free graph.*

In this section, we introduce axiom (wg) for the characterization of (W_4, gem)-free graphs. Axiom (wg), together with axioms (cw) and (oh), provides a characterization of line graphs of bipartite multigraphs.

(wg): For distinct vertices $a, b, c, d, e \in V$, if $R(a, c) = \{a, c\}, R(c, b) = \{c, b\}, R(b, e) = \{b, e\}$, $c, d \in R(a, b)$, $b \in R(c, e)$ then $d \notin R(c, e)$.

Proposition 3.10. *The interval function I of a connected graph G satisfies the axiom (wg) if and only if G is a (W_4, gem)-free graph.*

Proof. Let I be the interval function of a connected graph G. Assume that G contains one of the W_4 or gem as an induced subgraph with vertices as labeled in Figure 3. Then $I(a, c) = \{a, c\}, I(c, b) = \{c, b\}, I(b, e) = \{b, e\}$, $c, d \in I(a, b)$, $b \in I(c, e)$ but $d \in I(c, e)$. That is if G contains any one of the W_4 or gem graph as an induced subgraph, then I does not satisfy axiom (wg).

Conversely, assume that I does not satisfy axiom (wg) on G. That is $I(a, c) = \{a, c\}, I(c, b) = \{c, b\}, I(b, e) = \{b, e\}$, $c, d \in I(a, b)$, $b \in I(c, e)$ and $d \in I(c, e)$. From this assumption we have ac and cb are edges, together with $c, d \in I(a, b)$ implies that both ad and bd are also edges. Again by assumption be is an edge and $b \in I(c, e)$ imply that $d(c, e) = 2$. Hence $d \in I(c, e)$ implies that both cd and de are edges. Then the vertices a, b, c, d, e induce W_4 if $ae \in E(G)$ and induce a gem if $ae \notin E(G)$. That is if I does not satisfy axiom (wg), then G contain either a W_4 graph or gem graph. □

In light of Propositions 2.1,3.6, and 3.10, the following theorem follows.

Theorem 3.11. *The interval function I of a connected graph G satisfies the axioms (wg), (cw), and (oh) if and only if G is a line graph of a bipartite multigraph.*

Theorem 1.1 and 3.11, provide characterization of line graph of a bipartite multigraphs using arbitrary arbitrary transit function.

Theorem 3.12. *Let R be a transit function defined on a graph G satisfies the axioms (b2), (b3), (s1), (s2), (cw), (wg), and (oh) if and only if $R = I_{G_R}$ and G_R is a line graph of a bipartite multigraph.*

3.4 Line graphs of multigraphs without triangles

In this section, we characterize the line graphs of multigraphs without triangles, which is the superclass of both line graphs of bipartite multigraphs and line graphs of triangle-free graphs. The forbidden subgraph characterization of line graphs of multigraphs is given below.

Theorem 3.13. *[19] A graph G is a line graph of a multigraph without triangles if and only if it is (W_4, claw, gem)-free graph.*

The following theorem can be expressed in light of propositions 2.1, and 3.10 and characterization using arbitrary arbitrary transit function follows from Theorem 1.1 and 3.14.

Theorem 3.14. *The interval function I of a connected graph G satisfies the axioms (cw) and (wg) if and only if G is a line graph of multigraphs without triangles.*

Theorem 3.15. *Let R be a transit function defined on a graph G satisfies the axioms (b2), (b3), (s1), (s2), (cw), and (wg) if and only if $R = I_{G_R}$ and G_R is a line graph of a bipartite multigraph.*

3.5 Line graphs of linear hypergraphs of rank 3

The line graphs of linear hypergraphs of rank 3 is the superclass of line graphs of multigraphs without triangles. We characterize line graphs of linear hypergraphs of rank 3 using axioms (kl) and (dm) on I. The forbidden subgraph characterization of line graphs of linear hypergraphs of rank 3 is given below.

Theorem 3.16. *[19] A graph G is a line graph of a linear hypergraph of rank* 3 *if and only if it is (*$K_{1,4}$ *,diamond)-free graph.*

(kl): For distinct vertices $a, b, c, d, e \in V$, if $e \in R(a,b) \cap R(c,d)$ then $e \notin R(a,c) \cup R(c,b) \cup R(b,d) \cup R(a,d)$.

Proposition 3.17. *The interval function I of a connected graph G satisfies the axiom (kl) if and only if G is a (*$K_{1,4}$*,)-free graph.*

Proof. Let I be the interval function of a connected graph G. If G contains a $K_{1,4}$ as an induced subgraph. It is easily seen that the vertices a, b, c, d, e as shown in Fig. 3, $e \in I(a,b) \cap I(c,d)$, but $e \in I(a,c) \cap I(c,b) \cap I(b,d) \cap I(a,d)$. Therefore, if G contains $K_{1,4}$ as an induced subgraph, then I violates the axiom (kl).

Conversely, we need to prove that if G is $K_{1,4}$-free, then I satisfies axiom (kl). Suppose that I does not satisfies (kl). That is, $e \in I(a,b) \cap I(c,d)$, and $e \in I(a,c) \cap I(c,b) \cap I(b,d) \cap I(a,d)$. This implies that there is a shortest a,b-path P containing e and a shortest c,d-path Q containing e. The assumption $e \in I(a,c) \cap I(c,b) \cap I(b,d) \cap I(a,d)$ implies that e is in shortest a,c-path, shortest a,d-path, shortest c,b-path and shortest b,d-path. Let respectively a', b', c', d' be the neighbors of e in shortest e,a-path, shortest e,b-path, shortest e,c-path, shortest e,d-path respectively. Then the vertices a', b', c', d' together with e induces a $K_{1,4}$, a contradiction. □

The characterization of line graph of a linear hypergraph of rank 3 using I follows from propositions 3.2 and 3.17 and characterization using arbitrary transit function follows from Theorems 1.1 and 3.18.

Theorem 3.18. *The interval function I of a connected graph G satisfies the axioms (dm) and (kl) if and only if G is a line graph of a linear hypergraph of rank* 3.

Theorem 3.19. *Let R be a transit function defined on a graph G satisfies the axioms (b2), (b3), (s1), (s2), (dm), and (kl) if and only if $R = I_{G_R}$ and G_R is a line graph of linear hypergraphs of rank* 3.

The independence of axioms (b2), (b3), (s1), (s2) with other axioms used in this section are easily follows from Examples 2.7, 2.8, 2.9 and 2.10. Let G be the graph respectively diamond, C_5, gem, or $K_{1,4}$ and $R = I$, then R does not satisfy axioms (dm), (oh), (wg) or (kl) respectively on $V(G)$ and R satisfy all other axioms.

4 Conclusion

In this paper, we provide an axiomatic characterization of line graphs and their related graph classes. Using the forbidden subgraph characterization of line graphs and their related graphs, we characterized the line graphs and their related graphs from an axiomatic point of view using first-order axioms defined by the interval function.

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