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On a Class of Pathway Type General Operator

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Abstract This paper introduces and analyzes a new class of pathway type operator named the $\mathcal{Q}$-operator, along with its diverse properties. The $\mathcal{Q}$-operator encompasses a spectrum of operators with kernel functions that range from binomial to exponential forms. Sufficient conditions for the existence of $\mathcal{Q}$-operator along with the convolution and inversion theorems, when $\mathcal{Q}$-operator is considered as an integral transform, are established. Furthermore, the composition of $\mathcal{Q}$-operator with ordinary and partial differential operators are derived. It is demonstrated that the $\mathcal{Q}$-operator of an integral operator yields a $\mathcal{Q}$-operator. Finally, the composition of $\mathcal{Q}$-operator with generalized special functions, including the generalized hypergeometric function, the Wright hypergeometric function, the Mittag-Leffler function, H- and G-functions are presented.

1 Introduction

Integral operators are widely used across various STEM disciplines. They are used to transform differential equations to integral equations [10]. In image processing, they are used in techniques like image blurring, edge and noise detection, etc. [6]. Integral operators play a remarkable role in quantum mechanics, signal processing and probability theory [35, 39]. In certain experimental scenarios, one may require operators with kernels switching among two families of parametric distributions. This can also happen while dealing with two classes of distributions which need to be switched among themselves for a specific experimental setup. For dealing with such instances, Mathai introduced the pathway model [16, 17] in 2005, with the principle of switching among three different families of functional forms, namely type-1 beta, type-2 beta and gamma functional forms. The pathway model in the real scalar case is defined as follows:

$$f(x)=\begin{cases} c_1|x|^{\gamma-1}[1-a(1-\zeta)|x|^{\delta}]^{\frac{1}{1-\zeta}}, & 1-a(1-\zeta)|x|^{\delta}>0, \zeta<1 \quad \text{[type-1 beta form]} \\ c_2|x|^{\gamma-1}[1+a(\zeta-1)|x|^{\delta}]^{-\frac{1}{\zeta-1}}, & -\infty<x<\infty, \zeta>1 \quad \text{[type-2 beta form]} \\ c_3|x|^{\gamma-1}\mathrm{e}^{-a|x|^{\delta}}, & -\infty<x<\infty, \zeta\to 1 \quad \text{[gamma form],} \end{cases}$$

where $a>0, \gamma>0, \delta>0, \zeta$ is the pathway parameter and c_1, c_2, c_3 are the normalizing constants when each of them is considered to be a statistical density. As $\zeta \to 1$, the type-1 and type-2 beta forms reduce to the gamma form. Leveraging the concept of the pathway model, researchers have expanded a realm of integral transform operators [11, 12, 26, 27]. Kumar introduced $\mathcal{P}$-

transform [11, 13], which is defined as

$$(\mathcal{P}_{\nu}^{\rho,\beta,\alpha}f)(x)=\int_0^\infty D_{\rho,\beta}^{\nu,\alpha}(xt)f(t)\mathrm{d}t, x>0.$$

$\mathcal{P}$-transform is defined as type-1 or type-2 based on the kernel function $D_{\rho,\beta}^{\nu,\alpha}(x)$. $\mathcal{P}$-transform is a generalization of many integral transforms, such as Krätzel transform [9] and Meijer transform [22]. To address challenges commonly encountered in physical contexts, Kumar later introduced the pathway Laplace transform or P_α-transform [12], which was further utilized by many authors in their research [1, 27]. Pathway fractional integration operator introduced by Nair [26] is useful in various mathematical fields such as Fractional calculus and statistical distribution theory. It is denoted as $(P_{0+}^{z,\zeta}f)(t)$ and defined as

$$(P_{0+}^{z,\zeta}f)(t)=t^z\int_0^{\frac{t}{a(1-\zeta)}}\left[1-\frac{a(1-\zeta)x}{t}\right]^{\frac{z}{1-\zeta}}f(x)\mathrm{d}x \tag{1.1}$$

This paper presents a novel class of operator called the $\mathcal{Q}$-operator. The $\mathcal{Q}$-operator of a function $f(x)$, denoted by $\big(\mathcal{Q}_\xi^a f(x)\big)(z)$, is defined for $\xi>1$ and $a>0$ as

$$\big(\mathcal{Q}_\xi^a f(x)\big)(z)=\int_0^\infty[1+a(\xi-1)x]^{-\frac{z}{\xi-1}}f(x)\mathrm{d}x. \tag{1.2}$$

Here, ξ is the switching parameter which enables the operator to switch among different classes of operators. The $\mathcal{Q}$-operator is valid for all Lebesgue measurable complex-valued functions $\mathbb{L}_{\nu,p}(\mathbb{R}^+)$ with the norm

$$||f||_{\nu,p}=\left(\int_0^\infty|x^\nu f(x)|^p\frac{\mathrm{d}x}{x}\right)^{\frac{1}{p}}<\infty,\ 1\le p<\infty,\ \nu\in\mathbb{R}.$$

Various properties, existence conditions, convolution and inversion, and connections to special functions and operators are the prime focus of this paper.

The paper unfolds as follows: Section 2 explores the sufficient condition for the existence of $\mathcal{Q}$-operator and key properties like linearity, scalar multiplication, shifting, general derivative and analyse $\mathcal{Q}$-operator as an integral transform, discussing some of the major properties such as inversion and convolution theorems along with the connection with other integral transforms. Section 3 delves into the $\mathcal{Q}$-operator of ordinary and partial derivatives, along with integral operators. Section 4 tackles the derivation of $\mathcal{Q}$-operators for some well-known special functions. Finally, some concluding remarks are given in Section 5.

2 Existence theorem and Properties of $\mathcal{Q}$-operator

A sufficient condition for the existence of the $\mathcal{Q}$-operator is established in the following theorem:

Theorem 2.1. *Let $f(x)$ be an integrable function on any finite interval $[0,c]$, where c is an arbitrary positive real number. If there exists a positive real number b such that $\int_c^\rho x^{-b}f(x)\mathrm{d}x$ approaches a finite limit as ρ tends to infinity, then the $\mathcal{Q}$-operator of f converges absolutely for $\Re\big(\frac{z}{\xi-1}\big)>b$ where $\Re(\cdot)$ denotes the real part of $(\cdot)$.*

Proof. Let c and ϵ be any positive real numbers with $\epsilon<c$. Since $\xi>1, a>0$ and $\Re\big(\frac{z}{\xi-1}\big)>b$, we have $[1+a(\xi-1)x]^{\frac{z}{\xi-1}}>1$. Hence, we obtain

$$\left|\int_\epsilon^c[1+a(\xi-1)x]^{-\frac{z}{\xi-1}}f(x)\mathrm{d}x\right|\le\int_\epsilon^c|f(x)|\mathrm{d}x.$$

Similarly, we have $[1+a(\xi-1)x]^{\frac{z}{\xi-1}}>x^b$. Hence, it can be observed that

$$\int_c^\rho[1+a(\xi-1)x]^{-\frac{z}{\xi-1}}f(x)\mathrm{d}x\le\int_c^\rho x^{-b}|f(x)|\mathrm{d}x. \tag{2.1}$$

Now we can see that

$$\int_0^\rho [1+a(\xi-1)x]^{-\frac{z}{\xi-1}} f(x)\mathrm{d}x = \left(\int_0^c + \int_c^\rho\right)[1+a(\xi-1)x]^{-\frac{z}{\xi-1}} f(x)\mathrm{d}x.$$

The theorem follows by using the given conditions and letting $\rho \to \infty$. □

Corollary 2.2. *Let $f(x)$ be an integrable function on any finite interval $[0, c]$, where c is any positive real number. If there exists a real number $b > 0$ such that $f(x) = O(x^b)$ as $x \to \infty$ where $O(\cdot)$ denotes big O notation, which means the absolute value of $f(x)$ is at most a positive constant multiple of x^b for all sufficiently large values of x, then the $\mathcal{Q}$-operator of f, $\big(\mathcal{Q}_\xi^a f(x)\big)(z)$ exists in the region $\Re\big(\frac{z}{\xi-1}\big) > b$.*

Proof. The proof is a direct aftermath of Theorem 2.1. □

Corollary 2.3. *If $f(x)$ satisfies the conditions of Theorem 2.1, then $\big(\mathcal{Q}_\xi^a f(x)\big)(z)$ is an analytic function of z in the region $\Re\big(\frac{z}{\xi-1}\big) > b$.*

Proof. The proof is similar to the theorem in [36, sec 3-1, p. 138-140]. □

Remark 2.4. All the functions discussed in this paper satisfy the conditions of Theorem 2.1.

We now explore some fundamental properties of the $\mathcal{Q}$-operator. The following theorem establishes its linearity.

Theorem 2.5. *(Linearity): If f_1 and f_2 are any two functions of real variable x, then*

$$\big(\mathcal{Q}_\xi^a(c_1 f_1(x) + c_2 f_2(x))\big)(z) = c_1\big(\mathcal{Q}_\xi^a f_1(x)\big)(z) + c_2\big(\mathcal{Q}_\xi^a f_2(x)\big)(z),$$

where c_1 and c_2 are coefficients in $\mathbb{C}$.

Proof. The proof is straightforward from the application of the definition in (1.2). □

Theorem 2.6. *(Multiplication of function variable by a scalar): For any function f of real variable x and for any real number c, we have*

$$\big(\mathcal{Q}_\xi^a f(cx)\big)(z) = \frac{1}{c}\big(\mathcal{Q}_\xi^{\frac{a}{c}} f(x)\big)(z).$$

Proof. By changing the variable $x = \frac{u}{c}$, we obtain

$$\big(\mathcal{Q}_\xi^a f(cx)\big)(z) = \frac{1}{c}\int_0^\infty [1 + \frac{a}{c}(\xi-1)u]^{-\frac{z}{\xi-1}} f(u)\mathrm{d}u.$$

Then, the theorem follows from the definition (1.2). □

Theorem 2.7. *(Shifting property): If $f(x)$ is any function, then*

$$\big(\mathcal{Q}_\xi^a(\{1+a(\xi-1)x\}^{-\frac{c}{\xi-1}} f(x))\big)(z) = \big(\mathcal{Q}_\xi^a f(x)\big)(z+c).$$

Proof. Using (1.2), we get

$$\big(\mathcal{Q}_\xi^a(\{1+a(\xi-1)x\}^{-\frac{c}{\xi-1}} f(x))\big)(z) = \int_0^\infty [1+a(\xi-1)x]^{-\frac{(z+c)}{\xi-1}} f(x)\mathrm{d}x,$$

and the theorem follows. □

The next theorem gives the (n^{th}) derivative of $\mathcal{Q}$-operator.

Theorem 2.8. *(General derivative): If $F(z)$ denotes the $\mathcal{Q}$-operator of $f(x)$, then for $\xi > 1$,*

$$\frac{d^n}{dz^n}[F(z)] = \left(-\frac{1}{\xi-1}\right)^n \big(\mathcal{Q}_\xi^a\big(\big(\ln[1+a(\xi-1)x]\big)^n f(x)\big)\big)(z).$$

Proof. Using (1.2) and simplifying, we get

$$\begin{aligned}\frac{d}{dz}[F(z)] &= -\frac{1}{\xi-1}\int_0^\infty [1+a(\xi-1)x]^{-\frac{z}{\xi-1}}\ln[1+a(\xi-1)x]f(x)\mathrm{d}x \\ &= -\frac{1}{\xi-1}\left(\mathcal{Q}_\xi^a(\ln[1+a(\xi-1)x]f(x))\right)(z).\end{aligned} \tag{2.2}$$

The theorem follows by differentiating (2.2) $(n-1)$ times. □

Considering $\mathcal{Q}$-operator as an integral transform, it can be notated as $F(z) = \mathcal{Q}_\xi^a[f(x); z]$, where $f(x)$ is a function of a real variable and z is the transform variable. For dealing with the physical problems governed by differential and integral equations, integral transforms offer a powerful toolset for diverse scientific inquiries [31, 32, 33, 34, 36, 37]. The necessity of any integral transform hinges on the existence of a valid inversion formula which is provided in the next theorem.

Theorem 2.9. (Inversion theorem)*: Suppose that $F(z)$ is an analytic function in the region $\Re\left(\frac{z}{\xi-1}\right) > b$, then the integral*

$$\frac{a}{2\pi i}\int_{c-i\infty}^{c+i\infty}[1+a(\xi-1)x]^{\frac{z}{\xi-1}-1}F(z)\mathrm{d}z$$

along any line $\Re(z) = c > b(\xi-1)$ is a c-independent function $f(x)$ whose $\mathcal{Q}$-transform is $F(z)$. Furthermore, $f(x)$ is continuous for each $x \geq 0$ and is of order $O(x^b)$, where b is a positive real number, as $x \to \infty$.

Proof. Suppose that $f(x)$ has a continuous derivative and if $f(x) = O(x^b)$ for large positive values of x, where $b > 0$, then

$$F(z) = \int_0^\infty [1+a(\xi-1)x]^{-\frac{z}{\xi-1}}f(x)\mathrm{d}x, \tag{2.3}$$

which converges absolutely and uniformly in the region $\Re\left(\frac{z}{\xi-1}\right) > b$. The function $F(z)$ is analytic in $\Re\left(\frac{z}{\xi-1}\right) > b$. Substituting (2.3) in

$$\frac{a}{2\pi i}\int_{c-i\omega}^{c+i\omega}[1+a(\xi-1)x]^{\frac{z}{\xi-1}-1}F(z)\mathrm{d}z,$$

where $c > b(\xi-1)$, we obtain

$$\frac{a}{2\pi i}\int_{c-i\omega}^{c+i\omega}[1+a(\xi-1)x]^{\frac{z}{\xi-1}-1}\mathrm{d}z\int_0^\infty [1+a(\xi-1)u]^{-\frac{z}{\xi-1}}f(u)\mathrm{d}u.$$

By interchanging the integration order, we get

$$\frac{a}{2\pi i}[1+a(\xi-1)x]^{-1}\int_0^\infty f(u)\mathrm{d}u\int_{c-i\omega}^{c+i\omega}\left[\frac{1+a(\xi-1)x}{1+a(\xi-1)u}\right]^{\frac{z}{\xi-1}}\mathrm{d}z.$$

Replacing the variable $z = c - i\omega(1-2t)$, the integral becomes

$$\frac{a}{\pi}[1+a(\xi-1)x]^{-1}\int_0^\infty f(u)\frac{\xi-1}{\ln\left[\frac{1+a(\xi-1)x}{1+a(\xi-1)u}\right]}\left[\frac{1+a(\xi-1)x}{1+a(\xi-1)u}\right]^{\frac{c}{\xi-1}} \sin\left(\frac{\omega}{\xi-1}\ln\left[\frac{1+a(\xi-1)x}{1+a(\xi-1)u}\right]\right)\mathrm{d}u.$$

Replacing the variable $v = \frac{1}{\xi-1}\ln\left[\frac{1+a(\xi-1)x}{1+a(\xi-1)u}\right]$ and letting

$$g(v) = f\left(\frac{[1+a(\xi-1)x]\mathrm{e}^{-(\xi-1)v}-1}{a(\xi-1)}\right)\mathrm{e}^{(c-(\xi-1))v},$$

we get

$$\frac{a}{2\pi i}\int_{c-i\omega}^{c+i\omega}[1+a(\xi-1)x]^{\frac{z}{\xi-1}-1}F(z)\mathrm{d}z=\frac{1}{\pi}\int_{-\infty}^{\frac{\ln[1+a(\xi-1)x]}{\xi-1}}g(v)\frac{\sin(\omega v)}{v}\mathrm{d}v.$$

Letting $x\to\infty$, the integral becomes

$$\frac{1}{\pi}\int_0^{\infty}g(v)\frac{\sin(\omega v)}{v}\mathrm{d}v+\frac{1}{\pi}\int_0^{\infty}g(-v)\frac{\sin(\omega v)}{v}\mathrm{d}v.$$

Letting $\omega\to\infty$ and using corollary of localization lemma [36, sec 2-2, p. 34], we have

$$\frac{a}{2\pi i}\int_{c-i\infty}^{c+i\infty}[1+a(\xi-1)x]^{\frac{z}{\xi-1}-1}F(z)\mathrm{d}z=\frac{1}{2}f(x+)+\frac{1}{2}f(x-)=f(x),$$

which completes the proof.

□

While dealing with certain integral equations, one may require the convolution property for the operator/ transform. The next theorem gives the convolution property of the $\mathcal{Q}$-operator/ transform.

Theorem 2.10. (Convolution theorem): *Let g_1*g_2 denotes the convolution of two functions g_1 and g_2, which is defined by $g_1*g_2(x)=\int_0^x g_1(y)g_2(x-y)\mathrm{d}y$. If $g_1(x)$ and $g_2(x)$ are two positive functions of x whose $\mathcal{Q}$-operators/ transforms exist, then we have*

$$\mathcal{Q}_\xi^a\big[g_1*g_2(x);z\big]=\mathcal{Q}_\xi^a[g_1(x);z]\mathcal{Q}_\xi^a[\tilde{g}_2(x);z],$$

where $\tilde{g}_2(x)=[1+a(\xi-1)x]^{\frac{z}{\xi-1}}[1+a(\xi-1)y]^{\frac{z}{\xi-1}}[1+a(\xi-1)(x+y)]^{-\frac{z}{\xi-1}}g_2(x)$.

Proof. Using (1.2) and changing the order of integration, we see that

$$\begin{aligned}\mathcal{Q}_\xi^a\big[g_1*g_2(x);z\big]&=\int_0^{\infty}[1+a(\xi-1)x]^{-\frac{z}{\xi-1}}\Big\{\int_0^x g_1(y)g_2(x-y)\mathrm{d}y\Big\}\mathrm{d}x\\&=\int_0^{\infty}g_1(y)\mathrm{d}y\Big\{\int_y^{\infty}[1+a(\xi-1)x]^{-\frac{z}{\xi-1}}g_2(x-y)\mathrm{d}x\Big\}.\end{aligned}\tag{2.4}$$

Consider the inner integral and changing the variable $x-y=u$, we get

$$I_1=\int_0^{\infty}[1+a(\xi-1)(u+y)]^{-\frac{z}{\xi-1}}g_2(u)\mathrm{d}u.\tag{2.5}$$

Letting the function

$$\tilde{g}_2(u)=[1+a(\xi-1)u]^{\frac{z}{\xi-1}}[1+a(\xi-1)y]^{\frac{z}{\xi-1}}[1+a(\xi-1)(u+y)]^{-\frac{z}{\xi-1}}g_2(u)$$

and on substituting $g_2(u)$ in (2.5), we get

$$I_1=\int_0^{\infty}[1+a(\xi-1)y]^{-\frac{z}{\xi-1}}[1+a(\xi-1)u]^{-\frac{z}{\xi-1}}\tilde{g}_2(u)\mathrm{d}u.\tag{2.6}$$

Substituting for inner integral I_1, (2.4) reduces to

$$\mathcal{Q}_\xi^a\big[g_1*g_2(x);z\big]=\int_0^{\infty}[1+a(\xi-1)y]^{-\frac{z}{\xi-1}}g_1(y)\mathrm{d}y\int_0^{\infty}[1+a(\xi-1)u]^{-\frac{z}{\xi-1}}\tilde{g}_2(u)\mathrm{d}u.$$

The integrals on the right side are the $\mathcal{Q}$-transforms of $g_1(x)$ and $\tilde{g}_2(x)$ respectively, and the theorem follows. □

One can observe that

$$\lim_{\xi\to 1+} t^z \mathcal{Q}_\xi^{\frac{a}{t}}[f(x);z] = t^z\int_0^\infty \mathrm{e}^{-\frac{azx}{t}} f(x)\mathrm{d}x = t^z\mathcal{L}\left[f(x);\frac{az}{t}\right] = (P_{0+}^{z,1}f)(t), \tag{2.7}$$

where $\mathcal{L}[f(x);z]$ is the Laplace transform, which is defined as

$$\mathcal{L}[f(x);z] = \int_0^\infty \mathrm{e}^{-zx} f(x)\mathrm{d}x, \tag{2.8}$$

whenever the integral on the right exists and $(P_{0+}^{z,1}f)(t)$ is the pathway fractional integral operator defined in (1.1). (2.7) establishes a relation between $\mathcal{Q}$-operator, Laplace transform and pathway fractional integral operator. Also when $\xi = 2, a = 1$ and replacing x by $x-1$, we have

$$\mathcal{Q}_2^1[f(x-1);1-z] = \int_0^\infty x^{z-1}f(x-1)\mathrm{d}x = \mathcal{M}\left[f(x-1);z\right],$$

where $\mathcal{M}\left[f(x);z\right]$ is the Mellin operator of $f(x)$ with complex variable z, which is defined as

$$\mathcal{M}\left[f(x);z\right] = \int_0^\infty x^{z-1}f(x)\mathrm{d}x, \tag{2.9}$$

whenever integral on the right exists.

A comparative study of the behaviour of kernel function of Laplace transform and that of $\mathcal{Q}$-operator when the parameter ξ is changed is given in the following graph.

Figure 1. Graphical representation of the kernel functions of $\mathcal{Q}$-operator for different values of ξ when $a = 1, x = 1$

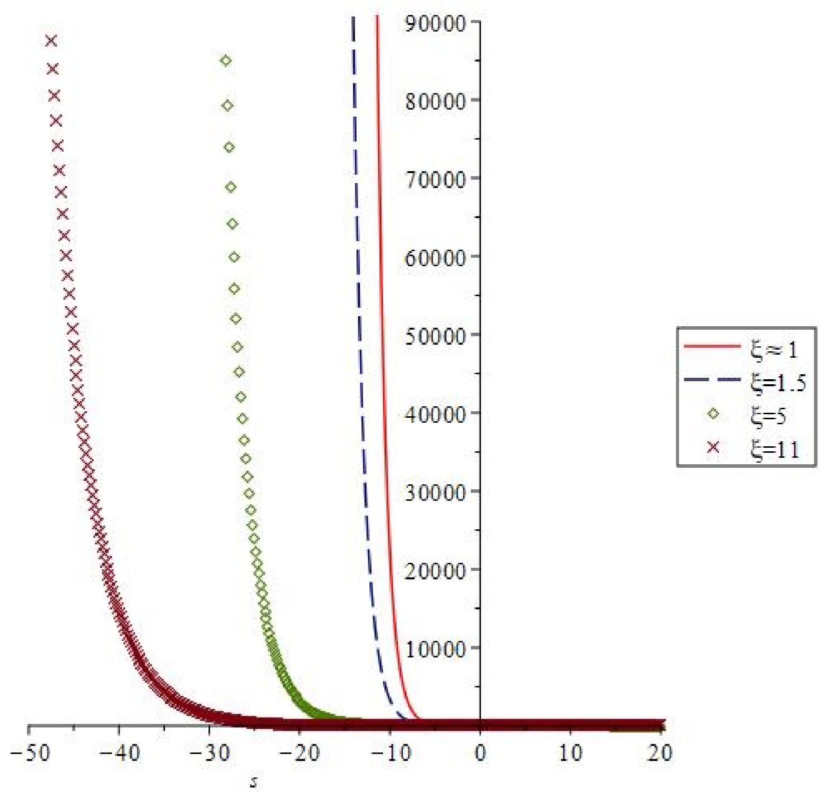

Figure 1: for different values of ξ when a=1,x=1

From the figure, we can see that the kernel function of $\mathcal{Q}$-operator assumes a negative exponential curve. As ξ decreases, the negative slope of the curve increases and the limiting curve, when $\xi\to 1$, is the kernel function of Laplace transform.

3 $\mathcal{Q}$-operator of differential and integral operators

This section tackles the $\mathcal{Q}$-operator of ordinary and partial derivatives, along with integral operators. We begin by presenting a theorem on the $\mathcal{Q}$-operator of an (n^{th}) order differential operator.

Theorem 3.1. *If $f(x)$ and its derivatives $f'(x), f''(x), \cdots, f^{(n-1)}(x)$ are continuous except at the origin and $f^{(n)}(x)$ is at least piecewise continuous, then*

$$\begin{aligned}\big(\mathcal{Q}_\xi^a f^{(n)}(x)\big)(z) = a^n \prod_{j=0}^{n-1}(z+j(\xi-1))\big(\mathcal{Q}_\xi^a f(x)\big)(z+n(\xi-1)) \\ - \sum_{k=0}^{n-1} a^{n-k-1}\prod_{j=0}^{n-k-2}(z+j(\xi-1))f^{(k)}(0+),\end{aligned} \tag{3.1}$$

where $\prod$ denotes the usual product, and the empty product is defined as unity.

Proof. Using (1.2), we have

$$\big(\mathcal{Q}_\xi^a f^{(n)}(x)\big)(z) = \int_0^\infty [1+a(\xi-1)x]^{-\frac{z}{\xi-1}} f^{(n)}(x)\mathrm{d}x. \tag{3.2}$$

Applying integration by parts on (3.2), we get

$$\big(\mathcal{Q}_\xi^a f^{(n)}(x)\big)(z) = az\big(\mathcal{Q}_\xi^a f^{(n-1)}(x)\big)(z+\xi-1) - f^{(n-1)}(0+). \tag{3.3}$$

Again, applying integration by parts on the right side of (3.3), we get

$$\big(\mathcal{Q}_\xi^a f^{(n)}(x)\big)(z) = a^2 z(z+\xi-1)\big(\mathcal{Q}_\xi^a f^{(n-2)}(x)\big)(z+2(\xi-1)) - azf^{(n-2)}(0+) - f^{(n-1)}(0+). \tag{3.4}$$

Continuing in this manner, the solution is obtained as

$$\begin{aligned}\big(\mathcal{Q}_\xi^a f^{(n)}(x)\big)(z) =& a^n z(z+\xi-1)\cdots(z+(n-1)(\xi-1))\big(\mathcal{Q}_\xi^a f(x)\big)(z+n(\xi-1)) \\ &- a^{n-1}z(z+\xi-1)\cdots(z+(n-2)(\xi-1))f(0+) - \cdots - f^{(n-1)}(0+),\end{aligned}$$

which on simpler terms can be written as (3.1). Hence, the theorem follows. □

The following theorem provides the composition of $\mathcal{Q}$-operator with an integral operator.

Theorem 3.2. *Let $f(x)$ be an integrable function, then*

$$\left(\mathcal{Q}_\xi^a\left[\int_0^t f(x)\mathrm{d}x\right]\right)(z) = \frac{1}{a(z-\xi+1)}\big(\mathcal{Q}_\xi^a f(x)\big)(z-\xi+1),\ \Re(z) > \xi - 1. \tag{3.5}$$

Proof. Using (1.2), we have

$$\left(\mathcal{Q}_\xi^a\left[\int_0^t f(x)\mathrm{d}x\right]\right)(z) = \int_0^\infty [1+a(\xi-1)t]^{-\frac{z}{\xi-1}}\left\{\int_0^t f(x)\mathrm{d}x\right\}\mathrm{d}t.$$

Changing the order of integration, we get

$$\left(\mathcal{Q}_\xi^a\left[\int_0^t f(x)\mathrm{d}x\right]\right)(z) = \int_0^\infty f(x)\mathrm{d}x\left\{\int_x^\infty [1+a(\xi-1)t]^{-\frac{z}{\xi-1}}\mathrm{d}t\right\}. \tag{3.6}$$

Then (3.5) follows by integrating the right-hand side of (3.6).

□

Theorem 3.3. *Let $f(x)$ be an integrable function on positive x-axis, then*

$$\left(\mathcal{Q}_\xi^a\left[\int_t^\infty f(x)\mathrm{d}x\right]\right)(z) = -\frac{1}{a(z-\xi+1)}\left\{\big(\mathcal{Q}_\xi^a f(x)\big)(z-\xi+1) - \int_0^\infty f(x)\mathrm{d}x\right\},\ t>0.$$

Proof. Using (1.2) and changing the order of integration, we have

$$\left(\mathcal{Q}_\xi^a\left[\int_t^\infty f(x)\mathrm{d}x\right]\right)(z) = \int_0^\infty [1+a(\xi-1)t]^{-\frac{z}{\xi-1}}\left\{\int_t^\infty f(x)\mathrm{d}x\right\}\mathrm{d}t$$
$$= \int_0^\infty f(x)\mathrm{d}x\left\{\int_0^x [1+a(\xi-1)t]^{-\frac{z}{\xi-1}}\mathrm{d}t\right\}.$$

On simplification, the theorem follows. □

The following theorem gives the $\mathcal{Q}$-operator of an (n^{th}) order integral.

Theorem 3.4. *Let $f(x)$ along with its integrals $f^{(-1)}(x), \cdots, f^{(-n)}(x)$ and their $\mathcal{Q}$-operators exist, then we have*

$$\left(\mathcal{Q}_\xi^a f^{(-n)}(x)\right)(z) = \frac{1}{a^n(z-n(\xi-1))\cdots(z-\xi+1)}\left(\mathcal{Q}_\xi^a f(x)\right)(z-n(\xi-1))$$
$$+\sum_{k=1}^{n}\frac{f^{(-k)}(0+)}{a^{n-k+1}\prod_{j=k}^{n}(z-(n-j+1)(\xi-1))},\ \Re(z) > n(\xi-1).$$

Proof. Consider

$$\left(\mathcal{Q}_\xi^a f(x)\right)(z-n(\xi-1)) = \int_0^\infty [1+a(\xi-1)x]^{-\frac{z-n(\xi-1)}{\xi-1}} f(x)\mathrm{d}x.$$

Applying integration by parts and simplifying, we get

$$\left(\mathcal{Q}_\xi^a f(x)\right)(z-n(\xi-1)) = a(z-n(\xi-1))\left(\mathcal{Q}_\xi^a f^{(-1)}(x)\right)(z-(n-1)(\xi-1)) - f^{(-1)}(0+). \tag{3.7}$$

Again, applying integration by parts $(n-1)$ times on the right side of (3.7), the theorem follows. □

The next theorem gives the $\mathcal{Q}$-operator of partial derivatives of a function of two variables.

Theorem 3.5. *If $f = f(x,y)$ along with all its partial derivatives exist and are continuous, then the following result holds:*

(i)

$$\left(\mathcal{Q}_\xi^a\left[\frac{\partial f}{\partial y}\right]\right)(y\to z) = az\left(\mathcal{Q}_\xi^a f(x,y)\right)(y\to z+\xi-1) - f(x,0).$$

(ii)

$$\left(\mathcal{Q}_\xi^a\left[\frac{\partial^n f}{\partial y^n}\right]\right)(y\to z) = a^n\prod_{j=0}^{n-1}(z+j(\xi-1))\left(\mathcal{Q}_\xi^a f(x,y)\right)(y\to z+n(\xi-1))$$
$$-\sum_{k=0}^{n-1}a^{n-k-1}\prod_{j=0}^{n-k-2}(z+j(\xi-1))\frac{\partial^k f}{\partial y^k}(x,0).$$

(iii)

$$\left(\mathcal{Q}_\xi^a\left[\frac{\partial f}{\partial x}\right]\right)(y\to z) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathcal{Q}_\xi^a f(x,y)\right)(y\to z).$$

(iv)

$$\left(\mathcal{Q}_\xi^a\left[\frac{\partial^m f}{\partial x^m}\right]\right)(y\to z) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\mathcal{Q}_\xi^a f(x,y)\right)(y\to z).$$

(v)

$$\left(\mathcal{Q}_\xi^a\left[\frac{\partial^{m+n}f}{\partial x^m\partial y^n}\right]\right)(y\to z)=\left(\mathcal{Q}_\xi^a\left[\frac{\partial^{n+m}f}{\partial y^n\partial x^m}\right]\right)(y\to z)$$
$$=a^n\prod_{j=0}^{n-1}(z+j(\xi-1))\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\mathcal{Q}_\xi^a f(x,y)\right)(y\to z+n(\xi-1))$$
$$-\sum_{k=0}^{n-1}a^{n-k-1}\prod_{j=0}^{n-k-2}(z+j(\xi-1))\frac{\partial^{m+k}f}{\partial x^m\partial y^k}(x,0),$$

where the empty product is defined as unity.

Proof. Using (1.2), we have

$$\left(\mathcal{Q}_\xi^a\left[\frac{\partial f}{\partial y}\right]\right)(y\to z)=\int_0^\infty[1+a(\xi-1)y]^{-\frac{z}{\xi-1}}\frac{\partial f}{\partial y}\mathrm{d}y.$$

Results (i) and (ii) can be proved by using the proof of Theorem 3.1, considering f as a function dependent on y only.
Using Leibniz integral rule [38], which is stated as

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b g(x,y)\mathrm{d}y=\int_a^b\frac{\partial}{\partial x}g(x,y)\mathrm{d}y, \tag{3.8}$$

where a and b are constants, the result (iii) follows.
(iv) follows by using Leibniz integral rule (3.8) m times.
Since all the partial derivatives $\frac{\partial^{i+j}f}{\partial x^i\partial y^j}; i=0,1,\cdots,m; j=0,1,\cdots,n$ exist and are continuous, we observe that

$$\frac{\partial^{m+n}f}{\partial x^m\partial y^n}=\frac{\partial^{n+m}f}{\partial y^n\partial x^m}=\frac{\partial^m}{\partial x^m}\left(\frac{\partial^n f}{\partial y^n}\right). \tag{3.9}$$

Hence by employing (3.9) and the result (iv), we have

$$\left(\mathcal{Q}_\xi^a\left[\frac{\partial^{m+n}f}{\partial x^m\partial y^n}\right]\right)(y\to z)=\left(\mathcal{Q}_\xi^a\left[\frac{\partial^{n+m}f}{\partial y^n\partial x^m}\right]\right)(y\to z)=\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left(\mathcal{Q}_\xi^a\left[\frac{\partial^n f}{\partial y^n}\right]\right)(y\to z).$$

Using the result (ii), we obtain

$$\left(\mathcal{Q}_\xi^a\left[\frac{\partial^{m+n}f}{\partial x^m\partial y^n}\right]\right)(y\to z)=\frac{\mathrm{d}^m}{\mathrm{d}x^m}\left\{a^n\prod_{j=0}^{n-1}(z+j(\xi-1))\left(\mathcal{Q}_\xi^a f(x,y)\right)(y\to z+n(\xi-1))\right.$$
$$\left.-\sum_{k=0}^{n-1}a^{n-k-1}\prod_{j=0}^{n-k-2}(z+j(\xi-1))\frac{\partial^k f}{\partial y^k}(x,0)\right\}.$$

On further simplification, the result (v) follows. □

4 $\mathcal{Q}$-operator of some elementary and generalized special functions

The following discussion focuses on the $\mathcal{Q}$-operator of some elementary and generalized special functions. The $\mathcal{Q}$-operator of the power function (x^ν) can be readily obtained as:

$$\mathcal{Q}_\xi^a\left[x^\nu;z\right]=\frac{\Gamma(\nu+1)\Gamma\left(\frac{z}{\xi-1}-\nu-1\right)}{[a(\xi-1)]^{\nu+1}\Gamma\left(\frac{z}{\xi-1}\right)};\ 0<\Re(\nu+1)<\Re\left(\frac{z}{\xi-1}\right). \tag{4.1}$$

As a special case, when $\nu=0$, we get

$$\mathcal{Q}_\xi^a[1;z]=\frac{1}{a(z-\xi+1)};\ \Re(z)>\xi-1. \tag{4.2}$$

By using (4.1), the $\mathcal{Q}$-operator of an exponential function can be obtained as

$$\mathcal{Q}_{\xi}^{a}\left[\mathrm{e}^{cx};z\right]=\frac{1}{a(z-\xi+1)}{}_{1}F_{1}\left(1;2-\frac{z}{\xi-1};-\frac{c}{a(\xi-1)}\right),\ \Re(z)>\xi-1, \tag{4.3}$$

where ${}_1F_1(a;b;w)$ is the confluent hypergeometric function [29], which is defined as

$${}_1F_1(a;b;w)=\sum_{n=0}^{\infty}\frac{(a)_n w^n}{(b)_n n!}, \tag{4.4}$$

where b is not a zero or negative integer.

Next, the $\mathcal{Q}$-operator of the Mittag-Leffler function is evaluated. The Mittag-Leffler function was introduced by *Magnus Gustaf (Gösta) Mittag-Leffler* in 1903 [24], denoted by $E_\rho(w)$, and defined as

$$E_\rho(w)=\sum_{n=0}^{\infty}\frac{w^n}{\Gamma(\rho n+1)};\ \rho\in\mathbb{C},\Re(\rho)>0,w\in\mathbb{C}. \tag{4.5}$$

There are many generalizations for this function. Wiman introduced the two-parameter Mittag-Leffler function [41], which is defined as

$$E_{\rho,\gamma}(w)=\sum_{n=0}^{\infty}\frac{w^n}{\Gamma(\rho n+\gamma)};\ \rho,\gamma\in\mathbb{C},\Re(\rho)>0,\Re(\gamma)>0,w\in\mathbb{C}. \tag{4.6}$$

Prabhakar introduced the generalized three-parameter Mittag-Leffler function [28], which is defined as

$$E_{\rho,\gamma}^{\sigma}(w)=\sum_{n=0}^{\infty}\frac{(\sigma)_n w^n}{\Gamma(\rho n+\gamma)n!};\ \rho,\gamma,\sigma\in\mathbb{C},\Re(\rho)>0,\Re(\gamma)>0,\Re(\sigma)>0,w\in\mathbb{C}. \tag{4.7}$$

Here, $(c)_n$ denotes the Pochhammer symbol which is defined for non-zero $c\in\mathbb{C}$ as

$$\begin{aligned}(c)_0=1,(c)_n&=c(c+1)\cdots(c+n-1)\\&=\frac{\Gamma(c+n)}{\Gamma(c)},\ n=1,2,\cdots,\end{aligned} \tag{4.8}$$

whenever $\Gamma(c)$ exists. The Mittag-Leffler functions have been found to have applications in physics, engineering, and applied sciences. The generalized Mittag-Leffler function can be represented [18] in the form

$$E_{\rho,\gamma}^{\sigma}(w)=\frac{1}{\Gamma(\sigma)}H_{1,2}^{1,1}\left[-w\middle|\begin{array}{c}(1-\sigma,1)\\(0,1),(1-\gamma,\rho)\end{array}\right], \tag{4.9}$$

where $H_{1,2}^{1,1}(w)$ is the H-function.

Introduced by C. Fox in 1961 [5], the H-function, also known as Fox's H-function, is a cornerstone of mathematical physics. This generalized special function unifies a wide range of special functions, including the Mittag-Leffler function, the hypergeometric function, and Meijer's G-function [21]. The H-function finds applications in diverse fields such as quantum physics [14, 15, 19], statistical mechanics [30], and electromagnetic theory [7]. Represented by the notation $H_{q,r}^{k,l}(w)$, it is defined via a Mellin-Barnes type integral [20] as:

$$\begin{aligned}H_{q,r}^{k,l}(w)=H_{q,r}^{k,l}\left[w\middle|\begin{array}{c}(m_1,M_1),\cdots,(m_q,M_q)\\(n_1,N_1),\cdots,(n_r,N_r)\end{array}\right]=\frac{1}{2\pi i}\\ \times\int_{\mathcal{L}}\frac{\left\{\prod_{j=1}^{k}\Gamma(n_j+N_js)\right\}\left\{\prod_{i=1}^{l}\Gamma(1-m_i-M_is)\right\}w^{-s}}{\left\{\prod_{j=k+1}^{r}\Gamma(1-n_j-N_js)\right\}\left\{\prod_{i=l+1}^{q}\Gamma(m_i+M_is)\right\}}\mathrm{d}s,\end{aligned} \tag{4.10}$$

where k, l, q, r are integers such that $1 \leq k \leq r, 0 \leq l \leq q; M_i, N_j \in \mathbb{R}^+, m_i, n_j \in \mathbb{C}; i = 1, 2, \cdots, q, j = 1, 2, \cdots, r$. Here,

$$w^{-s} = \exp[-s\{\ln|w| + i \ \arg w\}]; \ w \neq 0, i = \sqrt{-1},$$

with $\ln|w|$ is the natural logarithm of $|w|$ and $\arg w$ is the argument of w which is not necessarily the principal value. The contour $\mathcal{L}$ in (4.10) separates the poles of $\Gamma(n_j + N_j s), j = 1, \cdots, k$ from those of $\Gamma(1 - m_i - M_i s), i = 1, \cdots, l$. Let

$$\begin{aligned}
\sigma &= \sum_{i=1}^{l} M_i - \sum_{i=l+1}^{q} M_i + \sum_{j=1}^{k} N_j - \sum_{j=k+1}^{r} N_j,\\
\beta &= \left[\prod_{i=1}^{q} (M_i)^{-M_i}\right]\left[\prod_{j=1}^{r} (N_j)^{N_j}\right],\\
\eta &= \sum_{j=1}^{r} N_j - \sum_{i=1}^{q} M_i,\\
\Delta &= \sum_{j=1}^{r} n_j - \sum_{i=1}^{q} m_i + \frac{q-r}{2}.
\end{aligned} \tag{4.11}$$

H-function exists for the following cases [20]:

(i) $r \geq 1, \eta > 0$, for all $w, w \neq 0$.

(ii) $r \geq 1, \eta = 0$, for $0 < |w| < \beta$.

(iii) $r \geq 1, \eta = 0, \Re(\Delta) < -1$, for $|w| = \beta$.

(iv) $q \geq 1, \eta < 0$, for all $w, w \neq 0$.

(v) $q \geq 1, \eta = 0$, for $|w| > \beta$.

(vi) $q \geq 1, \eta = 0, \Re(\Delta) < -1$, for $|w| = \beta$.

The H-function reduces to G-function when $M_1 = \cdots = M_q = N_1 = \cdots = N_r = 1$ in (4.10). In 1936, C. S. Meijer studied the G-function [22], which is denoted by $G^{k,l}_{q,r}(w)$ and is defined in the form of a Mellin-Barnes type integral as

$$G^{k,l}_{q,r}\left[w\middle| \begin{array}{c} m_1 \cdots m_q \\ n_1 \cdots n_r \end{array}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\left\{\prod_{j=1}^{k}\Gamma(n_j+s)\right\}\left\{\prod_{i=1}^{l}\Gamma(1-m_i-s)\right\}w^{-s}}{\left\{\prod_{j=k+1}^{r}\Gamma(1-n_j-s)\right\}\left\{\prod_{i=l+1}^{q}\Gamma(m_i+s)\right\}}\mathrm{d}s, \tag{4.12}$$

where $\mathcal{L}$ is a contour seperating all the poles of $\Gamma(n_j + s), j = 1, \cdots, k$ from those of $\Gamma(1 - m_i - s), i = 1, \cdots, l$. The conditions of existence are given in [20].
The following result gives the $\mathcal{Q}$-operator of the Mittag-Leffler function $E_\rho(cx)$ defined in (4.5) in terms of H-function.

Theorem 4.1. *Let $\rho \in \mathbb{R}^+; c \neq 0, z \in \mathbb{C}$ and $\xi > 1$. Suppose $w \neq 1 - \frac{z}{\xi-1} - n$, for any complex number w and for any positive integer n, then*

$$\mathcal{Q}^a_\xi\left[E_\rho(cx); z\right] = \frac{1}{a(\xi-1)\Gamma\left(\frac{z}{\xi-1}\right)} H^{2,2}_{2,3}\left[-\frac{c}{a(\xi-1)}\middle| \begin{array}{c} (0,1),(0,1) \\ (0,1),\left(\frac{z}{\xi-1}-1,1\right),(0,\rho) \end{array}\right], \tag{4.13}$$

where $H^{2,2}_{2,3}(\cdot)$ is the H-function defined in (4.10).

Proof. Using (1.2), (4.1) and (4.5), we obtain

$$\begin{aligned}
\mathcal{Q}^a_\xi\left[E_\rho(cx); z\right] &= \sum_{n=0}^{\infty} \frac{c^n}{\Gamma(\rho n+1)} \mathcal{Q}^a_\xi\left[x^n; z\right]\\
&= \frac{1}{a(\xi-1)\Gamma\left(\frac{z}{\xi-1}\right)} \sum_{n=0}^{\infty} \frac{\Gamma(1+n)\Gamma\left(\frac{z}{\xi-1}-1-n\right)}{\Gamma(1+\rho n)}\left(\frac{c}{a(\xi-1)}\right)^n.
\end{aligned} \tag{4.14}$$

Since $w \neq 1 - \frac{z}{\xi-1} - n$, by considering appropriate contour $\mathcal{L}$ and using residue theorem [2], the sum on the right side of (4.14) can be represented as the Mellin-Barnes type integral

$$\frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(w)\Gamma(1-w)\Gamma(1-w)\Gamma\left(\frac{z}{\xi-1}-1+w\right)}{\Gamma(1-\rho w)}\left(-\frac{c}{a(\xi-1)}\right)^{-w}\mathrm{d}w,$$

which is the H-function

$$H^{2,2}_{2,3}\left[-\frac{c}{a(\xi-1)}\,\middle|\,\begin{matrix}(0,1),(0,1)\\(0,1),\left(\frac{z}{\xi-1}-1,1\right),(0,\rho)\end{matrix}\right].$$

Hence, the theorem follows. □

Remark 4.2. When $\rho = 1$, (4.13) becomes

$$\mathcal{Q}^a_\xi\left[E_1(cx);z\right] = \mathcal{Q}^a_\xi[\mathrm{e}^{cx};z] = \frac{1}{a(\xi-1)\Gamma\left(\frac{z}{\xi-1}\right)}\,G^{2,2}_{2,3}\left[-\frac{c}{a(\xi-1)}\,\middle|\,\begin{matrix}0,0\\0,\frac{z}{\xi-1}-1,0\end{matrix}\right], \tag{4.15}$$

where $G^{2,2}_{2,3}(\cdot)$ is the G-function defined in (4.12).

The following theorem gives the $\mathcal{Q}$-operator of the generalized Mittag-Leffler function defined in (4.7).

Theorem 4.3. *Let $\rho,\delta \in \mathbb{R}^+$; $c \neq 0, \gamma, \sigma, \lambda, z \in \mathbb{C}$ and $\xi > 1$. Suppose $w \neq (\lambda - \frac{z}{\xi-1} - n)/\delta$, for any complex number w and for any positive integer n, then the $\mathcal{Q}$-operator of the generalized Mittag-Leffler function is obtained as*

$$\begin{aligned}\mathcal{Q}^a_\xi\left[x^{\lambda-1}E^\sigma_{\rho,\gamma}(cx^\delta);z\right] &= \frac{1}{(a(\xi-1))^\lambda\Gamma(\sigma)\Gamma\left(\frac{z}{\xi-1}\right)}\\ &\quad\times H^{2,2}_{2,3}\left[-\frac{c}{(a(\xi-1))^\delta}\,\middle|\,\begin{matrix}(1-\sigma,1),(1-\lambda,\delta)\\(0,1),\left(\frac{z}{\xi-1}-\lambda,\delta\right),(1-\gamma,\rho)\end{matrix}\right],\end{aligned} \tag{4.16}$$

where $H^{2,2}_{2,3}(\cdot)$ is the H-function defined in (4.10).

Proof. Using (1.2), (4.1) and (4.7), we have

$$\begin{aligned}\mathcal{Q}^a_\xi\left[x^{\lambda-1}E^\sigma_{\rho,\gamma}(cx^\delta);z\right] &= \sum_{n=0}^{\infty}\frac{(\sigma)_n c^n}{n!\Gamma(\rho n+\gamma)}\mathcal{Q}^a_\xi\left[x^{\delta n+\lambda-1};z\right]\\ &= \frac{1}{(a(\xi-1))^\lambda\Gamma(\sigma)\Gamma\left(\frac{z}{\xi-1}\right)}\\ &\quad\times\sum_{n=0}^{\infty}\frac{\Gamma(\sigma+n)\Gamma(\lambda+\delta n)\Gamma\left(\frac{z}{\xi-1}-\lambda-\delta n\right)}{\Gamma(1+n)\Gamma(\gamma+\rho n)}\left(\frac{c}{(a(\xi-1))^\delta}\right)^n.\end{aligned} \tag{4.17}$$

Since $w \neq (\lambda - \frac{z}{\xi-1} - n)/\delta$, by considering appropriate contour $\mathcal{L}$ and using residue theorem [2], the sum on the right side of (4.17) can be represented as the Mellin-Barnes type integral

$$\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma(w)\Gamma(\sigma-w)\Gamma(\lambda-\delta w)\Gamma\left(\frac{z}{\xi-1}-\lambda+\delta w\right)}{\Gamma(\gamma-\rho w)}\left(-\frac{c}{(a(\xi-1))^\delta}\right)^{-w}\mathrm{d}w,$$

which is the H-function

$$H^{2,2}_{2,3}\left[-\frac{c}{(a(\xi-1))^\delta}\,\middle|\,\begin{matrix}(1-\sigma,1),(1-\lambda,\delta)\\(0,1),\left(\frac{z}{\xi-1}-\lambda,\delta\right),(1-\gamma,\rho)\end{matrix}\right],$$

and the theorem follows. □

E. M. Wright [42] introduced the Wright hypergeometric function in 1935. The generalized Wright hypergeometric function is denoted by ${}_q\Psi_r(w)$ and is defined [20] as

$$ {}_q\Psi_r\left[w\middle| \begin{array}{l} (m_1,M_1),\cdots,(m_q,M_q) \\ (n_1,N_1),\cdots,(n_r,N_r) \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{q}\Gamma(m_i+M_in)}{\prod_{j=1}^{r}\Gamma(n_j+N_jn)} \frac{w^n}{n!}, \tag{4.18} $$

where $m_i, n_j \in \mathbb{C}; M_i, N_j \in \mathbb{R}^+; i=1,2,\ldots,q; j=1,2,\ldots,r$ and $\sum_{j=1}^{r} N_j - \sum_{i=1}^{q} M_i > -1$. For conditions of existence, see [20]. The $\mathcal{Q}$-operator of the generalized Wright hypergeometric function defined in (4.18) is obtained in the following theorem.

Theorem 4.4. *Let* $\gamma, c, m_i, n_j, z \in \mathbb{C};\ M_i, N_j, \delta \in \mathbb{R}^+;\ i=1,2,\ldots,q; j=1,2,\ldots,r,$

$$ \sum_{j=1}^{r} N_j - \sum_{i=1}^{q} M_i > -1 $$

with $\xi > 1$. *Suppose* $w \neq (\gamma - \frac{z}{\xi-1} - n)/\delta$, *for any complex number* w *and for any positive integer* n, *then*

$$ \mathcal{Q}_\xi^a\left[x^{\gamma-1}{}_q\Psi_r(cx^\delta);z\right] = \frac{1}{(a(\xi-1))^\gamma\Gamma\left(\frac{z}{\xi-1}\right)} $$

$$ \times H_{q+1,r+2}^{2,q+1}\left[-\frac{c}{(a(\xi-1))^\delta}\middle| \begin{array}{l} (1-m_1,M_1),\cdots,(1-m_q,M_q),(1-\gamma,\delta) \\ (0,1),\left(\frac{z}{\xi-1}-\gamma,\delta\right),(1-n_1,N_1),\cdots,(1-n_r,N_r) \end{array}\right], $$

where $H_{q,r}^{k,l}(\cdot)$ *is the* H*-function defined in (4.10).*

Proof. Using (1.2), (4.1) and (4.18), we have

$$ \mathcal{Q}_\xi^a\left[x^{\gamma-1}{}_q\Psi_r(cx^\delta);z\right] = \frac{1}{(a(\xi-1))^\gamma\Gamma\left(\frac{z}{\xi-1}\right)} $$

$$ \times \sum_{n=0}^{\infty} \frac{\Gamma(m_1+M_1n)\cdots\Gamma(m_q+M_qn)\,\Gamma(\gamma+\delta n)\Gamma\left(\frac{z}{\xi-1}-\gamma-\delta n\right)}{\Gamma(n_1+N_1n)\cdots\Gamma(n_r+N_rn)\,n!} \left(\frac{c}{(a(\xi-1))^\delta}\right)^n. \tag{4.19} $$

The sum on the right side of (4.19) is the sum of residues of the function

$$ \frac{\Gamma(w)\Gamma(m_1-M_1w)\cdots\Gamma(m_q-M_qw)\,\Gamma(\gamma-\delta w)\Gamma\left(\frac{z}{\xi-1}-\gamma+\delta w\right)}{\Gamma(n_1-N_1w)\cdots\Gamma(n_r-N_rw)} \left(-\frac{c}{(a(\xi-1))^\delta}\right)^{-w}. $$

By residue theorem [2], this sum is the H-function

$$ H_{q+1,r+2}^{2,q+1}\left[-\frac{c}{(a(\xi-1))^\delta}\middle| \begin{array}{l} (1-m_1,M_1),\cdots,(1-m_q,M_q),(1-\gamma,\delta) \\ (0,1),\left(\frac{z}{\xi-1}-\gamma,\delta\right),(1-n_1,N_1),\cdots,(1-n_r,N_r) \end{array}\right], $$

and the theorem follows. □

The generalized hypergeometric function is denoted by ${}_qF_r(w)$ and is defined [20] as

$$ {}_qF_r(m_1,\cdots,m_q;n_1,\cdots,n_r;w) = \sum_{n=0}^{\infty} \frac{(m_1)_n\cdots(m_q)_n}{(n_1)_n\cdots(n_r)_n} \frac{w^n}{n!}, \tag{4.20} $$

where $m_i, n_j \in \mathbb{C}; n_j \neq 0,-1,\ldots; i=1,2,\ldots,q; j=1,2,\ldots,r$. Here $(m_i)_n$ and $(n_j)_n$ are Pochhammer symbols defined in (4.8). The convergence of the series (4.20) depends on q and r. More details on existence conditions can be seen in [18]. The following theorem gives the $\mathcal{Q}$-operator of the generalized hypergeometric function defined in (4.20).

Theorem 4.5. *Suppose $\delta \in \mathbb{R}^+; c \neq 0, \gamma, m_i, n_j, z \in \mathbb{C};\ i = 1, 2, \ldots, q; j = 1, 2, \ldots, r,$ with $q \leq r$ and $\xi > 1$. Let $w \neq (\gamma - \frac{z}{\xi-1} - n)/\delta$, for any complex number w and for any positive integer n, then we get*

$$\mathcal{Q}_\xi^a\left[x^{\gamma-1}{}_qF_r(cx^\delta); z\right] = \frac{\Gamma(n_1)\cdots\Gamma(n_r)}{(a(\xi-1))^\gamma \Gamma\left(\frac{z}{\xi-1}\right)\Gamma(m_1)\cdots\Gamma(m_q)} \times H^{2,q+1}_{q+1,r+2}\left[-\frac{c}{(a(\xi-1))^\delta}\middle| \begin{array}{l}(1-m_1,1),\cdots,(1-m_q,1),(1-\gamma,\delta)\\ (0,1),\left(\frac{z}{\xi-1}-\gamma,\delta\right),(1-n_1,1),\cdots,(1-n_r,1)\end{array}\right], \quad (4.21)$$

where $H^{k,l}_{q,r}(\cdot)$ is the H-function defined in (4.10). Furthermore when $\delta = 1$, we have

$$\mathcal{Q}_\xi^a\left[x^{\gamma-1}{}_qF_r(cx); z\right] = \frac{\Gamma(n_1)\cdots\Gamma(n_r)}{(a(\xi-1))^\gamma \Gamma\left(\frac{z}{\xi-1}\right)\Gamma(m_1)\cdots\Gamma(m_q)} \times G^{2,q+1}_{q+1,r+2}\left[-\frac{c}{(a(\xi-1))^\delta}\middle| \begin{array}{l}1-m_1,\cdots,1-m_q,1-\gamma\\ 0,\frac{z}{\xi-1}-\gamma,1-n_1,\cdots,1-n_r\end{array}\right], \quad (4.22)$$

where $G^{k,l}_{q,r}(\cdot)$ is the G-function defined in (4.12).

Proof. Using (1.2) and (4.20), we get

$$\mathcal{Q}_\xi^a\left[x^{\gamma-1}{}_qF_r(cx^\delta); z\right] = \sum_{n=0}^\infty \frac{(m_1)_n\cdots(m_q)_n}{(n_1)_n\cdots(n_r)_n}\frac{c^n}{n!}\mathcal{Q}_\xi^a\left[x^{\delta n+\gamma-1}; z\right]$$

$$= \frac{\Gamma(n_1)\cdots\Gamma(n_r)}{(a(\xi-1))^\gamma \Gamma\left(\frac{z}{\xi-1}\right)\Gamma(m_1)\cdots\Gamma(m_q)} \sum_{n=0}^\infty \frac{\Gamma(m_1+n)\cdots\Gamma(m_q+n)\,\Gamma(\gamma+\delta n)\Gamma\left(\frac{z}{\xi-1}-\gamma-\delta n\right)}{\Gamma(n_1+n)\cdots\Gamma(n_r+n)\,n!} \times \left(\frac{c}{(a(\xi-1))^\delta}\right)^n. \quad (4.23)$$

The sum on the right side of (4.23) is the sum of residues of the function

$$\frac{\Gamma(w)\Gamma(m_1-w)\cdots\Gamma(m_q-w)\,\Gamma(\gamma-\delta w)\Gamma\left(\frac{z}{\xi-1}-\gamma+\delta w\right)}{\Gamma(n_1-w)\cdots\Gamma(n_r-w)}\left(-\frac{c}{(a(\xi-1))^\delta}\right)^{-w}.$$

By residue theorem [2], this sum is the H-function

$$H^{2,q+1}_{q+1,r+2}\left[-\frac{c}{(a(\xi-1))^\delta}\middle| \begin{array}{l}(1-m_1,1),\cdots,(1-m_q,1),(1-\gamma,\delta)\\ (0,1),\left(\frac{z}{\xi-1}-\gamma,\delta\right),(1-n_1,1),\cdots,(1-n_r,1)\end{array}\right].$$

Hence, we get (4.21).
When $\delta = 1$, the sum on the right side of (4.23) reduces to G-function and (4.22) follows. □

Next, $\mathcal{Q}$-operator of the H-function defined in (4.10) is evaluated.

Theorem 4.6. *Let k, l, q, r be integers and let $c \neq 0, m_i, n_j, z, \sigma, \gamma \in \mathbb{C};\ M_i, N_j, \delta \in \mathbb{R}^+;\ i = 1, \ldots, q; j = 1, \ldots, r$ and $\xi > 1$, Suppose either $\sigma > 0, |\arg\ c| < \frac{\pi}{2}\sigma$ or $\sigma = 0$, and $\Re(\Delta) < -1$. Assume that $0 < \Re(\gamma - \delta s) < \Re\left(\frac{z}{\xi-1}\right)$, for any complex number s. Assuming that*

$$\Re(\gamma) + \delta \min_{1\leq j\leq k}\left[\frac{\Re(n_j)}{N_j}, \frac{\Re\left(\frac{z}{\xi-1}-\gamma\right)}{\delta}\right] > 0,$$

for $\sigma > 0$ or $\sigma = 0, \eta \geq 0$ and

$$\Re(\gamma) + \delta \min_{1\leq j\leq k}\left[\frac{\Re(n_j)}{N_j}, \frac{\Re\left(\frac{z}{\xi-1}-\gamma\right)}{\delta}, \frac{\Re(\Delta)+\frac{1}{2}}{\eta}\right] > 0,$$

for $\sigma = 0, \eta < 0$, we have

$$Q_\xi^a\left[x^{\gamma-1}H_{q,r}^{k,l}(cx^\delta); z\right] = \frac{1}{(a(\xi-1))^\gamma \Gamma\left(\frac{z}{\xi-1}\right)} \times H_{q+1,r+1}^{k+1,l+1}\left[\frac{c}{(a(\xi-1))^\delta}\middle| \begin{matrix} (m_1, M_1), \cdots, (m_q, M_q), (1-\gamma, \delta) \\ (n_1, N_1), \cdots, (n_r, N_r), \left(\frac{z}{\xi-1}-\gamma, \delta\right) \end{matrix}\right], \quad (4.24)$$

where $\sigma, \beta, \eta, \Delta$ are defined in (4.11).

Proof. Let us denote the constants $\sigma, \beta, \eta, \Delta$ for the H-function in right side of (4.24) by $\sigma_0, \beta_0, \eta_0, \Delta_0$. Then we have

$$\sigma_0 = \sigma + 2\delta, \beta_0 = \beta, \eta_0 = \eta, \Delta_0 = \Delta + \frac{z}{\xi - 1} - 1.$$

For the existence of H-functions on both sides of (4.24), we consider a contour beginning at $\rho - i\infty$ and ending at $\rho + i\infty$, denoted by $\mathcal{L} = \mathcal{L}_{i\rho\infty}$ and the choice of the real number ρ depends on the constants P, Q, R, M, where

$$P = \min_{1\leq i\leq l} \frac{1-\Re(m_i)}{M_i}, Q = -\min_{1\leq j\leq k} \frac{\Re(n_j)}{N_j}, R = -\frac{\Re(\Delta+1)}{\eta}, M = \frac{\Re(\gamma)}{\delta}.$$

Detailed notes on the convergence conditions are available in [8, 20]. Hence, H-functions on both sides exist. Now using (1.2) and (4.10), we get

$$Q_\xi^a\left[x^{\gamma-1}H_{q,r}^{k,l}(cx^\delta); z\right] = \int_0^\infty [1 + a(\xi-1)x]^{-\frac{z}{\xi-1}} x^{\gamma-1} \frac{1}{2\pi i} \times \int_{\mathcal{L}} \frac{\left\{\prod_{j=1}^k \Gamma(n_j + N_j s)\right\}\left\{\prod_{i=1}^l \Gamma(1 - m_i - M_i s)\right\}(cx^\delta)^{-s}}{\left\{\prod_{j=k+1}^r \Gamma(1 - n_j - N_j s)\right\}\left\{\prod_{i=l+1}^q \Gamma(m_i + M_i s)\right\}} \mathrm{d}x\mathrm{d}s,$$

where $0 < \Re(\gamma - \delta s) < \Re\left(\frac{z}{\xi-1}\right)$. Changing the order of integration, as the integral is uniformly convergent, and on simplification, the theorem follows. □

The following corollary gives the Q-operator of G-function, which is defined in (4.12).

Corollary 4.7. *Let k, l, q, r be integers and let $c \neq 0, m_i, n_j, z, \gamma \in \mathbb{C}; i = 1, \ldots, q; j = 1, \ldots, r; \delta \in \mathbb{R}^+$ and $\xi > 1$ with $|\arg\ c| < \frac{1}{2}(k + l + \frac{q-r}{2})$. Assume that $0 < \Re(\gamma - \delta s) < \Re\left(\frac{z}{\xi-1}\right)$, for any complex number s. Assuming that*

$$\Re(\gamma) + \delta \min_{1\leq j\leq k}\left[\Re(n_j), \frac{\Re\left(\frac{z}{\xi-1} - \gamma\right)}{\delta}\right] > 0,$$

we obtain the Q-operator of G-function as

$$Q_\xi^a\left[x^{\gamma-1}G_{q,r}^{k,l}(cx^\delta); z\right] = \frac{1}{(a(\xi-1))^\gamma \Gamma\left(\frac{z}{\xi-1}\right)} \times H_{q+1,r+1}^{k+1,l+1}\left[\frac{c}{(a(\xi-1))^\delta}\middle| \begin{matrix} (m_1, 1), \cdots, (m_q, 1), (1-\gamma, \delta) \\ (n_1, 1), \cdots, (n_r, 1), \left(\frac{z}{\xi-1}-\gamma, \delta\right) \end{matrix}\right]. \quad (4.25)$$

Furthermore, when $\delta = 1$, we get

$$Q_\xi^a\left[x^{\gamma-1}G_{q,r}^{k,l}(cx); z\right] = \frac{1}{(a(\xi-1))^\gamma \Gamma\left(\frac{z}{\xi-1}\right)} G_{q+1,r+1}^{k+1,l+1}\left[\frac{c}{a(\xi-1)}\middle| \begin{matrix} m_1, \cdots, m_q, 1-\gamma \\ n_1, \cdots, n_r, \frac{z}{\xi-1}-\gamma \end{matrix}\right]. \quad (4.26)$$

Proof. By putting $M_i = 1, i - 1, 2, \cdots, q$ and $N_j = 1, j = 1, 2, \cdots, r$ in Theorem 4.6, the result (4.25) follows. Further when $\delta = 1$, the H-function on the right reduces to G-function and (4.26) follows. □

5 Conclusion

The $\mathcal{Q}$-operator offers a remarkably versatile framework, encompassing a broader spectrum of operators compared to its counterparts. The $\mathcal{Q}$-operator smoothly transits between binomial and exponential forms, creating a pathway that connects various functional forms. The motivation of the $\mathcal{Q}$-operator lies in its well-defined inversion formula, granting it wider applicability compared to other pathway-type operators. Notably, Theorems 4.1–4.6 and Corollary 4.7 demonstrates that the H-function and G-function play a pivotal role in deriving the $\mathcal{Q}$-operator of generalized special functions. If considered as an integral transform, the $\mathcal{Q}$-operator emerges as a valuable tool with the potential to tackle diverse problems across STEM disciplines.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] R. Agarwal, S. Jain and R. P. Agarwal, *Solution of fractional Volterra integral equation and non-homogeneous time-fractional heat equation using integral transform of pathway type*, Progress in Frac. Differ. and Appl. **1(3)** , 145–155, (2015).

[2] L. V. Ahlfors, *Complex Analysis: An introduction to the theory of analytic functions of one complex variable*, McGraw-Hill Book Company, New York, (1966).

[3] F. Belgacem and A. Karaballi, *Sumudu transform fundamental properties investigations and applications*, J. Appl. Math. Stoch. Anal., **2006(1)**, 1–23, (2006).

[4] J. Bertrand, P. Bertrand and J. Ovarlez, *The Mellin Transform*, The Transforms and Applications Handbook: Second Edition, CRC Press LLC, Boca Raton, (2000).

[5] C. Fox, *The G and H functions as symmetrical Fourier kernels*, Trans. Am. Math. Soc., **98(3)**, 395–429, (1961).

[6] R.C.Gonzalez and R.E.Woods, *Digital Image Processing*, Pearson Education, New York, (2018).

[7] A. Iomin, *Fox H-functions in self-consistent description of a free-electron laser*, Fractal Fract., **5(4)**, 263, (2021).

[8] A. A. Kilbas and M. Saigo, *H-Transforms: Theory and Applications*, Chapman and Hall/CRC, Boca Raton, Fl., (2004).

[9] E. Krätzel, *Integral transformations of Bessel type*, In Generalized Functions and Operational Calculus, Proc. Conf. Verna, 1975 (pp. 148-155), Sofia: Bulg. Acad. Sci., (1979).

[10] R. Kress, *Linear Integral Equations*, Springer-Verlag, New York, (1999).

[11] D. Kumar, *P-Transform*, Integral Transforms Spec. Funct., **22(8)**, 603–616, (2011).

[12] D. Kumar, *Solution of fractional kinetic equation by a class of integral transform of pathway type*, J. Math. Phys., **54**, 043509, (2013).

[13] D. Kumar and A. A. Kilbas, *Fractional Calculus of P-transforms*, Fract. Calc. Appl. Anal., **13(3)**, 309–328, (2010).

[14] N. Laskin, *Fractional quantum mechanics*, Phys. Rev. E, **62(3)**, 3135–3145, (2000a).

[15] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A, **268(4)**, 298–305, (2000b).

[16] A. M. Mathai, *A pathway to matrix-variate gamma and normal densities*, Linear Algebra Appl., **396**, 317–328, (2005).

[17] A. M. Mathai and H. J. Haubold, *Pathway model, superstatistics, Tsallis statistics and a generalized measure of entropy*, Phys. A, **375**, 110–122, (2007).

[18] A. M. Mathai and H. J. Haubold, *Special Functions for Applied Scientists*, Springer-Verlag, New York, (2008).

[19] A. M. Mathai and H. J. Haubold, *A versatile integral in physics and astronomy and Fox's H-function*, Axioms, **8(4)**, (2019).

[20] A. M. Mathai, R. K. Saxena and H. J. Haubold, *The H-Function: Theory and Applications*, Springer, New York, (2010).

[21] A. M. Mathai and R. K. Saxena, *The H-Function with Applications in Statistics and Other Disciplines*, Halsted Press [John Wiley & Sons], New York-London-Sidney, (1978).

[22] C. S. Meijer, *Über Whittakersche bzw. Besselsche Funktionen und deren Produkte*, Nieuw Arch. Wisk. (2), **18(4)**, 10–39, (1936).

[23] C.S. Meijer, *Eine neue Erweiterung der Laplace Transformation I*, Proc. Koninkl. Ned. Akad. Wet. , **44**, 727–737, (1941).

[24] G. M. Mittag Leffler, *Sur la nouvelle fonction* $E_\alpha(x)$, C. R. Acad. Sci. Paris (2), **137**, 514–558, (1903).

[25] S. R. Naik and H. J. Haubold, *On the q-Laplace Transform and Related Special Functions*, Axioms, **5(3)**, 24, (2016).

[26] S. S. Nair, *Pathway Fractional Integration Operator*, Fract. Calc. Appl. Anal., **12(3)**, 237–252, (2009).

[27] R. K. Parmar, V. Rohira and A. K. Rathie, *On some new* P_δ*-transforms of Kummer's confluent hypergeometric functions*, Facta Universitatis (NIŠ), Ser. Math. Inform., **34(2)**, 373–380, (2019).

[28] T. R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in kernel*, Yokohama Math. J., **19**, 7–15, (1971).

[29] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, (1960).

[30] K. Rose, E. Gurewitz and G. Fox, *Statistical mechanics and phase transitions in clustering, Phys. Rev. Lett.,* **65(8)***, 945–948, (1990).*

[31] R. K. Saxena, A. M. Mathai and H. J. Haubold, *On fractional kinetic equations*, Astrophys. Space Sci., **282**, 281–287, (2002).

[32] R. K. Saxena, A. M. Mathai and H. J. Haubold, *On generalized fractional kinetic equations*, Physica A: Stat. Mech. Appl., **344(3-4)**, 657–664, (2004a).

[33] R. K. Saxena, A. M. Mathai and H. J. Haubold, *Unified fractional kinetic equation and a fractional diffusion equation*, Astrophys. Space Sci., **290**, 299–310, (2004b).

[34] R. K. Saxena, A. M. Mathai and H. J. Haubold, *Solution of certain fractional kinetic equations and a fractional diffusion equation*, J. Math. Phy., **51(10)**, 103506, (2010).

[35] J. J. Sakurai, *Modern Quantum Mechanics*, Addison-Wesley, Reading, MA, (1994).

[36] I. N. Sneddon, *The Use of Integral Transforms*, Tata McGraw Hill Publishers, New Delhi, (1972).

[37] M. R. Spiegel, *Schaum's outline of Theory and Problems of Laplace Transforms*, Schaum's outline series, McGraw Hill Education, New York, (1965).

[38] M. R. Spiegel, S. Lipschutz and J. Liu, *Mathematical Handbook of Formulas and Tables*, Schaum's outline series, McGraw Hill Education, New York, (1968).

[39] E. M. Stein, and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, New Jersey, (1971).

[40] G. K. Watugala, *Sumudu transform-a new integral transform to solve differential equations and control engineering problems*, Int. J. Math. Educ. Sci. Technol., **24(1)**, 35–43, (1993).

[41] A. Wiman, *Über den Fundamental salz in der Theorie der Funktionen* $E_a(x)$, Acta Math., **29**, 191–201, (1905).

[42] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, J. Lond. Math. Soc., **10**, 286–293, (1935).

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