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Spectrum of bipartite Kneser type-k graphs

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Abstract *Let n and k be integers with $n > 1$, $k \geq 1$. Let $[n] = \{1, 2, 3, \ldots, n\}$. Let $\phi([n])$ be the set of all non-empty subsets of $[n]$. Let V_1 be the set of all k-element subsets of $[n]$ and $V_2 = \phi([n]) - V_1$. A bipartite Kneser type-k graph $H_T(n, k)$ is a graph with vertex set $V = V_1 \cup V_2$ in which a vertex of V_1 is adjacent to a vertex of V_2 if and only if the set corresponding to one of the vertices is a subset of the set corresponding to the other vertex. The spectrum becomes an important molecular structure descriptor when thinking of chemical graphs. In this paper, we determined the spectrum of $H_T(n, k)$ and some of its characteristics. We also analysed the spectrum of the line graph of $H_T(n, k)$. It has been observed that the least eigenvalue of the line graph of $H_T(n, k)$ is -2.*

1 Introduction

Graph theory, chemical graph theory, and bipartite graphs intersect in intriguing ways, offering valuable applications across various scientific fields [4]. Chemical graph theory utilizes graph-theoretical concepts to model molecular structures and analyze their properties, while graph theory provides the mathematical framework to study interactions between entities.

In the context of chemistry, chemical graph theory is instrumental in understanding molecular composition and behaviour. Atoms are represented as vertices, and chemical bonds are depicted as edges in a graph. An important aspect of graph theory is the spectrum, which consists of the eigenvalues of the adjacency matrix and captures key structural information about the graph. The spectrum is a powerful tool for exploring network characteristics such as connectedness, symmetry, and isomorphism. Specifically, in chemical applications, bipartite graph analysis aids in examining atoms and bonds in molecular structures, uncovering patterns and helping to predict chemical properties.

The spectrum plays a key role in characterizing molecular structure within the framework of chemical graphs [9]. By linking specific molecular features to individual eigenvalues or patterns within the spectrum, it is possible to gain insights into molecular behaviour and stability. Additionally, shifts in the spectrum can be associated with chemical transformations and reactions, offering a deeper understanding of reaction mechanisms and facilitating the prediction of a compound's reactivity. In [7], several fundamental properties of the forgotten topological index were identified, highlighting its potential to greatly improve the physico-chemical relevance of the first Zagreb index.

Consider a simple graph $G = (V, E)$, where V represents the vertex set V and E represents the edge set. Let $A(G) = A$ denote the adjacency matrix of the graph G. Matrices derived from combinatorial structures are referred to as combinatorial matrices. A k-subset, or a subset with k elements, is a subset of a set with n elements that contains exactly k elements. These matrices find applications in block designs and coding theory. More generally, a combinatorial *matrix of type (m,t), as discussed in [14], where $m > t$, contains $\binom{m}{t}$ rows and columns, each indexed by t-element subsets of an m-element set S. For any two t-u and v, the value in the row corresponding to u and the column corresponding to v is determined solely by the size of $u \cap v$. A general combinatorial matrix of type (m,t) can have up to $t+1$ distinct entries.An association scheme based on such a combinatorial matrix was introduced and employed to analyse the spectrum of Johnson graphs [14]. The spectral radius of the signless Laplacian matrix and the sum-connectivity index of graphs were investigated in [11]. Furthermore, [12] identified graphs where the distance Laplacian eigenvalue λ has multiplicity $n-4$. In another study, Khan and Zhang examined the generalized distance spectral radius and generalized distance energy of graphs [13].*

The characteristic polynomial of a graph G is given by $\phi(G,x) = |xI - A|$, where I is the identity matrix of size $|V|$. The roots of $\phi(G,x)$ are referred to as the eigenvalues of G and the collection of all eigenvalues, along with their multiplicities, is known as the spectrum of G. The energy $E(G)$ of a simple graph G is defined as the sum of the absolute values of its eigenvalues, as described in [2].

Suppose the eigenvalues of G are $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ with multiplicities $m_1, m_2, \cdots, m_r$, respectively. Then we write

$$Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{pmatrix}$$

or

$$Spec(G) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \cdots, \lambda_r^{m_r}\}.$$

A graph G is referred to as an integral graph if all the eigenvalues in its spectrum are integers. Research on different spectra of graphs has been conducted in [1], [19], [18], [20]. If the vertex set of G can be partitioned into $V = V_1 \cup V_2$, $|V_1| = n_1$ and $|V_2| = n_2$, such that both V_1 and V_2 are independent sets and each vertex of V_1 (respectively V_2) has r_1 (respectively r_2) neighbours in V_2(respectively in V_1), then we say that G is a semi-regular bipartite graph with parameters (n_1, n_2, r_1, r_2).

Let $[n] = \{1, 2, 3, \cdots, n\}$, where n is a positive integer. Let n, k and l be integers with $1 \leq k < l \leq n-1$. The set-inclusion graph $G(n,k,l)$, [10], is defined as the graph with $V = \{v/v \subset [n], |v| \in \{k,l\}\}$, and two distinct vertices are adjacent if one of them is contained in another. The mapping $\tau : v \to \overline{v} = [n] - v$ gives an isomorphism between $G(n,k,l)$ and $G(n, n-l, n-k)$. Thus $k + l \leq n$ and $k \leq \frac{n-1}{2}$ as $k < l$.

Obviously, $G(n,k,l)$ is a connected semi-regular bipartite graph having parameters $\left(\binom{n}{k}, \binom{n}{l}, \binom{n-k}{l-k}, \binom{l}{k}\right)$, where the bipartition is given by $V = V_1' \cup V_2'$ with

$$V_1' = \{v/v \subseteq [n], |v| = k\} \text{ and } V_2' = \{v/v \subseteq [n], |v| = l\}. \tag{1.1}$$

Let V be the set of all k-subsets ($1 \leq k \leq n$) and $n-k$-subsets of $[n]$. The bipartite Kneser graph $H(n,k)$, [5], has V as its vertex set, and two vertices u, v are adjacent if and only if $u \subset v$ or $v \subset u$.

For $n = 2k$, the graph $H(n,k)$ is an empty graph, hence assume that $n \geq 2k+1$. It is evident from the definition of $H(n,k)$ that it has $2\binom{n}{k}$ vertices and each of its vertices has a degree of $\binom{n-k}{k} = \binom{n-k}{n-2k}$. The graph $H(n,k)$ is thus a bipartite regular graph. Also, we see that $H(n,k) = G(n,k,n-k)$ [10], which is very useful in computing its spectrum.

The distance between two distinct vertices u and v in G is denoted by $d(u,v)$. The distance matrix [19] of a graph G of order n, denoted by $D(G) = D$ is an $n \times n$ matrix whose (i,j)-entry is equal to $d(v_i, v_j)$ for $1 \leq i, j \leq n$. The characteristic polynomial of $D(G)$ is defined as $P_{D(G)}(x) = Det(xI - D(G))$, where I is the $n \times n$ identity matrix, and is called the distance *characteristic polynomial of G. Since $D(G)$ is a real symmetric matrix, all its eigenvalues called distance eigenvalues of G, are real. The distance spectrum of G is the spectrum of $D(G)$ denoted by $Spec(D) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.*

Let n, k and i be integers such that $0 \leq i \leq k \leq \frac{n}{2}$. The Johnson graph $J(n,k,i)$ has a vertex set consisting of all k-element subsets of $[n]$, where two vertices (k-element subsets) are adjacent if their intersection has size i. That is, $J(n,k,i)$ is a graph made up of $\binom{n}{k}$ isolated vertices, each of which is provided with one loop. The graphs in particular represent the Johnson graph and the Kneser graph $J(n,k) = J(n,k,k-1)$ and $K(n,k) = J(n,k,0)$, respectively.

For $0 \leq i \leq k$, let A_i be the adjacency matrix of $J(n,k,i)$. Due to the fact that the set of matrices $A_i : 0 \leq i \leq k$ forms a symmetric association scheme [14] and is known as the Johnson scheme. These matrices have the following properties

$$A_k = I, \ \sum_{i=0}^{k} A_i = J, \ A_i^T = A_i \ \text{ and}$$

$$A_iA_j = A_jA_i = \sum_{s=0}^{k}\left[\sum_{r=0}^{s}\binom{s}{r}\binom{k-s}{j-r}\binom{n-2k+s}{k-i-j+r}\right]A_s$$

for $0 \leq i \neq j \leq k$.

Cvetković et al. [3] conducted a study on graphs where -2 is the smallest eigenvalue, extending the concept of line graphs. The spectrum of the set inclusion graph was analysed in [10]. Additionally, K. C. Das, I. N. Cangül, and collaborators [5] derived results regarding the spectral radius of nearly complete bipartite graphs.

The bipartite Kneser type-k graph, [17], was defined as follows. Let $[n] = \{1, 2, 3, \ldots, n\}$ for a fixed integer $n > 1$ and an integer k such that $1 \leq k \leq n-1$. Let $\phi([n])$ be the set of all non-empty subsets of $[n]$. Let V_1 be the set of all k-element subsets of $[n]$, $k \geq 1$, and $V_2 = \phi([n]) - V_1$. A bipartite Kneser type-k graph $H_T(n,k)$ is defined with parts V_1 and V_2, and a vertex $A \in V_1$ is adjacent to a vertex $B \in V_2$ if and only if $A \subset B$ or $B \subset A$. The bipartite Kneser type-1 graph is isomorphic to the SM dyadic sum graph, [16], which deals with structure of dyadic fractions in the interval $(0,1)$.

In this study, we developed a system of matrices analogous to the association scheme defined for the adjacency matrices of $J(n,k)$. Leveraging this framework, we successfully derived the spectrum of the graph $H_T(n,k)$. Additionally, we extended our investigation to the line graph of $H_T(n,k)$, thoroughly analysing its spectral properties. Through this analysis, we established that all eigenvalues of the line graph of $H_T(n,k)$ are greater than or equal to -2. This result provides valuable insights into the spectral characteristics and structural properties of the line graph derived from $H_T(n,k)$.

2 Preliminaries

Some of the fundamental information that are relevant to the topics that follow are covered in this section. The following lemma provides insight into the spectrum-structure of the graph $J(n,k,i)$. Theorem 2.2 explains the spectrum of the graph $G(n,k,l)$ and they used Lemma 2.1 to prove the theorem below:

Lemma 2.1. *[10] Let n, k and i be integers with $0 \leq i \leq k \leq \frac{n}{2}$, then the spectrum of $J(n,k,i)$ is given by*

$$\{[\alpha_{i,s}]^{\binom{n}{s}-\binom{n}{s-1}} : 0 \leq s \leq k\},$$

$$\text{where } \ \alpha_{i,s} = \sum_{r=0}^{s}(-1)^{s-r}\binom{s}{r}\binom{k-r}{i-r}\binom{n-k-s+r}{k-i-s+r}.$$

Moreover, for any fixed s, there are $\left(\binom{n}{s}-\binom{n}{s-1}\right)$'s linearly independent vectors that serve as the common eigenvectors of A_i's (where A_i is the adjacency matrix of $J(n,k,i)$ corresponding to the eigenvalue $\alpha_{i,s}$ for $0 \leq i \leq k$).

As $G(n,k,l)$ is bipartite, its adjacency matrix can be written as

$$A = \begin{bmatrix} 0 & H \\ H^T & 0 \end{bmatrix} \begin{matrix} V_1' \\ V_2', \end{matrix}$$

where V_1' and V_2' are $V_1' = \{v/v \subseteq [n], |v| = k\}$ and $V_2' = \{v/v \subseteq [n], |v| = l\}$. Then $|V_1'| = \binom{n}{k}$ and $|V_2'| = \binom{n}{l}$. Therefore,

$$|xI - A^2| = |xI - HH^T||xI - H^TH| = x^{|V_2'|-|V_1'|}|xI - HH^T|^2 \tag{2.1}$$

Since V_1' and V_2' represents all k-element subsets and l-element subsets of $[n]$, $k > 1$, for any two vertices $u, v \in V_1'$, $(HH^T)_{u,v}$ represents the number of common neighbours of u, v in V_2' for each $k > 1$ [with u and v are not necessarily distinct]. According to the definition of $G(n,k,l)$, we get

$$(HH^T)_{u,v} = \begin{cases} \binom{n-2k+i}{l-2k+i} & if\, |u \cap v| = i\, with\ max\{2k-l, 0\} \leq i \leq k, \\ 0 & otherwise. \end{cases}$$

Then

$$HH^T = \sum_{i=max\{2k-l,0\}}^{k} \sum_{l=2}^{n} \binom{n-2k+i}{l-2k+i} A_i\ ,$$

where A_i is the adjacency matrix of $J(n,k,i)$. Therefore, by Lemma 2.1, each eigenvalue of HH^T is of the form

$$\gamma_s = \sum_{l=2}^{n} \beta_s = \sum_{i=max\{2k-l,0\}}^{k} \sum_{l=2}^{n} \binom{n-2k+i}{l-2k+i} \alpha_{i,s}$$

with multiplicity $\binom{n}{s} - \binom{n}{s-1}$, where

$$\alpha_{i,s} = \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \binom{k-r}{i-r} \binom{n-k-s+r}{k-i-s+r}.$$

Theorem 2.2. *[10] Let n, k and l be integers with $1 \leq k < l \leq n-1$ and $k + l \leq n$. Then the spectrum of $G(n,k,l)$ is given by*

$$\left\{ \left[\pm\sqrt{\beta_s}\right]^{\binom{n}{s}-\binom{n}{s-1}} : 0 \leq s \leq k \right\} \cup \left\{ [0]^{\binom{n}{l}-\binom{n}{k}} \right\},$$

where $\beta_s = \sum_{i=max\{2k-l,0\}}^{k} \binom{n-2k+i}{l-2k+i} \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \binom{k-r}{i-r} \binom{n-k-s+r}{k-i-s+r}$.

Note that spectrum of $G(n,k,l)$ is symmetric about 0 as the graph is bipartite. If $a_1, a_2, \cdots, a_{\binom{n}{k}}$ are all eigenvalues of HH^T, then from Equation. 2.1, the eigenvalues of $G(n,k,l)$ are $\pm\sqrt{a_1}, \pm\sqrt{a_2}, \cdots,$ $\pm\sqrt{a_{\binom{n}{k}}}$, and 0 of algebraic multiplicity $\binom{n}{l} - \binom{n}{k}$.

A graph is classified as an L-graph if its smallest eigenvalue is greater than or equal to -2. The concept of a line graph $L(G)$ is defined as follows: the vertices of $L(G)$ correspond to the edges of a given graph G and two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G share a common vertex. In the study conducted by Cvetković et al. [3], the investigation of graphs with -2 as the smallest eigenvalue led to the observation that all line graphs have their least eigenvalue greater than or equal to -2. This characteristic is not limited to line graphs alone; it is also shared by generalized line graphs and certain other specialized graph families. Specifically, this property extends to the line graph of bipartite Kneser type-k

graphs, demonstrating a broader applicability of the spectral behaviour observed in line graphs. These findings underscore the unique spectral characteristics of this class of graphs and their structural significance.

In the context of uni-monads, bipartite Kneser type-1 graphs have been defined. Next, we introduce the generalized bipartite Kneser type-k graphs as described below. The following three definitions are adapted from [15] and [17].

Definition 2.3. Let $[n] = 1, 2, 3, \dots, n$, where n is a fixed integer greater than 1. Define $\phi([n])$ as the set of all non-empty subsets of $[n]$. For $U \in \phi([n])$, let $U = \bigcup_{i=1}^{m} U_i$, where each U_i is a one-element subset of $[n]$, the sets U_i and U_j are disjoint ($U_i \cap U_j = \emptyset$), and $1 < m \leq n$. Each U_i is referred to as a uni-monad of U.

Definition 2.4. Let $\mathcal{M}_n$ represent the set of all uni-monads. Let $V = \phi([n])$ denote the vertex set, and consider $u, v \in V$. The adjacency in the graph $H_T(n, 1)$ is defined using uni-monads as follows: two vertices u and v are adjacent if u is a uni-monad of v or if v is a uni-monad of u.

For each $n \geq 2$, the bipartite Kneser type-1 graph consists of $2^n - 1$ vertices and $n(2^{n-1} - 1)$ edges. Additionally, it has been observed that these graphs are neither vertex-transitive nor edge-transitive. However, they are not asymmetric.

Let r-ELS represent the set of r-element subsets of $[n]$. If $V_1 = 1$-ELS $= \mathcal{M}_n$ and $V_2 = \bigcup r = 2^n r$-ELS, then V_1 and V_2 form the bipartition for the vertices of $H_T(n, 1)$. The definition of the bipartite Kneser type-k graph is provided below.

Definition 2.5. For integers $n > 1$ and $k \geq 1$, let $[n] = 1, 2, 3, \dots, n$. Let $\phi([n])$ represent the set of all non-empty subsets of $[n]$. Define V_1 as the set of all k-element subsets of $[n]$, and let $V_2 = \phi([n]) \setminus V_1$. A bipartite graph is constructed with vertex parts V_1 and V_2, where adjacency is defined as follows: a vertex $A \in V_1$ is adjacent to a vertex $B \in V_2$ if and only if $A \subset B$ or $B \subset A$. This graph is referred to as the bipartite Kneser type-k graph and is denoted by $H_T(n, k)$.

The bipartite Kneser type-k graph is a connected graph consisting of $2^n - 1$ vertices. The graph $H_T(n, k)$ is not asymmetric. The distances within the graph $H_T(n, k)$ for the case $k = 1$ are described in the following lemma:

Lemma 2.6. *[17] Let u and v be any two distinct vertices in the graph $H_T(n, 1)$, where $n \geq 4$. The distances between u and v are determined as follows:*

$$d(u, v) = \begin{cases} 1 & \textit{if } u \textit{ is a uni-monad of } v\textit{, or } v \textit{ is a uni-monad of } u, \\ 2 & \textit{if } u, v \in V_1 \textit{ or } u, v \in V_2,\ \ u \textit{ and } v \textit{ share at least one common uni-monad}, \\ 3 & \textit{if neither } u \textit{ nor } v \textit{ is a uni-monad of the other,} \\ & \ \textit{but exactly one of them belongs to } V_1, \\ 4 & \textit{if } u, v \in V_2,\ \ u \textit{ and } v \textit{ do not share a common uni-monads.} \end{cases}$$

This lemma will help in forming the distance matrix of the graph $H_T(n, 1)$. The diameter of $H_T(n, k)$ is 4. It is easy to see that the graph $H_T(n, k)$ is isomorphic to $H_T(n, n - k)$.

The nullity of a graph is defined as the multiplicity of the eigenvalue zero of graph G and is denoted by $\eta(G)$. The following is a known result for the nullity of a bipartite graphs of order n:

Theorem 2.7. *[6] Let $\mathcal{B}_n$ be the set of all bipartite graph of order n. The nullity set of $\mathcal{B}_n$ is $N = \{n - 2k : k = 0, 1, 2, \cdots, [\lfloor n/2 \rfloor]\}$.*

3 Main results-spectrum of $H_T(n, k)$

We discovered the spectrum of the graph $H_T(n, k)$ utilising the combinatorial matrix and findings related the spectrum of the set-inclusion graph and Johnson graph. Also, because $H_T(n, k)$ is a bipartite graph, its spectrum is symmetric about 0.

Let $(m, \tilde{t}\,)$ be a combinatorial matrix of the type (m, t) arising from the graph $H_T(n, k)$ with t , $t = 1, 2, \cdots, k-1, k, \cdots, n$. Let $u \in V_1$ and $v \in V_2$. Let $H_{m,\tilde{t}}$ be the combinatorial matrix of the type $(m, \tilde{t}\,)$ having 1 in a row u and column v when $u \subset v$ or $v \subset u$, and 0 otherwise. The combinatorial matrix $H_{m,\tilde{t}}$ is having $2^m - 1$ rows and columns.

Lemma 3.1. *Let $A_{(n,k,l)}$ be the adjacency matrix for the graph $G(n, k, l)$, $1 \leq k < l \leq n-1$. The combinatorial matrix $H_{m,\tilde{t}}$ is decomposable to block matrices of the type $A_{(n,k,l)}$.*

Proof. The combinatorial matrix of the type (m, t) has $\binom{m}{t}$ rows and columns. The combinatorial matrix of the type $(m, \tilde{t}\,)$ has $2^m - 1 = \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{m}$ rows and columns. □

The matrix $H_{m,\tilde{t}}$ contains block matrices which are the $A_{(n,k,l)}$ and one block of typical row-column matrix corresponding to the case when $l = n$ in $G(n, k, l)$. These blocks completely determines the adjacency matrix of the graph $H_T(n, k)$. So the eigenvalues are determined by these block matrices in either of the cases.

Proposition 3.2. *For the graph $H_T(n, k)$, $k = 1$, let $u, v \in V_1$, not necessarily distinct, then the number of common neighbours of vertices u, v, in V_2, $\aleph_{u,v}$ is given by*

$$\aleph_{u,v} = \begin{cases} \sum\limits_{b=2}^{n} \binom{n-2+i}{b-2+i} & if\ |u \cap v| = i \ \ with\ max\{2-b, 0\} \leq i \leq 1, \\ 0 & otherwise. \end{cases}$$

Proof. For the graph $H_T(n, k)$, $k = 1$, the proof follows from the counting principle. □

Here the restriction on i is taken in such a way to proceed for the general case of all possible values of k.

Remark 3.3. The same result in Proposition 3.2 holds for $k > 1$, but still with a greater value than $\aleph_{u,v}$. It means that a finite number is added with $\aleph_{u,v}$ when $k > 1$. Let $\aleph'_{u,v}$ denote the required number of common neighbours for $k > 1$. This case is explained in the following proposition.

Proposition 3.4. *For $k > 1$, we get*

$$\aleph'_{u,v} = \begin{cases} \sum\limits_{b=2k-i}^{n} \binom{n-2k+i}{b-2k+i} + \delta & if\ |u \cap v| = i \ \ with\ max\{2k-b, 0\} \leq i \leq k, \\ 0 & otherwise, \end{cases}$$

where $\delta - 1 = \sum\limits_{j=1}^{k-1} \binom{i}{j}$, $|u \cap v| = i$, $0 < i \leq k-1$.

Proof. The proof follows from Proposition 3.2 and by considering both of the inclusions. □

Theorem 3.5. *The spectrum of $H_T(n, k)$, $k = 1$ is $\left\{0^F, \pm\sqrt{\sum_{b=2}^{n} \Theta_0}^{\,1}, \pm\sqrt{\sum_{b=2}^{n-1} \Theta_1}^{\,n-1}\right\}$, where $F = \sum_{i=2, i \neq n-1}^{n} \binom{n}{i}$, $\Theta_s = \sum_{i=max\{2k-b,0\}}^{k} \binom{n-2k+i}{b-2k+i} \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \binom{k-r}{i-r} \binom{n-k-s+r}{k-i-s+r}$,*

$0 \leq s \leq k$ and $0 \leq b \leq n$.

Proof. Since $H_T(n, k)$ is bipartite, its adjacency matrix can be written as

$$A = \begin{bmatrix} 0 & H \\ H^T & 0 \end{bmatrix} \begin{matrix} V_1 \\ V_2 \end{matrix}$$

where $V_1 = \{v/v \subset [n], |v| = k\}$, and $V_2 = \phi([n]) - V_1$
Then $|V_1| = n$ and $|V_2| = 2^n - n - 1$. For each $b > k = 1$, we have

$$\begin{aligned} xI - A^2 &= \begin{bmatrix} xI & 0 \\ 0 & xI \end{bmatrix} - \begin{bmatrix} o & H \\ H^T & 0 \end{bmatrix} \begin{bmatrix} 0 & H \\ H^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} xI & 0 \\ 0 & xI \end{bmatrix} - \begin{bmatrix} HH^T & 0 \\ 0 & H^T H \end{bmatrix} \\ &= \begin{bmatrix} xI - HH^T & 0 \\ 0 & xI - H^T H \end{bmatrix}. \end{aligned}$$

Therefore,

$$|xI - A^2| = |xI - HH^T||xI - H^T H|$$

But

$$|xI - H^T H| = x^F |xI - HH^T| \tag{3.1}$$

$$|xI - A^2| = x^F |xI - HH^T|^2, \tag{3.2}$$

where $F = \sum_{i=2, i\neq n-1}^{n} \binom{n}{i}$ and $\binom{n}{k} \leq \binom{n}{b}$ as $1 < b \leq n$.

The spectrum of $H_T(n,k)$ is symmetric about 0 as the graph is bipartite. If $\mu_1, \mu_2, \cdots,$ $\mu_{\binom{n}{k}}$ are all eigenvalues of HH^T, then from equation (3.2), the eigenvalues of $H_T(n,k)$ are $\pm\sqrt{\mu_1}, \pm\sqrt{\mu_2}, \cdots, \pm\sqrt{\mu_{\binom{n}{k}}}$ and 0 of algebraic multiplicity $F = \sum_{i=2, i\neq n-1}^{n} \binom{n}{i}$. Since V_1 and V_2 are all 1-element subsets and k-element subsets of $[n]$, respectively, $k > 1$, for any two vertices $u, v \in V_1$ (not necessarily distinct), by Proposition 3.2 , $(HH^T)_{u,v}$ is just the number of common neighbours of u, v, in V_2 for each $k > 1$. By the definition of $H_T(n,1)$,

$$(HH^T)_{u,v} = \begin{cases} \sum\limits_{b=2}^{n} \binom{n-2k+i}{b-2k+i} & if\ |u \cap v| = i \quad with\ max\{2k-b, 0\} \leq i \leq k, \\ 0 & otherwise. \end{cases}$$

Then

$$HH^T = \sum_{i=max\{2k-b,0\}}^{k} \sum_{b=2}^{n} \binom{n-2k+i}{b-2k+i} A_i \ ,$$

where A_i is the adjacency matrix of $J(n,k,i)$. Therefore, by Proposition 3.2, each eigenvalue of HH^T is of the form

$$\gamma_s = \sum_{b=2}^{n} \Theta_s = \sum_{i=max\{2k-b,0\}}^{k} \sum_{b=2}^{n} \binom{n-2k+i}{b-2k+i} \alpha_{i,s}$$

having multiplicity $\binom{n-1}{s} - \binom{n-1}{s-2}$, where

$$\alpha_{i,s} = \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} \binom{k-r}{i-r} \binom{n-k-s+r}{k-i-s+r}$$

Thus we obtain the spectrum of $H_T(n,1)$.

Since $H_T(n,1)$ is a bipartite graph with $2^n - 1$ vertices, its largest eigenvalue must be $\sqrt{\sum\limits_{b=2}^{n} \binom{n-k}{b-k}\binom{b}{k}}$, that is exactly equal to $\sqrt{\sum\limits_{b=2}^{n} \Theta_0}$. For this, we consider $b = 2, 3, \cdots, n$. In each case we get

$$\begin{aligned}
\sum_{b=2}^{n}\Theta_0 &= \sum_{b=2}^{n}\sum_{i=max\{2k-b,0\}}^{k}\binom{n-2k+i}{b-2k+i}\binom{k}{i}\binom{n-k}{k-i}\\
&= \sum_{b=2}^{n}\sum_{i=max\{2k-b,0\}}^{k}\frac{(n-2k+i)!}{(b-2k+i)!(n-b)!}\binom{k}{i}\frac{(n-k)!}{(n-2k+i)!(k-i)!}\\
&= \sum_{b=2}^{n}\sum_{i=max\{2k-b,0\}}^{k}\frac{(n-k)!}{(b-2k+i)!(n-b)!(k-i)!}\binom{k}{i}\\
&= \sum_{b=2}^{n}\sum_{i=max\{2k-b,0\}}^{k}\frac{(n-k)!}{(n-b)!(b-k)!}\frac{(b-k)!}{(b-2k+i)!(k-i)!}\binom{k}{i}\\
&= \sum_{b=2}^{n}\binom{n-k}{b-k}\sum_{i=max\{2k-b,0\}}^{k}\binom{b-k}{k-i}\binom{k}{i}\\
&= \sum_{b=2}^{n}\binom{n-k}{b-k}\binom{b}{k}
\end{aligned}$$

Now, by taking the square root, we get the required eigenvalue as $\sqrt{\sum\limits_{b=2}^{n}\Theta_0}$. □

Now, we consider the spectrum of the graph $H_T(n,k)$, $k>1$. The nullity $\eta(G)$ of the graph $H_T(n,k)$ for $k>1$ is given as follows:

Lemma 3.6. *For the graph $H_T(n,k)$ for $k>1$, the nullity is given by*

$$\eta(H_T(n,k)) = \begin{cases} \sum\limits_{i=1,i\neq k,n-k}^{n}\binom{n}{i} & \text{if } 1<k<\frac{n}{2},\\ \eta(H_T(n,k-1)) & \text{if } n=2m, \text{ for some integer } m, \text{ and } k=m. \end{cases}$$

Proof. Since $H_T(n,k)$ is a bipartite graph, the adjacency matrix $A(G)=\begin{bmatrix}0 & H\\ H^T & 0\end{bmatrix}$. Then by Theorem 2.7, we get $\eta(G)=\sum\limits_{i=1}^{n}\binom{n}{i}-2\times rank(H)$. Here H is a matrix of order $|V_1|\times|V_2|$. It is not a difficult task to show that the rows of H are linearly independent, using elementary transformations. Therefore, the rank of the matrix H is $\binom{n}{k}$. The result follows. □

The upper and lower bounds for the eigenvalues are more important in spectral graph theory. The lower bound for the eigenvalues of the graph $H_T(n,k)$ is obtained as follows.

Theorem 3.7. *Let $\pm\lambda_1,\pm\lambda_2,\pm\lambda_3,\cdots,\pm\lambda_j$ be the non-zero and distinct eigenvalues of the graph $H_T(n,k)$. Then $\lambda_i\geq\sqrt{\sum_{b=k+1}^{n-1}\Theta_k}$, $1\leq i\leq j$.*

Proof. Let $\lambda_1<\lambda_2<\cdots<\lambda_j$, where $1\leq j\leq k+1$ be the eigenvalues of the graph $H_T(n,k)$. From the proof of Theorem 3.5, we know that if $c_1,c_2,\cdots,c_{\binom{n}{k}}$ are all eigenvalues of HH^T, then from equation (3.2), the eigenvalues of $H_T(n,k)$ are $\pm\sqrt{c_1},\pm\sqrt{c_2},\cdots,\pm\sqrt{c_{\binom{n}{k}}}$, and 0 of algebraic multiplicity $F=\sum_{i=2,i\neq n-1}^{n}\binom{n}{i}$. In the case of $H_T(n,k)$, each $c_s=\sum_{b=k+1}^{n-1}\Theta_s+\xi_s$, where ξ_s is a non-negative integer and $0\leq s\leq k$. From the values of Θ_s and ξ_s, we get that $c_0>c_1>\cdots>c_k$. This completes the proof. □

The next theorem gives the eigenvalues of the graph $H_T(n,k)$ for $k=2$. The eigenvalues of the graph $H_T(n,k)$ for $k>2$ can be obtained in the same way.

Theorem 3.8. *The spectrum of $H_T(n,k)$, $k=2$ is* $\left\{0^F,\pm\sqrt{\sum_{b=3}^{n}\Theta_0+2(n-1)}^{\,1}\right\}\cup\left\{\pm\sqrt{\sum_{b=3}^{n-1}\Theta_1+(n-2)}^{\,n-1},\pm\sqrt{\sum_{b=3}^{n-1}\Theta_2}^{\,\binom{n}{2}-n}\right\}$, *where* $F=\sum_{i=1,i\neq k,n-k}^{n}\binom{n}{i}$.

Proof. By the definition of the graph $H_T(n,k)$, $k>1$, we see that both the inclusions holds true. Let V_1 and V_2 be the partition with V_1 contains all k-element subsets($k>1$) and V_2 contains all non-empty subsets except the k-element subsets. So obviously, if $|u|=1$, then $u\in V_2$. This generates more adjacency in addition to the adjacencies for the vertices with size more than 2, and will yield to an additional term as given in the eigenvalues except in the eigenvalue which is the spectral radius. Let $u,v\in V_2$. Let $|u|=1$ and $|v|=j$, $3\le j\le n$. Then we get the number of ordered pairs of vertices (u,v) having distance $d(u,v)=2$ is $\sum\limits_{j=1}^{n}\binom{n}{j}\binom{j}{1}$. Also, the number of ordered pairs of vertices (u,v) having distance $d(u,v)=1$ is $(2^{n-2}+1)\binom{n}{2}$. The proof is then follows from the proof of Theorem 3.5 and Proposition 3.4. □

Theorem 3.9. *The spectrum of $H_T(n,k)$, $k=3$ is* $\left\{0^F, \pm\sqrt{\sum_{b=3}^{n}\Theta_0+3(\binom{n}{2}-1)}^{\,1}\right\}\cup$ $\left\{\pm\sqrt{\sum_{b=3}^{n-1}\Theta_1+(\binom{n}{2}-1)}^{\,n-1}, \pm\sqrt{\sum_{b=3}^{n-1}\Theta_2+(n-4)}^{\,\binom{n-1}{2}-1}, \pm\sqrt{\sum_{b=3}^{n-1}\Theta_3}^{\,\binom{n-1}{2}-n-1}\right\}$*, where* $F=\sum_{i=1,i\neq k,n-k}^{n}\binom{n}{i}$.

Proof. The proof is same as the proof of Theorem 3.5 and Theorem 3.8, mutatis mutandis. □

Lemma 3.10. *The number of vertices of the line graph of $H_T(n,k)$ is $(2^k+2^{n-k})\binom{n}{k}$.*

Theorem 3.11. *The line graph $\mathcal{L}(H_T(n,k))$, in general, is not an integral graph.*

Proof. The proof is clear from the Table 1 given below. □

Table 1 gives the eigenvalues of the graphs $H_T(n,k)$ for some values of k and n. For convenience, we wrote the multiplicities as right subscripts. From the table given below, we observed that the smallest eigenvalue of the line graph of $H_T(n,k)$ is -2. Thus, it must be verified that it is a generalised line graph. This, however, is not explored in this work.

n	*k*	*Eigenvalues with multiplicity (approximately)*
3	1	$-2_{(3)}, -0.618033_{(2)}, 0.585786_{(1)}, 1.618033_{(2)}, 3.414213_{(1)}$
4	1	$-2_{(14)}, -0.423622_{(3)}, 0_{(2)}, 0.432970_{(1)}, 0.846532_{(3)},$
		$1.779475_{(1)}, 5.577089_{(3)}, 7.787554_{(1)}$
4	2	$-2_{(16)}, -0.236067_{(3)}, 1_{(4)}, 3_{(3)}, 4.236067_{(3)}, 7_{(1)}$
5	1	$-2_{(45)}, -0.285340_{(4)}, 0_{(5)}, 0.318555_{(1)}, 0.801424_{(4)}, 1.617692_{(1)},$
		$1_{(5)}, 1.936767_{(4)}, 2.877242_{(1)}, 13.547148_{(4)}, 16.186508_{(1)}$
5	2	$-2_{(60)}, 0.115211_{(4)}, 0.837722_{(5)}, 1.482873_{(1)}, 1.693158_{(4)}, 2.879076_{(1)},$
		$3.565792_{(4)}, 6.930642_{(1)}, 7.162276_{(5)}, 8.625836_{(4)}, 12.707407_{(1)}$
6	1	$-2_{(124)}, -0.182154_{(5)}, 0_{(9)}, 0.230278_{(1)}, 0.796267_{(5)}, 1.483469_{(1)},$
		$1_{(14)}, 1.886067_{(5)}, 2_{(9)}, 2.740629_{(1)}, 2.971046_{(5)}$
		$3.929799_{(1)}, 29.528772_{(5)}, 32.615823_{(1)}$
6	2	$-2_{(193)}, 0.401846_{(5)}, 1_{(5)}, 0.855351_{(9)}, 1.685451_{(1)}, 2.567249_{(5)},$
		$3.246214_{(1)}, 3.552945_{(5)}, 3.915045_{(9)}, 6.459784_{(1)}, 7.654269_{(5)},$
		$12.204439_{(1)}, 15.229602_{(9)}, 16.823688_{(5)}, 22.404111_{(1)}$
6	3	$-2_{(198)}, 0.627718_{(5)}, 2_{(15)}, 1.575571_{(9)}, 4.246255_{(1)}, 6.372281_{(5)},$
		$8_{(6)}, 11.424428_{(9)}, 11_{(5)}, 14_{(5)}, 14.945366_{(1)}, 21.808377_{(1)}$

Table 1. *Eigenvalues of the graph $\mathcal{L}(H_T(n,k))$*

4 Distance spectrum

The distance matrix $\mathcal{D}(G)=[d_{ij}]$ of a graph G of order n is the matrix indexed by the vertices $\{v_1,v_2,\cdots,v_n\}$ of G where $d_{ij}=d(v_i,v_j)$ is the distance between the vertices v_i and v_j, i.e., *the length of a shortest path between the vertices v_i and v_j. Distance matrices and its spectrum were used in the study of a data communication problem, [8].*

Consider the graph $H_T(n,k))$ for $n \geq 2$. The distance matrix of the graph $H_T(n,k))$ having $2^n - 1$ vertices is a symmetric matrix $\mathcal{D}_p = [d_{ij}]$ of order $p = 2^n - 1$ whose entries d_{ij} are defined by

$$d_{ij} = \begin{cases} d(v_i, v_j) & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

where $d(v_i, v_j)$ is given in Lemma 2.6.

Example 4.1. Consider the graph $H_T(n,k))$. When $n = 3$ and $k = 1$, the distance matrix is

$$\mathcal{D}_7 = \begin{bmatrix} 0 & 2 & 1 & 2 & 1 & 3 & 1 \\ 2 & 0 & 1 & 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & 3 & 2 & 2 & 2 \\ 2 & 2 & 3 & 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 1 & 0 & 2 & 2 \\ 3 & 1 & 2 & 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 1 & 2 & 2 & 0. \end{bmatrix}$$

The characteristic polynomial of this matrix is $x^7 - 72x^5 - 320x^4 - 384x^3$ and the eigenvalues are $4 \pm 2\sqrt{10},\ -4,\ 0$. Since $\mathcal{D}(G)$ is a real symmetric matrix, its eigenvalues, called distance eigenvalues of G, are all real.

Conclusion

In this paper, we established the formula for finding the spectrum of bipartite Kneser type-k graphs $H_T(n,k)$ and studied its algebraic features. We first analysed the structure of this graph for all n, k, where $2k < n$ and $2k > n$. The nullity of the adjacency matrix related to $H_T(n,k)$ is obtained later. More specifically, we obtained a formula for finding the eigenvalues of the graph $H_T(n,k)$ for different values of n and k. It may be interesting to investigate the relationships between the combinatorial matrix of $J(n,k,i)$ and the adjacency matrix of $H_T(n,k)$ in line with the association scheme. Also, we could observe that the least eigenvalue of the line graph of $H_T(n,k)$ is -2. The spectrum is an effective tool for examining a graph's structural characteristics, and using it in chemical graph theory improves the comprehension of reactivity and molecular structures. This application of graph theory to capture fundamental interactions between atoms and bonds in chemical compounds is further enhanced by the use of bipartite graphs.

Conflict of Interest

The authors hereby declare that there is no potential conflict of interest.

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