

Cross Connections of UR – Categories

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Abstract Normal categories arising from unit regular semigroups and strong unit regular semigroups were categorized as UR – categories with appropriate variations. In this article we describe and characterize the cross connection semigroups arising from UR – categories and strong UR – categories.

1 Introduction

Normal categories arise in the study of regular semigroups using cross connection and was introduced by K. S. S. Nambooripad [7]. The normal categories associated with unit regular semigroups and strongly unit regular semigroups were characterized as UR – categories and UR – categories with appropriate variations as seen in [15]. Also properties of semigroups of normal cones and normal duals associated with UR – category and its variations were studied [16].

In this article we consider cross connection semigroups associated with UR – categories and its variations. This paper aims to study the structure of cross connection semigroups associated with UR – categories and strong UR – categories. We see that the cross connection semigroup associated with weak UR – categories is a unit regular semigroup with group of units isomorphic to $G(TC)$. The cross connection semigroup associated with UR – categories is an \mathcal{R} –strongly unit regular semigroup. The cross connection semigroup associated with weak UR – categories with connected retractions is an \mathcal{L} –strongly unit regular semigroup. The cross connection semigroup associated with strongly connected weak UR – categories is a strongly unit regular semigroup.

2 Preliminaries

Some basic definitions and properties of unit regular semigroups, normal categories associated with unit regular semigroups and strongly unit regular semigroups are discussed here. We can see these definitions and properties in [7], [10], [14], [15], [2], [3], [11].

2.1 Normal Categories and Cross Connection

The concept of Normal categories are introduced by Nambooripad [7] as an abstraction of the category of principal left ideals of regular semigroups. For a small category \mathcal{C} , $v\mathcal{C}$ denote the set of all objects and for $a, b \in v\mathcal{C}$, $f : a \rightarrow b$ represents a morphism from a to b . For a morphism f , with normal factorization $f = quj$ where q is a retraction, u is an isomorphism and j an inclusion, $f^\circ = qu$ represents the ephimorphic part of f .

In the following we use 1 to denote the identity morphism on any object. Also for $f : a \rightarrow b$ writing $f = 1$ means that $a = b$ and $f = 1_a$.

Proposition 2.1. [7] Let \mathcal{C} be a normal category and $f : a \rightarrow b$ be a morphism in \mathcal{C} . Let $f = quj$ be a normal factorization of f . Then

- (i) If f is an isomorphism and an inclusion then $f = 1$.
- (ii) If f is an isomorphism and a retraction then $f = 1$.

- (iii) If $f = q_1 u_1 j_1$ is another normal factorization of f then $qu = q_1 u_1$ and $j = j_1$.
- (iv) If $f = quj$ is a monomorphism then $q = 1$.
- (v) If $f = quj$ is an epimorphism then $j = 1$.

In the following TC represents the semigroup of all normal cones γ , whereas a normal cone γ with vertex c_γ is the collection of all morphisms $\{\gamma(a) : a \rightarrow c_\gamma, a \in v\mathcal{C}\}$ such that whenever \leq is a partial order and $b \leq a$ in $v\mathcal{C}$ then $\gamma(b) = j(b, a)\gamma(a)$, $j(b, a)$ is the inclusion morphism and there exist an object $c \in v\mathcal{C}$ such that $\gamma(c)$ is an isomorphism.

For $\gamma \in TC$, $M\gamma = \{a \in v\mathcal{C} : \gamma(a) \text{ is an isomorphism}\}$.

Definition 2.2. For each $\gamma \in TC$, the H - functor $H(\gamma, -)$ is defined on objects and morphisms of \mathcal{C} as follows. For each $c \in v\mathcal{C}$ and for each morphism $g : c \rightarrow c'$ in \mathcal{C} ,

$$H(\gamma, c) = \{\gamma * f^\circ : f \in C(c_\gamma, c)\}$$

$H(\gamma, g) : H(\gamma, c) \rightarrow H(\gamma, c')$ is defined by

$$\gamma * f^\circ \rightarrow \gamma * (fg)^\circ$$

for all $\gamma * f^\circ \in H(\gamma, c)$.

Proposition 2.3. For $\gamma, \gamma' \in TC$, define $H(\gamma, -) \leq H(\gamma', -)$ if $H(\gamma, c) \subseteq H(\gamma', c)$ for all $c \in v\mathcal{C}$. Then \leq is a partial order on $vN^*\mathcal{C}$.

Proposition 2.4. For $\gamma, \gamma' \in TC$, we have the following.

- (i) $H(\gamma; -) \subseteq H(\gamma'; -)$ if and only if there exists an epimorphism h from $c_{\gamma'} \rightarrow c_\gamma$ such that $\gamma = \gamma' * h$. Moreover, $H(\gamma; -) = H(\gamma'; -)$ if and only if there is an isomorphism $h : c_{\gamma'} \rightarrow c_\gamma$ such that $\gamma = \gamma' * h$.
- (ii) $\gamma \mathcal{R} \gamma'$ if and only if $H(\gamma; -) = H(\gamma'; -)$.

Definition 2.5. If \mathcal{C} is a normal category then the normal dual of \mathcal{C} denoted by $N^*\mathcal{C}$ is the category whose objects are H - functors $H(\epsilon, -)$ for $\epsilon \in E(TC)$, the set of all idempotents in TC and whose morphisms are natural transformations between the H - functors.

Definition 2.6. Let \mathcal{C} and \mathcal{D} be normal categories. A cross connection between these categories is a 4-tuple $(\mathcal{C}, \mathcal{D}, \Gamma, \Delta)$ where $\Gamma : \mathcal{D} \rightarrow N^*\mathcal{C}$ and $\Delta : \mathcal{C} \rightarrow N^*\mathcal{D}$ are local isomorphisms satisfying the condition

$$c \in M\Gamma(d) \iff d \in M\Delta(c)$$

for all $c \in v\mathcal{C}$ and $d \in v\mathcal{D}$.

We often denote a cross connection $(\mathcal{C}, \mathcal{D}, \Gamma, \Delta)$ by (Γ, Δ) if the categories involved are clear in the context.

Remark 2.1. By the definition of normal cone we see that $M\gamma \neq \emptyset$ for any normal cone γ . So if (Γ, Δ) is a cross connection between \mathcal{C} and \mathcal{D} then for every $c \in v\mathcal{C}$ there is some $d \in v\mathcal{D}$ such that $c \in M\Gamma(d)$.

The cross connection semigroup associated with a cross connection (Γ, Δ) is described as follows:

The functors Γ and Δ are realized as functors from the product category $\mathcal{C} \times \mathcal{D} \rightarrow Set$ as given above. Now for $c \in v\mathcal{C}$ and $d \in v\mathcal{D}$ we have

$$\Gamma(c, d) = \Gamma(d)(c) = \{\gamma * f^\circ; f \in C(c_\gamma, c)\}$$

$$\Delta(c, d) = \Delta(c)(d) = \{\delta * g^\circ : g \in D(c_\delta, d)\}$$

where γ is a normal cone in \mathcal{C} such that $\Gamma(d) = H(\gamma, -)$ and δ is a normal cone in \mathcal{D} such that $\Delta(c) = H(\delta, -)$.

The cross connection semigroup is defined as:

$$S(\Gamma, \Delta) = \{(\gamma, \delta) : \gamma \in \Gamma(c, d), \delta = \chi(c, d)(\gamma) \in \Delta(c, d)\}$$

for some $(c, d) \in v\mathcal{C} \times v\mathcal{D}$. The semigroup structure of $S(\Gamma, \Delta)$ arises as a subsemigroup of $TC \times T^\circ\mathcal{D}$ where $T^\circ\mathcal{D}$ is the semigroup of all normal cones in \mathcal{D} with product $\delta\delta' = \delta' * \delta$ where $*$ denotes the usual product of normal cones.

Theorem 2.7. Let (Γ, Δ) be a cross connection between \mathcal{C} and \mathcal{D} . Then $S(\Gamma, \Delta)$ is a regular semigroup with binary operation defined by $(\gamma, \delta)(\gamma', \delta') = (\gamma\gamma', \delta'\delta)$ for all $(\gamma, \delta), (\gamma', \delta') \in S(\Gamma, \Delta)$.

Remark 2.2. When $(\gamma, \delta) \in S(\Gamma, \Delta)$ we often say that (γ, δ) is a linked pair of normal cones or that γ and δ are linked. When (γ, δ) is a linked pair we sometimes write $\delta = \gamma^*$.

2.2 Unit Regular Semigroups

We consider semigroups S with an identity, denoted by 1. An element $u \in S$ is said to be a unit in S if there exists an inverse $u^{-1} \in S$ such that $uu^{-1} = 1 = u^{-1}u$. It is easy to see that the set of all units in S is a group and we call it the group of units of S .

Definition 2.8. If for each element x in regular monoid S there exists a unit u in S such that $x = xux$, then S is called a unit regular monoid.

Now we describe stronger versions of unit regularity.

Definition 2.9. Let S be a unit regular monoid. S is said to be \mathcal{R} -strongly unit regular if for any idempotents e, f

$$e \mathcal{R} f \text{ implies } f = eu \text{ for a unit } u.$$

S is said to be \mathcal{L} -strongly unit regular if for any idempotents e, f

$$e \mathcal{L} f \text{ implies } f = ve \text{ for a unit } v.$$

S is said to be strongly unit regular if for any idempotents e, f

$$e \mathcal{D} f \text{ implies } f = u^{-1}eu \text{ for some unit } u.$$

2.3 Normal Categories of Unit Regular Semigroups and Strongly Unit Regular Semigroups

Here we recall the definitions and properties of \mathcal{UR} -categories and various classes of \mathcal{UR} -categories as given in [10], [14] and [15].

Definition 2.10. [14] A normal category \mathcal{C} is said to be a \mathcal{UR} -category if the following hold.

UR1 There is an object m such that $a \leq m$ for all $a \in v\mathcal{C}$. m is called the maximum object in \mathcal{C} or the greatest object in \mathcal{C} .

UR2 Every isomorphism $u : a \rightarrow b$ in \mathcal{C} is the restriction of an isomorphism $\alpha : m \rightarrow m$. That is,

$$j(a, m)\alpha = uj(b, m).$$

In this case we say that α is an extension of u .

Definition 2.11. A normal category \mathcal{C} is said to be a weak \mathcal{UR} -category if the following hold.

UR1 There is a maximum object m such that $a \leq m$ for all $a \in v\mathcal{C}$.

WUR2 For every isomorphism $u : a \rightarrow b$ in \mathcal{C} and every retraction $q : m \rightarrow a$ there is an isomorphism $\alpha : m \rightarrow m$ and a retraction $q' : m \rightarrow b$ such that

$$qu = \alpha q'.$$

We describe some properties of normal cones in a weak \mathcal{UR} -category with maximum object m . These can be seen in [15].

Lemma 2.12. For any weak \mathcal{UR} -category \mathcal{C} with maximum object m , we have

- (i) For any $\gamma \in TC$, $\gamma(m)$ is an epimorphism.
- (ii) For any $\gamma \in TC$, $\gamma(m)$ is an isomorphism if and only if $c_\gamma = m$.

- (iii) The semigroup \mathcal{TC} of all normal cones in \mathcal{C} is a monoid with the identity $\varepsilon : \varepsilon(m) = 1_m$.
- (iv) For any $\gamma \in \mathcal{TC}$, γ is a unit in \mathcal{TC} if and only if $c_\gamma = m$.
- (v) For any $\gamma \in \mathcal{TC}$, γ is an idempotent in \mathcal{TC} if and only if $\gamma(m)$ is a retraction.

Now we describe weak \mathcal{UR} -category with connected retractions and strongly connected weak \mathcal{UR} -category, which are normal categories associated with \mathcal{L} -strongly unit regular semigroup and strongly unit regular semigroup respectively.

Definition 2.13. A weak \mathcal{UR} -category \mathcal{C} with maximum object m is said to have connected retractions if the following hold.

WUR3 If q, q' are retractions from m to c for some $c \in v\mathcal{C}$ then $q' = \alpha q$ for an isomorphism $\alpha : m \rightarrow m$.

Definition 2.14. Two retractions $q : m \rightarrow a$ and $q' : m \rightarrow b$ in a weak \mathcal{UR} -category \mathcal{C} are said to be conjugates of each other if there is an isomorphism $\alpha : m \rightarrow m$ such that $q' = \alpha^{-1}q(j(a, m)\alpha)^\circ$ denoted by q^α .

Definition 2.15. A weak \mathcal{UR} -category \mathcal{C} is said to be strongly connected if any two retractions with domain m and isomorphic codomains are conjugates.

The following Theorem connects the above four variations of \mathcal{UR} -categories to the unit regularity of semigroups of normal cones.

Theorem 2.16. Let \mathcal{C} be a normal category. Then

- (i) \mathcal{TC} is unit regular if and only if \mathcal{C} is a weak \mathcal{UR} -category.
- (ii) \mathcal{TC} is \mathcal{R} -strongly unit regular if and only if \mathcal{C} is a \mathcal{UR} -category.
- (iii) \mathcal{TC} is \mathcal{L} -strongly unit regular if and only if \mathcal{C} is a weak \mathcal{UR} -category with connected retractions.
- (iv) \mathcal{TC} is strongly unit regular if and only if \mathcal{C} is a weak \mathcal{UR} -category which is strongly connected.

3 Cross connection of unit regular semigroups

Here we consider the natural cross connection between the category $\mathcal{L}(S)$ of principal left ideals and the category $\mathcal{R}(S)$ of principal right ideals of a unit regular semigroup S . Recall from [7] that $\Gamma : \mathcal{D} = \mathcal{R}(S) \rightarrow N^*\mathcal{C} = N^*\mathcal{L}(S)$ and $\Delta : \mathcal{C} = \mathcal{L}(S) \rightarrow N^*\mathcal{D} = N^*\mathcal{R}(S)$ are given as follows. $\Gamma(eS) = H(\epsilon, -)$ where $\epsilon = \rho^e$ is an idempotent normal cone with vertex Se such that $\rho^e(Sg) = \rho(g, ge, e)$ and $\Delta(Se) = H(\delta, -)$ where $\delta = \lambda^e$ where $\lambda^e(kS) = \lambda(k, ek, e)$.

In the case when S is a unit regular semigroup we show that the linked pairs of normal cones in the cross connection semigroup can be determined by the component at the maximum object.

Theorem 3.1. Let S be a unit regular semigroup and let γ be a normal cone in $\mathcal{L}(S)$. Then $\gamma = \rho^x$ for some $x \in S$ where ρ^x is the normal cone with vertex $Sx = Sf$ such that $\rho^x(Se) = \rho(e, ex, f)$.

Proof. Let γ be a normal cone with vertex Sf .

Let $\gamma(S) = \rho(1, x, f) : S \rightarrow Sf$.

Then clearly $Sx = Sf$. Now for any $Se \in v\mathcal{L}(S)$,

$$\begin{aligned} \gamma(Se) &= j(Se, S)\gamma(S) \\ &= \rho(e, e, 1)\rho(1, x, f) \\ &= \rho(e, ex, f) \\ &= \rho^x(Se) \end{aligned}$$

It follows that $\gamma = \rho^x$. □

For convenience we denote the maximum object in $\mathcal{L}(S)$ by m and the maximum object in $\mathcal{R}(S)$ by p . Observe that $m = S$ and $p = S$.

4 Cross connection semigroup associated with weak \mathcal{UR} – categories

We consider cross connections of weak \mathcal{UR} – categories. We identify the relation between the maximum objects in the categories involved. Also some special properties of the cross connection functors are described.

Theorem 4.1. Let S be a unit regular semigroup and (Γ, Δ) be the natural cross connection between $\mathcal{C} = \mathcal{L}(S)$ and $\mathcal{D} = \mathcal{R}(S)$. Then

- i. For every $c \in v\mathcal{C}$ and $d \in v\mathcal{D}$

$$\Gamma(c, d) \subseteq \Gamma(m, p).$$

- ii. For every $\gamma \in \Gamma(c, d)$ we have

$$\chi(c, d)(\gamma) = \chi(m, p)(\gamma).$$

- iii. $S(\Gamma, \Delta) = \{(\gamma, \gamma^*) \in \Gamma(m, p) \times \Delta(m, p) : \gamma^* = \chi(m, p)(\gamma)\}$.

Proof. i. Since $d \leq p$ and Γ is a local isomorphism $\Gamma(d) \subseteq \Gamma(p)$ and so $\Gamma(c, d) \subseteq \Gamma(c, p)$. Now $\Gamma(c, p) = H(\epsilon, c)$ since $\Gamma(p) = H(\epsilon, -)$ where ϵ is the identity of $T\mathcal{C}$ by definition of Γ . Let $\gamma \in \Gamma(c, p)$. Then $\gamma = \epsilon * f^o$ where $f : m \rightarrow c$. But $f^o = (fj(c, m))^o$ and so $\gamma \in H(\epsilon, m)$. Therefore $\gamma \in \Gamma(m, p)$.

- ii. Since $\chi : \Gamma \rightarrow \Delta$ is a natural transformation, the following diagram is commutative for any $c \in v\mathcal{C}$ and $d \in v\mathcal{D}$. Here $j = j(c, m)$ in \mathcal{C} and $j_1 = j(d, p)$ in \mathcal{D} .

$$\begin{array}{ccc} \Gamma(c, d) & \xrightarrow{\chi(c, d)} & \Delta(c, d) \\ \downarrow \Gamma(j, j_1) & & \downarrow \Delta(j, j_1) \\ \Gamma(m, p) & \xrightarrow{\chi(m, p)} & \Delta(m, p) \end{array}$$

Since $\Gamma(j, j_1)$ and $\Delta(j, j_1)$ are inclusions (ii.) follows.

- iii. By definition

$$S(\Gamma, \Delta) = \{(\gamma, \gamma^*) : \gamma \in \Gamma(c, d) \text{ and } \gamma^* = \chi(c, d)(\gamma)\}.$$

Now by (i) and (ii.)

$$S(\Gamma, \Delta) = \{(\gamma, \gamma^*) : \gamma \in \Gamma(m, p) \text{ and } \gamma^* = \chi(m, p)(\gamma)\}.$$

□

Lemma 4.2. Let \mathcal{C}, \mathcal{D} be weak \mathcal{UR} – categories and $\Gamma : \mathcal{D} \rightarrow N^*\mathcal{C}$ and $\Delta : \mathcal{C} \rightarrow N^*\mathcal{D}$ be local isomorphisms. Let m be the maximum in $v\mathcal{C}$ and p be the maximum in $v\mathcal{D}$. Then $M\Gamma(p) = \{m\}$.

Proof. First we will see that $m \in M\Gamma(p)$. Let $d \in v\mathcal{D}$ be such that $m \in M\Gamma(d)$. Let ϵ be the unique idempotent normal cone with $c_\epsilon = m$ and $\Gamma(d) = H(\epsilon, -)$. Clearly ϵ is the identity normal cone in $T\mathcal{C}$ and $\epsilon(m) = 1_m$. Now let $c \in v\mathcal{C}$ be such that $p \in M\Delta(c)$. Then $c \in M\Gamma(p)$ and let $\delta \in T\mathcal{C}$ be the unique idempotent normal cone with $c_\delta = c$ and $\Gamma(p) = H(\delta, -)$. Since Γ is a local isomorphism and $d \subseteq p$, we have

$$\begin{aligned} \Gamma(d) &= H(\epsilon, -) \\ &\subseteq H(\delta, -) \\ &= \Gamma(p). \end{aligned}$$

Then by Proposition 2.4 there exists a unique epimorphism $h : c \rightarrow m$ such that $\epsilon = \delta * h$. Then $h = qu$ for some retraction $q : c \rightarrow c'$ and an isomorphism $u : c' \rightarrow m$. Let q_1 be the retraction from $m \rightarrow c$ so that $q_2 = q_1q$ is a retraction from $m \rightarrow c'$. Since \mathcal{C} is a weak \mathcal{UR} – category there exists an isomorphism $\alpha : m \rightarrow m$ and a retraction $q' : m \rightarrow m$ such that $q_2u = \alpha q'$.

Thus $q_2 = \alpha u^{-1}$ is an isomorphism and so $q_2 = 1_m$ and so $u = \alpha$. Thus $c' = c = m$ and so $m \in M\Gamma(p)$.

Now let $c \in v\mathcal{C}$ be such that $c \in M\Gamma(p)$. Then there exists a unique idempotent normal cone θ with vertex c and $\Gamma(p) = H(\theta, -)$. Also since $m \in M\Gamma(p)$, $\Gamma(p) = H(\epsilon, -)$, where ϵ is the idempotent normal cone with vertex m . But the idempotent normal cone with vertex m is the identity in $T\mathcal{C}$. Then $\Gamma(p) = H(\epsilon, -)$ where ϵ is the identity in $T\mathcal{C}$. Hence $H(\epsilon, -) = H(\theta, -)$ and so by Proposition 2.4 there exists a unique isomorphism $h' : m \rightarrow c$ such that $\theta = \epsilon * h'$. By the condition for a weak \mathcal{UR} -category there exists an isomorphism $\alpha : m \rightarrow m$ and a retraction $q : m \rightarrow c$ such that $q(1)h' = \alpha q$. Hence $h' = \alpha q$ and $q = \alpha^{-1}h'$. Thus q is an isomorphism and hence $q = 1$ and so $c = m$. Therefore, $M\Gamma(p) = \{m\}$. \square

Remark 4.1. Let \mathcal{C}, \mathcal{D} be weak \mathcal{UR} -categories and $\Gamma : \mathcal{D} \rightarrow N^*\mathcal{C}$ and $\Delta : \mathcal{C} \rightarrow N^*\mathcal{D}$ be local isomorphisms. Let m be the maximum in $v\mathcal{C}$ and p be the maximum in $v\mathcal{D}$. Then as in the above Lemma we can see that $M\Delta(m) = \{p\}$.

Now we proceed to describe different choices of cross connections between given \mathcal{C} and \mathcal{D} . First we see all possible choices for $\Gamma(p)$.

Lemma 4.3. Let \mathcal{C}, \mathcal{D} be weak \mathcal{UR} -categories and $\Gamma : \mathcal{D} \rightarrow N^*\mathcal{C}$ and $\Delta : \mathcal{C} \rightarrow N^*\mathcal{D}$ be local isomorphisms. Let m be the maximum in $v\mathcal{C}$ and p be the maximum in $v\mathcal{D}$. If $\Gamma(p) = H(\epsilon, -)$, then ϵ is a unit in $T\mathcal{C}$.

Proof. By Lemma 4.2, we have $m \in M\Gamma(p)$. Let θ be the unique idempotent normal cone with vertex m and $\Gamma(p) = H(\theta, -)$. Now we have $H(\epsilon, -) = H(\theta, -)$. Thus by Proposition 2.4 there exists a unique isomorphism $h : m \rightarrow c_\epsilon$ such that $\epsilon = \theta * h$. Then by the condition of a weak \mathcal{UR} -category there exists an isomorphism $\alpha : m \rightarrow m$ and retraction $q : m \rightarrow c_\epsilon$ such that $q(1)h = \alpha q$. Hence $h = \alpha q$ and $q = \alpha^{-1}h$. Thus q is an isomorphism and so by Proposition 2.1 $q = 1$ and $c_\epsilon = m$. Thus by Lemma 2.12 we get, $\epsilon(m)$ is an isomorphism and so by Lemma 2.12 ϵ is a unit in $T\mathcal{C}$. \square

In the following lemmas \mathcal{C} and \mathcal{D} are weak \mathcal{UR} -categories and (Γ, Δ) is a cross connection between them.

From, here onwards ϵ denote the identity cone in $T\mathcal{C}$ so that $\epsilon(m) = 1_m$ and δ denote the identity cone in $T\mathcal{D}$ so that $\delta(p) = 1_p$.

Lemma 4.4. Let \mathcal{C}, \mathcal{D} be weak \mathcal{UR} -categories. (Γ, Δ) be a cross connection between \mathcal{C} and \mathcal{D} . Let m be the maximum in $v\mathcal{C}$ and p be the maximum in $v\mathcal{D}$. Then $\Gamma(p) = H(\epsilon, -)$ and $\Delta(m) = H(\delta, -)$. Further $\Gamma(m, p) = \{\epsilon * f^\circ : f : m \rightarrow m\}$ and $\Delta(m, p) = \{\delta * g^\circ : g : p \rightarrow p\}$.

Proof. By the previous Lemma $\Gamma(p) = H(\gamma, -)$ where γ is a unit in $T\mathcal{C}$. Since ϵ is the identity in $T\mathcal{C}$ we have $\gamma\mathcal{H}\epsilon$. In particular $\gamma\mathcal{R}\epsilon$ and so by Proposition 2.4 $H(\gamma, -) = H(\epsilon, -)$. Thus $\Gamma(p) = H(\epsilon, -)$. Similarly $\Delta(m) = H(\delta, -)$. By the definition of $\Gamma(-, -)$ and $\Delta(-, -)$ we have

$$\begin{aligned} \Gamma(m, p) &= \Gamma(p)(m) \\ &= H(\epsilon, m) \\ &= \{\epsilon * f^\circ; f : m \rightarrow m\} \end{aligned}$$

Similarly

$$\begin{aligned} \Delta(m, p) &= \Delta(m)(p) \\ &= H(\delta, -) \quad \text{by the first part of the Lemma} \end{aligned}$$

So

$$\begin{aligned} \Delta(m, p) &= H(\delta, p) \\ &= \{\delta * g^\circ; g : p \rightarrow p\}. \end{aligned}$$

\square

Lemma 4.5. Let \mathcal{C}, \mathcal{D} be weak \mathcal{UR} - categories and $\Gamma : \mathcal{D} \rightarrow N^*\mathcal{C}$ and $\Delta : \mathcal{C} \rightarrow N^*\mathcal{D}$ be local isomorphisms. Let m be the maximum in $v\mathcal{C}$ and p be the maximum in $v\mathcal{D}$. If α is a unit in $T\mathcal{C}$ then $\alpha \in \Gamma(m, p)$. That is $G(T\mathcal{C}) \subseteq \Gamma(m, p)$ where $G(T\mathcal{C})$ is the group of units of $T\mathcal{C}$.

Proof. We have by Lemma 4.4 $\Gamma(m, p) = \{\epsilon * f^\circ : f : m \rightarrow m\}$, Now given α is a unit in $T\mathcal{C}$ and so by Lemma 2.12 $\alpha(m) : m \rightarrow m$ is an isomorphism. Also,

$$\begin{aligned}\alpha(m) &= 1_m \alpha(m) \\ &= (\epsilon * \alpha(m))(m).\end{aligned}$$

Thus $\alpha = \epsilon * \alpha(m)$. Hence $\alpha \in \Gamma(m, p)$. \square

Lemma 4.6. Let m be the maximum in $v\mathcal{C}$ and p be the maximum in $v\mathcal{D}$ where \mathcal{C} and \mathcal{D} are weak \mathcal{UR} - categories. Let $\Gamma : \mathcal{D} \rightarrow N^*\mathcal{C}$ and $\Delta : \mathcal{C} \rightarrow N^*\mathcal{D}$ be local isomorphisms. Then $\Gamma(m, p) = T\mathcal{C}$.

Proof. Clearly $\Gamma(m, p) \subseteq T\mathcal{C}$. Now let $\gamma \in T\mathcal{C}$. Let $c_\gamma = a \in v\mathcal{C}$. Then $\gamma(m) : m \rightarrow a$ is an epimorphism by Lemma 2.12. So

$$\begin{aligned}\gamma &= \epsilon * \gamma(m) \\ &= \epsilon * (\gamma(m)j(a, m))^\circ.\end{aligned}$$

So by Lemma 4.4 $\gamma \in \Gamma(m, p)$. Thus $\Gamma(m, p) = T\mathcal{C}$. \square

Similarly we have the following Lemma.

Lemma 4.7. Let m be the maximum in $v\mathcal{C}$ and p be the maximum in $v\mathcal{D}$ where \mathcal{C} and \mathcal{D} are weak \mathcal{UR} - categories. Let $\Gamma : \mathcal{D} \rightarrow N^*\mathcal{C}$ and $\Delta : \mathcal{C} \rightarrow N^*\mathcal{D}$ be local isomorphisms. Then $\Delta(m, p) = T\mathcal{D}$.

Now consider the natural isomorphism $\chi : \Gamma \rightarrow \Delta$ associated with the cross connection. Now for $c \in v\mathcal{C}$ and $d \in v\mathcal{D}$, $\chi(c, d) : \Gamma(c, d) \rightarrow \Delta(c, d)$ is the component of the natural isomorphism. In particular $\chi(m, p) : \Gamma(m, p) \rightarrow \Delta(m, p)$ is a bijection. Moreover since χ is a natural isomorphism $\chi(c, d) : \Gamma(c, d) \rightarrow \Delta(c, d)$ is determined by $\chi(m, p)$ and $\chi(c, d) = \chi(m, p)|_{\Gamma(c, d)}$. This will in turn provide that the cross connection semigroup is completely determined by $\Gamma(m, p)$, $\Delta(m, p)$ and $\chi(m, p)$.

We proceed to describe the special features of $\chi(m, p)$. First we consider units in $T\mathcal{C}$. Recall from Lemma 4.5 that all units belong to $\Gamma(m, p)$.

Lemma 4.8. Let α be a unit in $T\mathcal{C}$. Then $\chi(m, p)(\alpha) = \alpha^*$ is a unit in $T\mathcal{D}$.

Proof. Since α is a unit in $T\mathcal{C}$ by Lemma 4.5, we have $\alpha \in \Gamma(m, p)$. Let $\alpha = \epsilon * u$, where $\epsilon(m) = 1_m$ and $u : m \rightarrow m$. Also $\alpha(m) = u$ is an isomorphism by Theorem 2.12. Now we can consider

$$\begin{aligned}\alpha^* &= \chi(m, p)\alpha \\ &= \delta * g^\circ.\end{aligned}$$

Now

$$\begin{aligned}\alpha^*(p) &= (\delta * g^\circ)(p) \\ &= \delta(p)g^\circ \\ &= g^\circ\end{aligned}$$

which is an epimorphism from p to p and hence $\alpha^*(p)$ is an isomorphism and hence by Theorem 2.12 α^* is a unit in $T\mathcal{D}$. \square

Lemma 4.9. Let $G(T\mathcal{C})$ be the group of units in $T\mathcal{C}$ and $G(T\mathcal{D})$ be the group of units in $T\mathcal{D}$. Then $\chi(m, p)$ restricted to $G(T\mathcal{C})$ is an anti isomorphism of $G(T\mathcal{C})$ onto $G(T\mathcal{D})$.

Proof. Since χ is a natural isomorphism $\chi^* : \Delta \rightarrow \Gamma$ is also a natural isomorphism. It follows that if δ is a unit in $T\mathcal{D}$ then $\delta = \gamma^*$ for a unit $\gamma \in TC$. It follows that $\chi(m, p)$ restricted to $G(TC)$ is an anti isomorphism onto $G(T\mathcal{D})$. \square

Now we describe the cross connection semigroup.

Theorem 4.10. Let \mathcal{C}, \mathcal{D} be weak \mathcal{UR} - categories and (Γ, Δ) be a cross connection between them. Then the cross connection semigroup $S(\Gamma, \Delta)$ is a unit regular semigroup with group of units isomorphic to $G(TC)$.

Proof. Since \mathcal{C}, \mathcal{D} are weak \mathcal{UR} - categories, by Theorem 2.16 we have TC and $T\mathcal{D}$ are unit regular semigroups. Now $S(\Gamma, \Delta) = \{(\gamma, \gamma^*) : \gamma^* = \chi(c, d)\gamma\}$. By Theorem 2.7 $S(\Gamma, \Delta)$ is a regular semigroup with the binary operation $(\gamma, \gamma^*)(\delta, \delta^*) = (\gamma\delta, \delta^*\gamma^*)$. Since $\gamma \in TC$ and TC is unit regular, there exist a unit $\alpha \in TC$ such that $\gamma = \gamma\alpha\gamma$. Now since α is a unit, by Lemma 4.8 we have $\alpha \in \Gamma(m, p)$ and $\chi(m, p)\alpha = \alpha^*$ is a unit in $T\mathcal{D}$. Now

$$\begin{aligned} (\gamma, \gamma^*)(\alpha, \alpha^*)(\gamma, \gamma^*) &= (\gamma\alpha\gamma, \gamma^*\alpha^*\gamma^*) \\ &= (\gamma, \gamma^*) \end{aligned}$$

since $\gamma^*\alpha^*\gamma^* = (\gamma\alpha\gamma)^* = \gamma^*$. Hence $S(\Gamma, \Delta)$ is a unit regular semigroup.

Now let (γ, γ^*) be a unit in $S(\Gamma, \Delta)$. Then γ is a unit in TC and γ^* is a unit in $T\mathcal{D}$. It follows that the group of units of $S(\Gamma, \Delta)$ is isomorphic to $G(TC)$. \square

5 Cross connection structure of strongly unit regular semigroups

Here we consider the cross connections associated with strongly unit regular semigroups. We have already shown that the corresponding normal categories are weak \mathcal{UR} - categories with strongly connected retractions.

Since the condition for weak \mathcal{UR} - category are weaker than that for \mathcal{UR} - category the results discussed in the previous section holds when the categories considered are \mathcal{UR} - category, weak \mathcal{UR} - category with connected retractions and strongly connected \mathcal{UR} - categories.

Theorem 5.1. Let \mathcal{C}, \mathcal{D} be \mathcal{UR} - categories and (Γ, Δ) be a cross connection between them. Then the cross connection semigroup $S(\Gamma, \Delta)$ is an \mathcal{R} - strongly unit regular semigroup.

Proof. Since \mathcal{C}, \mathcal{D} are \mathcal{UR} - categories, we have by Theorem 2.16 TC and $T\mathcal{D}$ are \mathcal{R} - strongly unit regular semigroups. Now $\gamma\mathcal{R}\delta$ in TC if and only if $(\gamma, \gamma^*)\mathcal{R}(\delta, \delta^*)$ in $S(\Gamma, \Delta)$. By Theorem 2.7 we know that $S(\Gamma, \Delta)$ is a regular semigroup. Now let $(\gamma, \gamma^*)\mathcal{R}(\delta, \delta^*)$ then $\gamma\mathcal{R}\delta$ and since TC is \mathcal{R} - strongly unit regular there exist a unit $\alpha \in TC$ such that $\gamma = \delta\alpha$. Now

$$\begin{aligned} (\gamma, \gamma^*) &= (\delta\alpha, (\delta\alpha)^*) \\ &= (\delta\alpha, \alpha^*\delta^*) \\ &= (\delta, \delta^*)(\alpha, \alpha^*). \end{aligned}$$

Here (α, α^*) is a unit in $S(\Gamma, \Delta)$ and so the cross connection semigroup associated with \mathcal{UR} - categories are \mathcal{R} - strongly unit regular. \square

Theorem 5.2. Let \mathcal{C}, \mathcal{D} be weak \mathcal{UR} - categories with connected retractions and (Γ, Δ) be a cross connection between them. Then the cross connection semigroup $S(\Gamma, \Delta)$ is an \mathcal{L} - strongly unit regular semigroup.

Proof. Since \mathcal{C}, \mathcal{D} are weak \mathcal{UR} - categories with connected retractions, we have by Theorem 2.16 TC and $T\mathcal{D}$ are \mathcal{L} - strongly unit regular semigroups. Now $\gamma\mathcal{L}\delta$ in TC if and only if $(\gamma, \gamma^*)\mathcal{L}(\delta, \delta^*)$ in $S(\Gamma, \Delta)$. We know by Theorem 2.7 that $S(\Gamma, \Delta)$ is a regular semigroup. Now let $(\gamma, \gamma^*)\mathcal{L}(\delta, \delta^*)$ then $\gamma\mathcal{L}\delta$ and since TC is \mathcal{L} - strongly unit regular there exist a unit $\alpha \in TC$ such that $\gamma = \alpha\delta$. Now

$$\begin{aligned} (\gamma, \gamma^*) &= (\alpha\delta, (\alpha\delta)^*) \\ &= (\alpha\delta, \delta^*\alpha^*) \\ &= (\alpha, \alpha^*)(\delta, \delta^*). \end{aligned}$$

Here (α, α^*) is a unit in $S(\Gamma, \Delta)$ and so the cross connection semigroup associated with weak \mathcal{UR} - categories with connected retractions are \mathcal{L} - strongly unit regular. \square

Theorem 5.3. Let \mathcal{C}, \mathcal{D} be strongly connected weak \mathcal{UR} - categories and (Γ, Δ) be a cross connection between them. Then the cross connection semigroup $S(\Gamma, \Delta)$ is a strongly unit regular semigroup.

Proof. Since \mathcal{C}, \mathcal{D} are strongly connected weak \mathcal{UR} - categories, we have by Theorem 2.16 $T\mathcal{C}$ and $T\mathcal{D}$ are strongly unit regular semigroups. Now $\gamma\mathcal{D}\delta$ in $T\mathcal{C}$ if and only if $(\gamma, \gamma^*)\mathcal{D}(\delta, \delta^*)$ in $S(\Gamma, \Delta)$. We know by Theorem 2.7 that $S(\Gamma, \Delta)$ is a regular semigroup. Now let $(\gamma, \gamma^*)\mathcal{D}(\delta, \delta^*)$ with $(\gamma, \gamma^*), (\delta, \delta^*)$ idempotents in $S(\Gamma, \Delta)$ then $\gamma\mathcal{D}\delta$ and since $T\mathcal{C}$ is strongly unit regular there exist a unit $\alpha \in T\mathcal{C}$ such that $\gamma = \alpha^{-1}\delta\alpha$. Now

$$\begin{aligned}(\gamma, \gamma^*) &= (\alpha^{-1}\delta\alpha, (\alpha^{-1}\delta\alpha)^*) \\ &= (\alpha^{-1}\delta\alpha, \alpha^*\delta^*\alpha^{-1*}) \\ &= (\alpha^{-1}, \alpha^{-1*})(\delta, \delta^*)(\alpha, \alpha^*).\end{aligned}$$

Here $(\alpha, \alpha^*), (\alpha^{-1}, \alpha^{-1*})$ are inverses of each other in $S(\Gamma, \Delta)$ and so the cross connection semigroup associated with strongly connected weak \mathcal{UR} - categories are strongly unit regular. \square

6 Conclusion remarks

This article studies the structure of cross connection semigroups associated with \mathcal{UR} - category and its variations. It is observed that the cross connection semigroup associated with weak \mathcal{UR} - category is a unit regular semigroup. The cross connection semigroup associated with \mathcal{UR} - category is an \mathcal{R} - strongly unit regular semigroup and those associated with weak \mathcal{UR} - category with connected retractions and strongly connected weak \mathcal{UR} - category are respectively \mathcal{L} - strongly unit regular semigroup and strongly unit regular semigroups.

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