Palestine Journal of Mathematics

Vol 14 (Special Issue III), (2025) , 72–80

NORMAL CATEGORIES ASSOCIATED WITH $*$-RINGS

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MSC 2010 Classifications: Primary 20M25; Secondary 20M50, 20M12.

Keywords and phrases: Semi-simple $*$-rings, Normal category, Normal cone.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract *This paper explores the interplay between semi-simple $*$-rings and their categories of left and right ideals. The concept of normal categories facilitates the study of their connection. It is shown that the collection of normal cones in the category $\mathbb{LI}(R)$ of left ideals and in the category $\mathbb{RI}(R)$ of right ideals form a ring. Additionally, the isomorphism between these rings is established. Key results show that R is embedded in the rings of normal cones of its left and right ideals.*

1 Introduction

This paper explores the structure of the category of left and right ideals in rings that are equipped with an involution operation, with a particular focus on semi-simple $*$-rings. The foundational work of Irving Kaplansky [2] on $*$-rings lays the groundwork for our investigation. In these rings, ideals can be paired with their $*$-conjugates, creating a natural framework for examining the relationships between left and right ideals, which can be understood as modules [1].

In constructing categories of left and right ideals over a ring R, we consider these ideals as left and right modules over R with module homomorphisms serving as the morphisms within each category. This framework allows us to show that an ideal A is a left ideal if and only if A^* is a right ideal, and vice versa. Furthermore, we demonstrate that right module homomorphisms induce corresponding left module homomorphisms under involution, thus defining a covariant functor between these categories. We use the concept of normal cones, factorization, and normal categories, following the framework in [3] and [4]. In semi-simple $*$-rings, any ideal can be decomposed as a direct sum of complementary sub-ideals. This decomposition leads to a construction of normal cones for each ideal. We establish that the collection of normal cones has a ring structure, building on the work in [5]. Finally, we demonstrate that the rings $\mathscr{T}\mathbb{LI}(R)$ and $\mathscr{T}\mathbb{RI}(R)$ of normal cones in the categories of left and right ideals are isomorphic. Key results include embedding the ring R into the rings of normal cones in the categories of its left and right ideals.

2 Preliminaries

A $*$-ring is a ring R with an involution $x \to x^*$ such that $(x^*)^* = x$, $(x+y)^* = x^* + y^*$, and $(xy)^* = y^*x^*$. Let R be a $*$-ring. Here, we extend the unary operation $*$ on the elements of R to the ideals of R. For an ideal $I \subseteq R$,

$$I^* = \{a^* \colon a \in I\}.$$

Definition 2.1. A subset A of R is a left(right) ideal of R provided that it is closed under addition and $RA \subseteq A$ ($AR \subseteq A$).

Let $\mathscr{I}$ be the collection of all ideals in R. Define

$$* : \mathscr{I} \to \mathscr{I} \text{ by } A \to A^*, \text{ where } A \in \mathscr{I}.$$

Remark 2.2. The involution on $\mathscr{I}$ has the following properties.

For $A, B \in \mathscr{I}$, $(A^*)^* = A$, $(A+B)^* = A^* + B^*$ and $(AB)^* = B^*A^*$.

It is worth noting that both right and left ideals over a ring R can be viewed as right and left R-modules, respectively.

Proposition 2.3. *Let R be a $*$-ring. Then A is a left(right) ideal of R if and only if A^* is a right(left) ideal of R.*

To establish the foundational relationships within our module structure, we introduce definitions for left and right module homomorphisms.

Definition 2.4. Let R be a ring and $A, B \subseteq R$ be left ideals in R. A map $\lambda : A \to B$ is a left module homomorphism if for all $a, a' \in A$ and $r \in R$,

(i) $(a + a')\lambda = a\lambda + a'\lambda$ and

(ii) $(ra)\lambda = r(a\lambda)$.

Definition 2.5. Let R be a ring and $A, B \subseteq R$ be right ideals in R. A map $\rho : A \to B$ is a right module homomorphism if $\forall a, a' \in A$ and $r \in R$,

(i) $(a + a')\rho = a\rho + a'\rho$ and

(ii) $(ar)\rho = (a\rho)r$.

A left(right) module homomorphism is an isomorphism if it is both one-to-one and onto.

To establish additional structure within our framework, we define an involution $*$ on module homomorphisms, which will play a key role in examining symmetry and duality properties.

Proposition 2.6. *Let A and B be two left ideals in R and $f : A \to B$ be a left module homomorphism. Define $f^* : A^* \to B^*$ by*

$$a^*f^* = (af)^*$$

Then f^* is a right module homomorphism.

Proof. For all $a, b \in A$,

$$\begin{aligned}(a^* + b^*)f^* &= (a+b)^*f^* = ((a+b)f)^* \\ &= (af + bf)^* = (af)^* + (bf)^* \\ &= a^*f^* + b^*f^*\end{aligned}$$

and for all $r \in R$,

$$(a^*r)f^* = (r^*a)^*f^* = ((r^*a)f)^* = (r^*(af))^* = (af)^*r = (a^*f^*)r$$

□

Throughout this paper, unless otherwise specified, we adopt the convention that R is a $*$-ring, $0^* = 0$ *and* $1^* = 1$.

Remark 2.7. In the conventional representation of a right module homomorphism ρ from a right module A to a right module B over a ring R, we often write for $a \in A$ and $r \in R$, $\rho(ar) = \rho(a)r$. However, in this work, we choose to write the expression $(ar)\rho = (a\rho)r$. This alternative approach allows for a more structured treatment when dealing with the involution on ideals.

We now proceed to demonstrate that the composition of the involuted homomorphisms satisfies $(\rho\rho')^* = \rho^*\rho'^*$, showing that involution distributes directly over the composition of these maps without reversing their order.

Theorem 2.8. *Let A, B and C be right ideals in a $*$-ring R and $\rho : A \to B, \rho' : B \to C$ are right module homomorphisms. Then $(\rho\rho')^* = \rho^*\rho'^*$.*

Proof. For $a \in A$,

$$\begin{aligned} a^*(\rho^* \circ \rho'^*) &= (a^*\rho^*)\rho'^* \\ &= (a\rho)^*\rho'^* \quad \text{(by definition of } \rho^*\text{)} \\ &= ((a\rho)\rho')^* \quad \text{(by definition of } \rho'^*\text{)} \\ &= (a(\rho \circ \rho'))^* \\ &= a^*(\rho \circ \rho')^* \end{aligned}$$

Thus $\quad (\rho \circ \rho')^* = \rho^* \circ \rho'^*$. □

Corollary 2.9. *Let A, B and C be left ideals in a $*$-ring R and $\lambda : A \to B, \lambda' : B \to C$ are left module homomorphisms. Then $(\lambda\lambda')^* = \lambda^*\lambda'^*$.*

Next, we give the definitions of category and functor as following [3].

Definition 2.10. A category $\mathscr{C}$ is defined by the following data:

(i) A class $\mathbf{v}\mathscr{C}$, referred to as the class of vertices or objects.

(ii) A class of disjoint sets $\mathscr{C}(a, b)$ is associated with each pair $(a, b) \in \mathbf{v}\mathscr{C} \times \mathbf{v}\mathscr{C}$. Each element $f \in \mathscr{C}(a, b)$ is called a morphism (or arrow) from a to b, denoted by $f : a \to b$.
The element $a =$ dom f is referred to as the domain of f, and $b =$ cod f is referred to as the codomain of f.

(iii) For $a, b, c \in \mathbf{v}\mathscr{C}$, there is a map $\circ : \mathscr{C}(a, b) \times \mathscr{C}(b, c) \to \mathscr{C}(a, c)$, defined by $(f, g) \mapsto f \circ g$. This map $\circ$ is referred to as the composition of morphisms in $\mathscr{C}$.

(iv) For each $a \in \mathbf{v}\mathscr{C}$, a unique $1_a \in \mathscr{C}(a, a)$ called the identity morphism on a.
These must satisfy the following axioms :
(cat 1) The composition is associative; that is, for $f \in \mathscr{C}(a, b), g \in \mathscr{C}(b, c)$ and $h \in \mathscr{C}(c, d)$, we have

$$f \circ (g \circ h) = (f \circ g) \circ h$$

(cat 2) For each $a \in \mathbf{v}\mathscr{C}$, $f \in \mathscr{C}(a, b)$ and $g \in \mathscr{C}(c, a)$,

$$1_a \circ f = f \text{ and } g \circ 1_a = g.$$

Definition 2.11. A covariant functor F between two categories $\mathscr{C}$ and $\mathscr{D}$ consists of two mappings:

A vertex map, $vF : v\mathscr{C} \to v\mathscr{D}$, which assigns to every object $a \in v\mathscr{C}$ an object $vF(a) \in v\mathscr{D}$.

A morphism map, $F : \mathscr{C} \to \mathscr{D}$, which assigns to each morphism $f : a \to b$ in $\mathscr{C}$ a morphism $F(f) : vF(a) \to vF(b)$ in $\mathscr{D}$.

These assignments satisfy the following properties:

(Fn 1) For every object $a \in v\mathscr{C}$, F maps the identity morphism in $\mathscr{C}$ to the identity morphism in $\mathscr{D}$:
$F(1_a) = 1_{vF(a)}$.

(Fn 2) For any pair of composable morphisms f, g in $\mathscr{C}$, the functor F preserves their composition:
$F(f)F(g) = F(fg)$

F is a contravariant functor if vF is as above and the morphism map assigns to each $f : a \to b$ in $\mathscr{C}$, a morphism

$$F(f) : vF(b) \to vF(a) \in \mathscr{D}$$

such that they satisfy axiom (Fn 1) and the following:
(Fn^*2) $F(g)F(f) = F(fg)$ *for all morphisms* $f, g \in \mathscr{C}$ *for which the composite* fg *exists.*
In this paper, unless otherwise stated, a functor will mean a covariant functor.

Definition 2.12. A ring R is semi-simple if every module over R can be decomposed into a direct sum of simple modules.

The following result establishes a decomposition of ideals within these rings.

Proposition 2.13. *Let* R *be a semi-simple ring,* A *and* B *be left(right) ideals of* R *such that* $A \subset B$. *Then there exists a left(right) ideal* A' *of* B *such that* $B = A \oplus A'$.

Proposition 2.14. *Let* R *be a semi-simple ring and* A *be a right ideal of* R *such that* $A = A^0 \oplus A^{0'}$. *Then the projection map* $q : A \to A^0$ *is a right module homomorphism.*

KSS Nambooripad ([3]) developed the theory of normal categories by first introducing the notion of a category with subobjects. This framework naturally establishes a partial order on the set of objects $v\mathscr{C}$, *where inclusion morphisms are defined from* a *to* b *whenever* $a \leq b$.
In a modified version of the the normal category, we begin by directly assigning a partial order on the object set of the category $v\mathscr{C}$ *and define inclusion morphisms from a smaller object* a *to a larger object* b, *corresponding to the relation* $a \leq b$.
Category equipped with this partial order and a specific factorization, is defined as a category with normal factorization. A formal definition is given as follows.

Definition 2.15. ([4]) A small category $\mathscr{C}$ with normal factorization is characterized by the following properties.

(i) The vertex set $v\mathscr{C}$ of the category $\mathscr{C}$ is a partially ordered set. For any $a, b \in v\mathscr{C}$ with $a \leq b$, there exists a monomorphism $j(a, b) : a \to b$ in $\mathscr{C}$, referred to as the inclusion morphism from a to b.

(ii) $j : (v\mathscr{C}, \leq) \to \mathscr{C}$ is a functor from the preorder $(v\mathscr{C}, \leq)$ to the category $\mathscr{C}$, assigning each pair (a, b) with $a \leq b$ in $v\mathscr{C}$ to the morphism $j(a, b)$ in $\mathscr{C}$.

(iii) If $a, b \leq c$ in $v\mathscr{C}$ and $j(a, c) = fj(b, c)$ for some morphism $f : a \to b$, then it follows that $a \leq b$ and $f = j(a, b)$.

(iv) Each morphism $j(a, b) : a \to b$ has a right inverse $q : b \to a$ satisfying $j(a, b)q = 1_a$. This morphism q is referred to as a retraction in $\mathscr{C}$.

(v) Every morphism f in $\mathscr{C}$ has a factorization $f = quj$, where q is a retraction, u is an isomorphism and j is an inclusion. Such a factorization is called normal factorization in $\mathscr{C}$.

The following proposition gives epimorphic component and image.

Proposition 2.16 ([4]). *In a category* $\mathscr{C}$ *with normal factorization, if a morphism* $f \in \mathscr{C}$ *admits two normal factorizations,* $f = quj$ *and* $f = q'u'j'$, *then it follows that* $j = j'$ *and* $qu = q'u'$.

In this case $f^0 = qu$ *is called epimorphic component of* f.
The codomain of f^0 *is acalled the image of* f *and is denoted by* imf.

One notable feature of normal categories is the existence of collections of morphisms, called normal cones. A normal cone is characterized as follows.

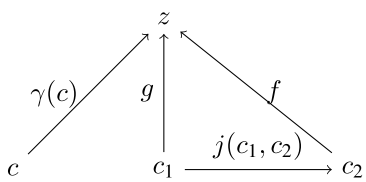
Definition 2.17. ([4])

(1) A cone γ with vertex z is a function from $v\mathscr{C}$ to $m\mathscr{C}$ that satisfies the following conditions.

  (i) $\gamma(c) \in \mathscr{C}(c, z)$ for all $c \in v\mathscr{C}$

  (ii) If $c_1 \subseteq c_2$ then $\gamma(c_1) = j(c_1, c_2)\gamma(c_2)$

(2) If there exist $d \in v\mathscr{C}$ such that $\gamma(d)$ is an isomorphism, then γ is called a normal cone.



Remark 2.18. ([3]) It may be observed that if $\gamma(b)$ is an epimorphism, for some $b \in v\mathscr{C}$, then there exists an $a \in A$ such that $\gamma(a)$ is an isomorphism.

Now we define normal categories.

Definition 2.19. ([4]) A normal category is a category $\mathscr{C}$ with normal factorization, where for each $a \in v\mathscr{C}$, there exists a normal cone γ satisfying $\gamma(a) = 1_a$.

Proposition 2.20. ([3]) *Consider a category $\mathscr{C}$ and let $\mathscr{T}\mathscr{C}$ denote the collection of all normal cones in $\mathscr{C}$. Suppose $\gamma \in \mathscr{T}\mathscr{C}$ and let $f : c_\gamma \to d$ be an epimorphism, where c_γ is a vertex of the normal cone γ. Then the map $\gamma \star f : v\mathscr{C} \to \mathscr{C}$ defined by*

$$(\gamma \star f)(a) = \gamma(a) f$$

is a normal cone with vertex d.

Then for $\gamma_1, \gamma_2 \in \mathscr{T}\mathscr{C}$, define

$$\gamma_1 . \gamma_2 = \gamma_1 \star (\gamma_2(c_{\gamma_1}))^0. \tag{2.1}$$

It follows that this operation defines a binary operation in $\mathscr{T}\mathscr{C}$ such that $\mathscr{T}\mathscr{C}$ is a regular semigroup.

3 Normal Categories of Ideals

Here we introduce the category $\mathbb{LI}(R)$ of left ideals of a $*$-ring and describe various properties of this category. We prove this $\mathbb{LI}(R)$ is a normal category and dually the category $\mathbb{RI}(R)$ of right ideals is also a normal category.

3.1 Category of Left Ideals

Let R be a $*$-ring. Consider the category $\mathbb{LI}(R)$, where v$\mathbb{LI}(R)$ = $\{A \subseteq R : A$ is a left ideal of $R\}$ as vertices and mophisms as left module homomorphisms between these ideals. We call this, **the category of left ideals**.

Remark 3.1. Let R be a semi-simple $*$-ring and $\mathbb{LI}(R)$ be the category of left ideals over R. For $A, B \in v\mathbb{LI}(R)$, consider a left module homomorphism $\lambda : A \to B$ in $\mathbb{LI}(R)$. We have Kerλ = $\{a \in A : a\lambda = 0$ in $B\}$ is an ideal of R contained in A. Then by Proposition 2.13, we can write A as the direct sum of $Ker\lambda$ ($A^{0'}$ say) and its complement A^0 (say). Then $A = A^0 \oplus A^{0'}$. Since $Im\lambda$ = $\{b \in B : a\lambda = b$ for some $a \in A\}$ is a submodule of B, we choose $B^0 = Im\lambda$. Then $A = A^0 \oplus A^{0'}$ such that $A^0 \cap A^{0'} = \emptyset$ and $B = B^0 \oplus B^{0'}$ such that $B^0 \cap B^{0'} = \emptyset$. Here exists an inclusion map $j : B^0 \to B$.

Proposition 3.2. *Let R be a semi-simple $*$-ring and $\mathbb{LI}(R)$ be the category of left ideals over R. For $A, B \in v\mathbb{LI}(R)$, consider a left module homomorphism $\lambda : A \to B$ in $\mathbb{LI}(R)$, in which $A = A^0 \oplus A^{0'}$, where $A^{0'} = Ker\lambda$, and $B = B^0 \oplus B^{0'}$, where $B^0 = Im\lambda$. Then $u : A^0 \to B^0$ defined by*

$$x \mapsto (x)\lambda$$

is an isomorphism from A^0 onto B^0.

Proof. Let $x_{a_0}, y_{a_0} \in A^0$. Then

$$\begin{aligned} x_{a_0}u = y_{a_0}u &\implies x_{a_0}\lambda = y_{a_0}\lambda \\ &\implies (x_{a_0} - y_{a_0})\lambda = 0 \\ &\implies x_{a_0} - y_{a_0} \in Ker\lambda = A^{0'}. \end{aligned}$$

Also $x_{a_0} - y_{a_0} \in A^0$, since A^0 is an ideal.
Thus $x_{a_0} - y_{a_0} \in A^0 \cap A^{0'} = \{0\} \implies x_{a_0} = y_{a_0}$ Hence u is one to one.
Since $B^0 = Im\lambda$, by definition of u, it is onto.
It is clear that u is a module homomorphism. Thus u is an isomorphism from A^0 onto B^0. □

Theorem 3.3. *Let R be a semi-simple $*$-ring and $\mathbb{LI}(R)$ be the category of left ideals of R. For $A, B \in v\mathbb{LI}(R)$, any left module homomorphism $\lambda : A \to B$ in $\mathbb{LI}(R)$, has a factorization as $\lambda = quj$, where q is a retraction, u is an isomorphism and j is an inclusion.*

Proof. Let $A = A^0 \oplus A^{0'}$, where $A^{0'} = Ker\lambda$ and
$B = B^0 \oplus B^{0'}$, where $B^0 = Im\lambda$.

Let $q : A \to A^0$ be the projection given in Proposition 2.14 and $u : A^0 \to B^0$ be the isomorphism given in Proposition 3.2.

We show that $\lambda = quj$, where $j : B^0 \to B$ is the inclusion.

Now for $a \in A$,

$$\begin{aligned} a(quj) &= (aq)uj, \text{where } aq = a_0 \\ &= (a_0u)j \\ &= (a\lambda)j, \text{by Proposition 3.2} \\ &= a\lambda, \text{since } j \text{ is an inclusion map.} \end{aligned}$$

Thus $\lambda = quj$. □

Lemma 3.4. *Let R be a semi-simple $*$-ring. Then $\mathbb{LI}(R)$ is a category with normal factorization with set inclusion as the partial order on $v\mathbb{LI}(R)$ and inclusion map as inclusion.*

Proof. Clearly the mapping $j : (\mathbb{LI}(R), \subseteq) \to \mathbb{LI}(R)$ which assigns to each pair (A, B) with $B \subseteq A$ to the morphism $j(A, B) : B \to A$ given by the inclusion is a functor from the preorder $(\mathbb{LI}(R), \subseteq)$ to $\mathbb{LI}(R)$.

Now we prove condition (iii) of Definition 2.15. Consider ideals $B_1, B_2 \subseteq A$ in $\mathbb{LI}(R)$. Let $f : B_1 \to B_2$ be such that $j(B_1, A) = fj(B_2, A)$. Then it follows that $B_1 \subseteq B_2$ and $f = j(B_1, B_2)$ ([4]).

Then we prove the existence of retractions. Let $j(B_1, B_2) : B_1 \to B_2$, be the inclusion, where $B_1 \subseteq B_2$. Then $B_2 = B_1 \oplus B_1'$, for some $B_1' \in v\mathbb{LI}(R)$. Then the projection $q : B_2 \to B_1$ is a right inverse of the inclusion $j : B_1 \to B_2$.

The existence of normal factorization follows from Theorem 3.4. Hence $\mathbb{LI}(R)$ is a category with normal factorization. □

Theorem 3.5. *Let R be a semi-simple $*$-ring. Then the category $\mathbb{LI}(R)$ is a normal category.*

Proof. It is sufficient to prove that for each $A \in v\mathbb{LI}(R)$ there is a normal cone with vertex A. By the semi-simple property $R = A \oplus A'$, for some left ideal A'. Let $q : R \to A$ be the projection onto kernel A'. Define $\gamma(B) = j(B, R)q$ for each $B \in v\mathbb{LI}(R)$. Clearly γ is a normal cone and $\gamma(A) = j(A, R)q = 1_A$. Hence $\mathbb{LI}(R)$ is a normal category. □

3.2 Category of Right Ideals

Let R be a $*$-ring. The category $\mathbb{RI}(R)$ of right ideals over R is defined by taking $v\mathbb{RI}(R)$ = $\{A' \subseteq R : A'$ is a right ideal of $R\}$ as vertices and mophisms as right module homomorphisms between these ideals.

Since A^* is a right ideal of R for every left ideal A of R, we get a one - one correspondence between $v\mathbb{LI}$ and $v\mathbb{RI}$ through unary operation $*$ by $A \to A^*$. Corresponding to the morphism

$\lambda : A \to B$ in the category $\mathbb{LI}(R)$, we have a morphism $\lambda^* : A^* \to B^*$ in the category $\mathbb{RI}(R)$ defined by $a^*\lambda^* = (a\lambda)^*$. This correspondence induces a functor from $\mathbb{LI}(R)$ to $\mathbb{RI}(R)$ as in the following theorem.

Theorem 3.6. *Let A be a vertex in the category of left ideals $\mathbb{LI}(R)$ and λ be the left module homomorphism in $\mathbb{LI}(R)$. Then there is a covariant functor $F : \mathbb{LI}(R) \to \mathbb{RI}(R)$ defined by $F(A) = A^*$ and $F(\lambda) = \lambda^*$.*

Proof. Clearly F is well defined. Also for left module homomorphisms $\lambda_1, \lambda_2 \in \mathbb{LI}(R)$,

$$\begin{aligned} F(\lambda_1\lambda_2) &= (\lambda_1\lambda_2)^* \\ &= \lambda_1^*\lambda_2^*, \quad \text{(by Theorem 2.8)} \\ &= F(\lambda_1)F(\lambda_2). \end{aligned}$$

Hence F is a functor. □

Remark 3.7. The constructions and results described in the remark 3.1 for the category of left ideals, $\mathbb{LI}(R)$, apply analogously to the category of right ideals, $\mathbb{RI}(R)$, by replacing left module homomorphisms with right module homomorphisms and adjusting kernel and Image accordingly. Further the functor described above, maps inclusions to inclusions and retractions to retractions. Thus we have the following theorem on $\mathbb{RI}(R)$.

Theorem 3.8. *Let R be a semi-simple $*$-ring. Then the category $\mathbb{RI}(R)$ of right ideals of R is a normal category.*

4 Ring of Normal cones in $\mathbb{LI}(R)$ and $\mathbb{RI}(R)$

For a semi-simple $*$-ring R, the categories $\mathbb{LI}(R)$ and $\mathbb{RI}(R)$ are normal categories and so the corresponding sets of normal cones form regular semigroup. Using the addition of module homomorphism, we can give a ring structure on the set of normal cones. Further we show that the involution on R induces an isomorphism of the ring of cones $\mathscr{T}\mathbb{LI}(R)$ and $\mathscr{T}\mathbb{RI}(R)$.

First we observe that in $\mathbb{LI}(R)$, every normal cone is determined by a module homomorphism from R.

Proposition 4.1. *Let $D \in v\mathbb{LI}(R)$ and let $f : R \to D$ be a left module morphism, define $\gamma : v\mathbb{LI}(R) \to \mathbb{LI}(R)$ by $\gamma(B) = f_{\restriction B}$ for all $B \in v\mathbb{LI}(R)$, which is onto. Then γ is a normal cone with vertex D. Further every normal cone with vertex D is given by an onto module map $f : R \to D$.*

Proof. It is easily observe that γ is a normal cone.

Conversely, let γ be a normal cone with vertex D. Choose $f = \gamma(R)$. Then f is an epimorphism. Further $\gamma(B) = j(B, R)f = f_{\restriction B}$ is an epimorphism. □

Now we proceed to show that $\mathscr{T}\mathbb{LI}(R)$ and $\mathscr{T}\mathbb{RI}(R)$ are rings. In the following, we denote the normal cone γ induced by f, given in Proposition 4.1, by $\tilde{f}$. It is easy to see that if $f : R \to D$ and $g : R \to D'$ are onto module maps then

$$\tilde{f}\tilde{g} = (\widetilde{f \circ g})^0, \tag{4.1}$$

where $f \circ g : R \to D'$ is the composition of f with $g_{\restriction D}$.

Next we proceed to introduce the addition of Normal Cones.

Theorem 4.2. *Let $\tilde{f}$, $\tilde{g}$ be normal cones in $\mathbb{LI}(R)$, where $f : R \to A$ and $g : R \to B$ be onto module homomorphisms. Define*

$$\tilde{f} + \tilde{g} = (\widetilde{f + g})^0. \tag{4.2}$$

where $f + g : R \to A + B$ is defined by $(x)(f + g) = (x)f + (x)g$. Then $\tilde{f} + \tilde{g}$ is a normal cone, ($\mathbb{LI}(R)$, +) is an abelian group and ($\mathbb{LI}(R)$, +, .) is a ring.

Proof. For every $\tilde{f}, \tilde{g}, \tilde{h} \in \mathscr{T}\mathbb{LI}(R)$,

$$\begin{aligned}(\tilde{f} + \tilde{g}) + \tilde{h} &= (\widetilde{f + g}) + \tilde{h} \\ &= ((\widetilde{f + g}) + h) = (f + \widetilde{(g + h)}) \\ &= \tilde{f} + (\widetilde{g + h}) = \tilde{f} + (\tilde{g} + \tilde{h})\end{aligned}$$

The zero map O is the additive identity in $\mathscr{T}\mathbb{LI}(R)$. Thus $\tilde{O} = O$.

For each $\tilde{f} \in \mathscr{T}\mathbb{LI}(R)$, $-(\tilde{f}) = (\widetilde{-f})$.

$\tilde{f} + \tilde{g} = (\widetilde{f + g}) = (\widetilde{g + f})) = \tilde{g} + \tilde{f}$.

Hence $\mathscr{T}\mathbb{LI}(R)$ is an additive abelian group.

$$\begin{aligned}(\tilde{f} + \tilde{g})\tilde{h} &= (\widetilde{f + g})\tilde{h} \\ &= ((f + g) \circ h)^0, \quad \text{(by Equation 4.1)} \\ &= ((f \circ h) + (g \circ h))^0 \\ &= (f \circ h)^0 + (g \circ h)^0 \\ &= \tilde{f}\tilde{h} + \tilde{g}\tilde{h}.\end{aligned}$$

Hence ($\mathscr{T}\mathbb{LI}(R)$, +, .) is a ring. □

Corollary 4.3. *$\mathscr{T}\mathbb{RI}(R)$ is a ring with the addition defined in Equation (4.2) and the multiplication defined in Equation (4.1).*

4.1 The unary operation ∗ on normal comes

In this section we explore the role of the involution on normal cones. We show that the involution induces an isomorphism of rings from $\mathscr{T}\mathbb{LI}(R)$ to $\mathscr{T}\mathbb{RI}(R)$.

Theorem 4.4. *Let R be a semi-simple ∗-ring. For each normal cone γ in $\mathbb{LI}(R)$, define $\gamma^* : v\mathbb{RI}(R) \to \mathbb{RI}(R)$ by $\gamma^*(B^*) = (\gamma(B))^*$, where $B^* \in v\mathbb{RI}(R)$. Then the map $\gamma \mapsto \gamma^*$ is an isomorphism from the ring $\mathscr{T}\mathbb{LI}(R)$ to $\mathscr{T}\mathbb{RI}(R)$.*

Proof. Consider the left module homomorphism $f : R \to A$. Then $f^* : R \to A^*$ is an R-module homomorphism.

Now every right ideal of R can be written as B^*, where B is a left ideal of R. Let $\tilde{f^*}$ be the normal cone in $\mathbb{RI}(R)$ determined by f^*. Then for any $B^* \in v\mathbb{RI}(R)$

$$\tilde{f^*}(B^*) = (\tilde{f}(B))^*.$$

So

$$\tilde{f^*} = (\tilde{f})^*.$$

Since $f \mapsto f^*$ is an isomorphism of $\mathbb{LI}(R)$ to $\mathbb{RI}(R)$, it follows that the map $\gamma \mapsto \gamma^*$ is an isomorphism of the map $\mathscr{T}\mathbb{LI}(R)$ to $\mathscr{T}\mathbb{RI}(R)$. □

The following theorems show that a semi-simple ∗-ring R can be embedded into the rings $\mathscr{T}\mathbb{LI}(R)$ and $\mathscr{T}\mathbb{RI}(R)$.

Theorem 4.5. *Let R be a semi-simple ∗-ring. Then there is an isomorphism of R into $\mathscr{T}\mathbb{LI}(R)$.*

Proof. For each $a \in R$, we choose $\rho_a : R \to Ra$, the right translation defined by $x\rho_a = xa$. It is a left module homomorphism and thus belongs to $\mathbb{LI}(R)$. Then

$$\begin{aligned}j \circ \rho_{a+b} &= (j \circ \rho_a) + (j \circ \rho_b) \\ \implies \widetilde{\rho_{a+b}} &= \tilde{\rho}_a + \tilde{\rho}_b\end{aligned}$$

and

$$\begin{aligned}\widetilde{\rho_a \rho_b} &= (\widetilde{\rho_a \circ \rho_b})^0, \quad (equation(4.1)) \\ &= \tilde{\rho_a}\tilde{\rho_b}\end{aligned}$$

implies that

$$\phi : R \to \mathscr{T}\mathbb{LI}(R) \text{ defined by } a\phi = \tilde{\rho_a}$$

is an into ring isomorphism. □

Corollary 4.6. *Let R be a semi-simple $*$-ring. Then there is an isomorphism of R into $\mathscr{T}\mathbb{RI}(R)$.*

Conclusion

This study establishes a categorical and algebraic framework for semi-simple $*$-rings by embedding the ring R into the ring of normal cones $\mathscr{T}\mathbb{LI}(R)$ and $\mathscr{T}\mathbb{RI}(R)$. This framework proposes a structured approach to studying module homomorphisms or morphisms in a category and realising the $*$-ring from the structure of the category.

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