

# INVARIANTS OF BIPARTITE KNESER B TYPE-k GRAPHS

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MSC 2010 Classifications: 05C07, 05C12, 05C69.

Keywords and phrases: Circuit Rank, Cyclomatic Number , bipartite Kneser B type-k graph

**Abstract** We study a class of bipartite graphs, termed *bipartite Kneser B-type-k graphs*, denoted by  $H_B(n, k)$ . Let  $\mathcal{B}_n = \{\pm a_1, \pm a_2, \dots, \pm a_{n-1}, a_n\}$  be a set where  $n > 1$  is a fixed integer, and each  $a_i \in \mathbb{R}^+$  such that  $a_1 < a_2 < \dots < a_n$ . Define  $\phi(\mathcal{B}_n)$  as the collection of all non-empty subsets  $S = \{x_1, x_2, \dots, x_t\}$  of  $\mathcal{B}_n$  satisfying the ordering condition  $|x_1| < |x_2| < \dots < |x_{t-1}| < |x_t|$ , where  $x_t \in \mathbb{R}^+$ . Let  $\mathcal{B}_n^+ = \{a_1, a_2, \dots, a_n\}$  be the strictly positive elements of  $\mathcal{B}_n$ . The vertex set of  $H_B(n, k)$  consists of two partitions:  $V_1$ , the collection of all  $k$ -element subsets of  $\mathcal{B}_n^+$ , where  $1 \leq k < n$ , and  $V_2$ , defined as  $V_2 = \phi(\mathcal{B}_n) - V_1$ . For any  $X \in V_2$ , its transformed set is given by  $X^\dagger = \{|x| : x \in X\}$ . Edges exist between a vertex  $A \in V_1$  and  $B \in V_2$  if and only if  $A \subset B^\dagger$  or  $B^\dagger \subset A$ . We analyze fundamental graph invariants of  $H_B(n, k)$ , including its degree sequence. Our results provide insights into the combinatorial nature of this special class of bipartite graphs.

## 1 Introduction

Named after the German mathematician Martin Kneser, Kneser graphs are an interesting family of combinatorial structures in the field of graph theory. Numerous fields, such as combinatorics, topology, coding theory, and combinatorial optimisation have Kneser graph applications. These are fundamental building blocks of combinatorial theory and can lead to interesting problems and conjectures. In this subject, questions about their chromatic number [13] and other graph-theoretic properties continue to be crucial to research. Kneser graphs are related to topological problems, for example, by helping to understand the homotopy type of some spaces [8]. Kneser graphs are used in coding theory [5] to design codes with efficient error-correcting features.

The Kneser graph  $K(n, k)$  consists of vertices corresponding to all  $k$ -element subsets of  $[n]$ . Two vertices share an edge if the corresponding subsets are disjoint. For integers  $k \geq 1$  and  $n \geq 2k + 1$ , the bipartite Kneser graph  $H(n, k)$  is defined with vertices representing all  $k$ -element and  $(n - k)$ -element subsets of  $[n]$ . A connection exists between two vertices, corresponding to sets  $A$  and  $B$ , whenever  $A \subseteq B$ . The algebraic properties of various types of bipartite Kneser graphs have been explored in previous studies [1, 9].

Consider the set  $\mathcal{S}_n = \{1, 2, 3, \dots, n\}$  for a fixed integer  $n > 1$ . Let  $\phi(\mathcal{S}_n)$  represent the collection of all non-empty subsets of  $\mathcal{S}_n$ . Define  $V_1$  as the set of all 1-element subsets of  $\mathcal{S}_n$ , and let  $V_2 = \phi(\mathcal{S}_n) - V_1$ . Define a bipartite graph with the following adjacency rule: a vertex  $A \in V_1$  is neighbour to a vertex  $B \in V_2$  if and only if  $A \subseteq B$ . This graph is referred to as a bipartite Kneser type-1 graph, denoted by  $H_T(n, 1)$ , as described by Sreekumar K. G. et al. [10]. In their work [11, 7, 12], they introduced the more generalized bipartite Kneser B type- $k$  graph, denoted by  $G = H_B(n, k)$ .

This paper determines the following invariants of the bipartite Kneser B type- $k$  graph  $H_B(n, k)$ : Order, size, independence number, covering number, domination number, vertex connectivity, edge connectivity, girth, circuit rank, distance between two vertices, eccentricity, periphery, centre, median, and degree sequence.

## 2 Preliminaries

Unless otherwise stated, definitions are from [2]. The diameter of a graph is given by  $\text{diam}(G) = \max\{d(a, b) \mid a, b \in V(G)\}$ . The eccentricity of a vertex  $a$  is  $e(a) = \max\{d(a, b) \mid b \in V(G)\}$ .

The radius of  $G$  is  $r(G) = \min\{e(a) \mid a \in V(G)\}$ . The center of  $G$  is  $C(G) = \{a \mid e(a) = r(G)\}$ . The periphery of a graph  $G$  is  $P(G) = \{a \in V(G) \mid e(a) = \text{diam}(G)\}$ . The girth of a graph refers to the smallest number of edges forming a cycle within the graph. For any vertex,  $a \in V(G)$ , the status of  $a$  is  $s(a) = \sum_{b \in V(G)} d(a, b)$ . The median  $M(G)$  of a graph is the set

of vertices with the smallest status. A subset  $S$  of the vertex set  $V$  of a graph  $G$  is called a dominating set of  $G$  if every vertex  $u \in V \setminus S$  has a neighbour  $v \in S$ . The domination number of  $G$  is the cardinality of a minimum dominating set of  $G$  and is denoted by  $\gamma(G)$ . A subset  $S$  of the vertex set  $V$  of a graph  $G$  is called independent if no two vertices of  $S$  are adjacent in  $G$ . The number of vertices in a maximum independent set of  $G$  is called the independence number of  $G$  and is denoted by  $\alpha(G)$ . The circuit rank, which is also called the cyclomatic number of a graph, is  $m - n + c$ , where  $n$ ,  $m$ , and  $c$  are the order, size, and the number of connected components, respectively. Let  $D = \{1^{(a_1)}, 2^{(a_2)}, 3^{(a_3)}, \dots, \Delta^{(a_\Delta)}\}$  be a realizable degree sequence and its realization be the graph  $G$ . The omega invariant of  $G$  [3] is defined as  $\Omega(G) = \sum_{i=1}^{\Delta} (i-2)a_i$ .

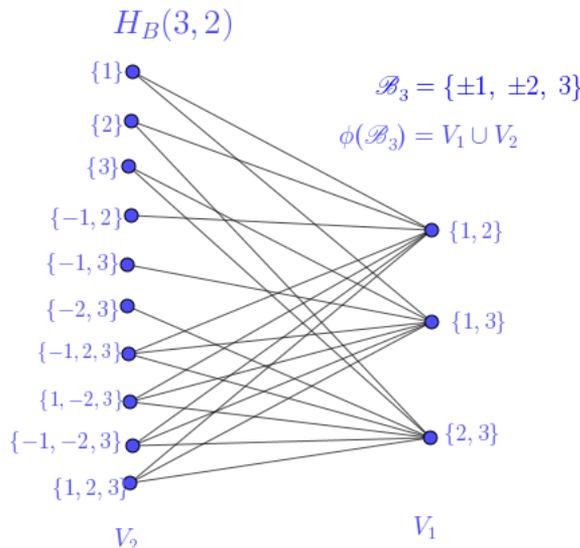
### 3 Basic definitions and examples

#### Definition 3.1. [11]

Consider the set  $\mathcal{B}_n = \{\pm a_1, \pm a_2, \pm a_3, \dots, \pm a_{n-1}, a_n\}$ , where  $n > 1$  is a fixed integer, and each  $a_i$  belongs to  $\mathbb{R}^+$  such that  $a_1 < a_2 < a_3 < \dots < a_n$ . Define  $\phi(\mathcal{B}_n)$  as the collection of all non-empty subsets  $S = \{x_1, x_2, \dots, x_t\}$  of  $\mathcal{B}_n$  that satisfy the ordering condition  $|x_1| < |x_2| < \dots < |x_{t-1}| < |x_t|$ , where  $x_t \in \mathbb{R}^+$ . Let  $\mathcal{B}_n^+ = \{a_1, a_2, a_3, \dots, a_{n-1}, a_n\}$ , which consists of the strictly positive elements of  $\mathcal{B}_n$ . For a given  $k$ , define  $V_1$  as the set of all  $k$ -element subsets of  $\mathcal{B}_n^+$ , where  $1 \leq k < n$ . The complementary set is given by  $V_2 = \phi(\mathcal{B}_n) - V_1$ . For any  $X \in V_2$ , define its transformed set as  $X^\dagger = \{|x| : x \in X\}$ . Now, construct a bipartite graph with vertex sets  $V_1$  and  $V_2$ , where a vertex  $A \in V_1$  is adjacent to a vertex  $B \in V_2$  if and only if  $A \subset B^\dagger$  or  $B^\dagger \subset A$ . A graph of this structure is referred to as a bipartite Kneser B-type- $k$  graph and is denoted by  $H_B(n, k)$ .

**Definition 3.2.** In  $H_B(n, k)$ , an  $s$ -vertex is any member of  $\phi(\mathcal{B}_n)$  consisting of exactly  $s$  elements, where  $1 \leq s \leq n$ . The elements of  $\phi(\mathcal{B}_n)$  are collectively referred to as  $s$ -vertices.

The graph  $H_B(n, k)$  for  $n = 3$  and  $k = 2$  is illustrated here.



### 4 Some parameters of bipartite Kneser B type- $k$ graphs

**Theorem 4.1.** *The order of  $H_B(n, k)$ ,  $|V| = \frac{3^n - 1}{2}$ .*

*Proof.* Every vertex of  $H_B(n, k)$  is a set formed using the elements of  $\mathcal{B}_n$ .

Here  $\mathcal{B}_n = \{\pm a_1, \pm a_2, \pm a_3, \dots, \pm a_{n-1}, a_n\}$ , where  $a_1 < a_2 < \dots < a_n$  and  $-a_n \notin \mathcal{B}_n$ .  $H_B(n, k)$  has vertices as subsets of cardinality 1 to  $n$ . Let  $|V|$  be the total number of sets of cardinality 1 to  $n$ . Let  $N_i$  be the number of subsets of  $\mathcal{B}_n$  of cardinality  $i$ , where  $1 \leq i \leq n$ .  $N_1 = 1 \binom{n}{1} = 2^0 \binom{n}{1}$ ,  $N_2 = 2 \binom{n}{2}$ ,  $N_3 = 2^2 \binom{n}{3}$ ,  $\dots$ ,  $N_n = 2^{n-1} \binom{n}{n}$ .

Thus,  $|V| = \sum_{i=1}^n N_i = 2^0 \binom{n}{1} + 2 \binom{n}{2} + 2^2 \binom{n}{3} + \dots + 2^{n-1} \binom{n}{n} = \frac{3^n - 1}{2}$ , by binomial theorem.  $\square$

**Corollary 4.2.** *For  $G = H_B(n, k)$ ,  $n \geq 2$ ,*

(i) *The vertex independence number,*

$$\alpha(G) = \sum_{i=1}^n 2^{i-1} \binom{n}{i} - \binom{n}{k}.$$

(ii) *The domination number,  $\gamma(H_B(n, k)) = \binom{n}{k}$ .*

*Proof.* By the construction of  $H_B(n, k) = V_1 \cup V_2$ , the partite set  $V_2$  forms a maximum independent set and  $|V_2| = |\phi(\mathcal{B}_n)| - \binom{n}{k}$ . Therefore,  $\alpha(G) = \sum_{i=1}^n 2^{i-1} \binom{n}{i} - \binom{n}{k}$ .

In  $H_B(n, k)$ , no two vertices in  $V_1$  are adjacent, and every vertex in it is adjacent to some other vertex in  $V_2$ . Thus,  $V_1$  forms a dominating set for  $H_B(n, k)$ . The partite set  $V_1$ , being the smallest dominating set, we get  $\gamma(H_B(n, k)) = \binom{n}{k}$ , for all  $n \geq 2$ .  $\square$

**Corollary 4.3.** *For  $G = H_B(n, k)$ , the vertex covering number,  $\beta(G) = \binom{n}{k}$ .*

*Proof.* As  $\alpha(G) + \beta(G) = |G|$ , we get  $\beta(G) = \binom{n}{k}$ .  $\square$

**Theorem 4.4.** *The size of  $H_B(n, k)$  is  $|E| = \binom{n}{k} \left( \frac{3^k - 3}{2} + 2^{k-1}(3^{n-k} - 1) \right)$ .*

*Proof.* Let  $u$  be a  $k$ -vertex in  $V_1$ . The number of 1-vertices adjacent to  $u$  in  $V_2$  is  $2^0 \binom{k}{1}$ , the number of 2-vertices adjacent to  $u$  in  $V_2$  is  $2^1 \binom{k}{2}$ , and the number of 3-vertices adjacent to  $u$  in  $V_2$  is  $2^2 \binom{k}{3}$ . Proceeding like this, the number of  $(k-1)$ -vertices adjacent to  $u$  in  $V_2$  is  $2^{k-2} \binom{k}{k-1}$ . Moreover, the number of  $k$ -vertices adjacent to  $u$  in  $V_2$  is  $2^{k-1} \binom{k}{k} - 1$ .

For  $s = k, k+1, k+2, \dots, n$ , the number of  $s$ -vertices adjacent to  $u$  in  $V_2$  is  $2^{s-1} \binom{n-k}{s-k}$ .

The degree of  $u$  is then the sum of these contributions:

$$\begin{aligned} d(u) &= 2^0 \binom{k}{1} + 2^1 \binom{k}{2} + \dots + 2^{k-1} \binom{k}{k} - 1 \\ &\quad + 2^k \binom{n-k}{1} + 2^{k+1} \binom{n-k}{2} + \dots + 2^{n-1} \binom{n-k}{n-k} \\ &= \left( \frac{3^k - 3}{2} + 2^{k-1}(3^{n-k} - 1) \right). \end{aligned}$$

Since all vertices in  $V_1$  have the same degree, and  $d(u)$  is the maximum degree, we conclude that the total number of edges in  $H_B(n, k)$  is:

$$|E| = \binom{n}{k} \left( \frac{3^k - 3}{2} + 2^{k-1}(3^{n-k} - 1) \right).$$

$\square$

**Theorem 4.5.** *For  $G = H_B(n, k)$ ,  $k > 1$*

(i) *The vertex connectivity,  $\kappa(G) = 1$ .*

(ii) The edge connectivity,  $\lambda(G) = 1$ .

*Proof.* As every  $k$ -vertex,  $v$  in  $V_2$ , is of degree 1, when a  $k$ -vertex  $u$  in  $V_1$  adjacent to  $v$  is removed,  $v$  becomes isolated. Accordingly, we get  $\kappa(G) = 1$ .

Let  $v$  be any  $k$ -vertex in  $V_2$ . Since  $d(v) = 1$ , the graph becomes disconnected when the only edge adjacent to it is removed. Consequently, we get  $\lambda(G) = 1$ . □

**Theorem 4.6.** The circuit rank of  $G = H_B(n, k)$  is  $\binom{n}{k} \left( \frac{3^k-3}{2} + 2^{k-1}(3^{n-k} - 1) \right) - \left( \frac{3^n-1}{2} \right) + 1$

*Proof.* Since  $H_B(n, k)$  is connected and the order and size of the graph have already been determined, the result will follow. □

**Theorem 4.7.** The degree of each vertex in the graph  $G = H_B(n, k)$  is determined, along with the number of vertices having a specific degree. The degree sequence is then obtained by arranging the set of degrees  $\left\{ d_{V_2}(k)^{N_{V_2}(k)}, d_{V_1}(k)^{N_{V_1}(k)}, d_{V_2}(k+1)^{N_{V_2}(k+1)}, d_{V_2}(2)^{N_{V_2}(2)}, d_{V_2}(n)^{N_{V_2}(n)}, d_{V_2}(1)^{N_{V_2}(1)}, d_{V_2}(k-1)^{N_{V_2}(k-1)}, d_{V_2}(3)^{N_{V_2}(3)}, \dots \right\}$  together with their corresponding multiplicities in a monotonic non-increasing order.

*Proof.* Let  $d_{V_2}(r)$  represent the degree of an  $r$ -vertex in  $V_2$  for values of  $r$  in the range  $r = 1, 2, 3, \dots, k-1, k+1, \dots, n$ . Similarly, let  $d_{V_1}(k)$  denote the degree of any  $k$ -vertex belonging to  $V_1$ .

Let the multiplicities of degrees of any  $k$ -vertex in  $V_1$ , and  $r$ -vertices in  $V_2$  be denoted by  $N_{V_1}(k)$  and  $N_{V_2}(r)$  respectively.

The degree and multiplicity of each vertex in  $V_2$  are given by:

$$\begin{aligned} d_{V_2}(1) &= \binom{n-1}{k-1}, & N_{V_2}(1) &= 2^0 \binom{n}{1}, \\ d_{V_2}(2) &= \binom{n-2}{k-2}, & N_{V_2}(2) &= 2^1 \binom{n}{2}, \\ d_{V_2}(k-1) &= \binom{n-(k-1)}{k-(k-1)}, & N_{V_2}(k-1) &= 2^{k-2} \binom{n}{k-1}, \\ d_{V_2}(k+1) &= \binom{k+1}{k}, & N_{V_2}(k+1) &= 2^k \binom{n}{k+1}, \\ &\dots & & \\ d_{V_2}(n) &= \binom{n}{k}, & N_{V_2}(n) &= 2^{n-1} \binom{n}{n}. \end{aligned}$$

Let  $u$  be an arbitrary  $k$ -vertex in  $V_1$ . For  $s = 1, 2, 3, \dots, k-1$ , the count of  $s$ -vertices adjacent to  $u$  is given by  $2^{s-1} \binom{k}{s}$ . The number of adjacent  $k$ -vertices is  $2^{k-1} \binom{k}{k} - 1$ .

For  $s = k+1, k+2, \dots, n$ , the number of adjacent  $s$ -vertices is  $2^{s-1} \binom{n-k}{s-k}$ , where  $t = 1, 2, \dots, n-k$ .

Thus, the degree of any  $k$ -vertex  $u$  in  $V_1$  is given by:  $d_{V_1}(k) = 2^0 \binom{k}{1} + 2^1 \binom{k}{2} + \dots + 2^{k-1} \binom{k}{k} - 1 + 2^k \binom{n-k}{1} + 2^{k+1} \binom{n-k}{2} + \dots + 2^{n-1} \binom{n-k}{n-k}$ .

The total number of  $k$ -vertices in  $V_1$  is given by  $N_{V_1}(k) = \binom{n}{k}$ . Each  $k$ -vertex in  $V_2$  has a degree of  $d_{V_2}(k) = 1$ . Consequently, the number of  $k$ -vertices in  $V_2$  is determined as  $N_{V_2}(k) = (2^{k-1} - 1) \binom{n}{k}$ .

This establishes the degree of every vertex in  $H_B(n, k)$ , as well as the count of vertices possessing a particular degree. The degree sequence is then formed by arranging the set of degrees with their respective multiplicities, ensuring a monotonic non-increasing order. □

**Example 4.8.** The degree sequence for  $H_B(4, 2)$  is obtained by arranging the sequence,  $\left\{ d_{V_2}(1)^{N_{V_2}(1)}, d_{V_1}(2)^{N_{V_1}(2)}, d_{V_2}(2)^{N_{V_2}(2)}, d_{V_2}(3)^{N_{V_2}(3)}, d_{V_2}(4)^{N_{V_2}(4)} \right\} = \{3^4, 19^6, 1^6, 3^{16}, 6^8\}$  of degrees with corresponding multiplicities as a monotonic, non-increasing sequence. That is, the degree sequence is  $\{19^6, 6^8, 3^{16}, 3^4, 1^6\}$ .

The following result gives the Omega invariant of  $G = H_B(n, k)$ :

**Theorem 4.9.** The Omega invariant of  $G = H_B(n, k)$  is

$$\Omega(H_B(n, k)) = \binom{n}{k} \sum_{i=1}^n \binom{k}{i} 2^{i-1} - \sum_{i=1}^n \binom{n}{i} 2^i.$$

In [3, 4], it was shown that the number of closed regions of a graph is given by

$$r(G) = \frac{\Omega(G)}{2} + c(G). \tag{4.1}$$

Here  $c(G)$  is the number of components of  $G$ . Hence the number of faces of the graph  $H_B(n, k)$  is given by the following result:

**Theorem 4.10.** *The number of faces of the graph  $H_B(n, k)$  is*

$$r(H_B(n, k)) = \binom{n}{k} \sum_{i=1}^n \binom{k}{i} 2^{i-2} - \sum_{i=1}^n \binom{n}{i} 2^{i-1} + c$$

*Proof.* It follows by the formula of  $r$  given in Eqn.(4.1). □

**Theorem 4.11.** *Consider the graph  $G = H_B(n, k)$ ,  $n > 2$ ,  $1 < k < n$  with  $V(G) = V_1 \cup V_2$  and  $u, v \in V(G)$ , then*

$$d(u, v) = \begin{cases} 1 & \text{if } u \in V_1 \text{ and } v \in V_2 \text{ are adjacent,} \\ 2 & \text{if } u \text{ and } v \text{ are in } V_1, \\ 3 & \text{if } u \in V_1 \text{ and } v \in V_2 \text{ are not adjacent,} \\ 2, 4 & \text{if } u \text{ and } v \text{ are in } V_2. \end{cases}$$

*Proof.* Let  $u \in V_1$  and  $v \in V_2$ . If  $u$  and  $v$  are adjacent, then  $d(u, v) = 1$ . Suppose that  $v$  is not adjacent to  $u$ . Since the degree of  $v$  is at least 1, it must be adjacent to some vertex  $w \in V_1$ . Let  $x$  be an  $n$ -vertex in  $V_2$ . As  $x$  is a common neighbour of  $u$  and  $w$ ,  $u - x - w - v$  is the shortest path from  $u$  to  $v$ , and hence  $d(u, v) = 3$ . Let  $u, v \in V_1$ . As any  $n$ -vertex  $x$  is a common neighbour of  $u$  and  $v$ ,  $u - x - v$  is the shortest path from  $u$  to  $v$ , and hence  $d(u, v) = 2$ . Let  $u, v \in V_2$ . Then,  $d(u, v) = 2$  in one of the following three cases.

**Case 1** There exists  $x \in V_1$  such that  $x$  is a superset of both  $u^\dagger$  and  $v^\dagger$ .

**Case 2** There exists  $y \in V_1$  such that  $y$  is a subset of both  $u^\dagger$  and  $v^\dagger$ .

**Case 3** There exists  $w \in V_1$  such that  $w$  is a subset of  $u^\dagger$  and a super set of  $v^\dagger$  or  $w$  is a subset of  $v^\dagger$  and a superset of  $u^\dagger$ .

In other words,  $d(u, v) = 2$  if either  $|u^\dagger \cup v^\dagger| \leq k$  or  $|u^\dagger \cap v^\dagger| \geq k$  or  $|u^\dagger \cap v^\dagger| = |u^\dagger| \leq k$ . If none of these three conditions are satisfied, then  $d(u, v) \neq 2$ . Choose  $x, y \in V_1$  such that  $d(x, u) = 1$  and  $d(y, v) = 1$ . Let  $w$  be any  $n$ -vertex in  $V_2$ . Then  $w$  is a common neighbour of  $x$  and  $y$ . Thus, we get the shortest path  $u - x - w - y - v$  of length 4 and hence  $d(u, v) = 4$ . □

**Corollary 4.12.** *If  $n > 2$  and  $1 < k < n$ , for the graph  $H_B(n, k)$ , the eccentricity is given by*

$$e(v) = \begin{cases} 3 & \text{if } v \in V_1, \\ 2 & \text{if } v \in V_2 \text{ is an } n\text{-vertex,} \\ 4 & \text{if } v \in V_2 \text{ is an } r\text{-vertex, } 1 \leq r < n. \end{cases}$$

*Proof.* Let  $v \in V_1$ . For any  $u \in V_1$ ,  $d(v, u) = 2$ . If  $w \in V_2$  such that it is adjacent to  $v$ , then  $d(v, w) = 2$ . If  $w$  is not adjacent to  $v$ , then  $d(v, w) = 3$ . Thus,  $e(v) = 3$ . Let  $v \in V_2$  be an  $n$ -vertex. Since  $v$  is adjacent to all vertices in  $V_1$  and  $d(v, u) = 2$  for any  $u \in V_2$ , we get  $e(v) = 2$ . Let  $v \in V_2$  be an  $r$ -vertex,  $1 \leq r < n$ . The distance from  $v$  to any vertex in  $V_1$  is either 1 or 3, and the distance from  $v$  to any vertex in  $V_2$  is either 2 or 4. Thus,  $e(v) = 4$ . □

**Corollary 4.13.** *The diameter of  $G = H_B(n, k)$ ,  $n > 2$ ,  $1 < k < n$  is  $\text{diam}(G) = 4$  and radius is  $\text{rad}(G) = 2$ .*

**Corollary 4.14.** *For  $G = H_B(n, k)$ ,  $n > 2$ ,  $1 < k < n$ ,*

(i) Periphery,  $P(G) = \{v \in V_2 | v \text{ is an } r\text{-vertex}, 1 \leq r < n\}$ .

(ii) Center,  $C(G) = \{v \in V_2 | v \text{ is an } n\text{-vertex}\}$ .

(iii) Median,  $M(G) = C(G)$ .

*Proof.* As the eccentricity of any  $r$ -vertex  $v$ , where  $1 \leq r < n$  is  $e(v) = 4 = \text{diam}(G)$ , we get, Periphery,  $P(G) = \{v \in V_2 | v \text{ is an } r\text{-vertex}, 1 \leq r < n\}$ . As the eccentricity of any  $n$ -vertex  $v$  is  $e(v) = 2 = \text{rad}(G)$ , we get,  $C(G) = \{v \in V_2 | v \text{ is an } n\text{-vertex}\}$ .

For finding the median of  $G$ , we find the status of vertices in  $V$ . We have the status of any vertex  $v \in G$ ,  $s(v) = \sum_{u \in V(G)} d(v, u)$ . First, we find the status of any  $n$ -vertex  $v$  in  $V_2$ . The

sum of the distances from  $v$  to  $\binom{n}{k}$  vertices in  $V_1$  is  $\binom{n}{k}$ . The sum of the distance from  $v$  to  $\frac{3^n-1}{2} - \binom{n}{k} - 2^{n-1}$ ,  $r$ -vertices, where  $1 \leq r < n$  in  $V_2$  is  $2 \left( \frac{3^n-1}{2} - \binom{n}{k} - 2^{n-1} \right)$ . Therefore, the status of  $v$  is  $2 \left( \frac{3^n-1}{2} - \binom{n}{k} - 2^{n-1} \right) + \binom{n}{k}$ . There are vertices at distances 1, 2, and 3 from vertices in  $V_1$ . There are vertices at distances 1, 2, 3, and 4 from  $r$ -vertices in  $V_2$ . This leads to the conclusion that the status of any  $n$ -vertex is minimum compared to other vertices in  $V$ . Thus, Median,  $M(G) = C(G) = \{v \in V_2 | v \text{ is an } n\text{-vertex}\}$ .  $\square$

**Corollary 4.15.** *The Girth of  $G = H_B(n, k)$  when  $k > 1$  is  $\text{Girth}(G) = 4$ .*

*Proof.* Let  $u$  be a  $k$ -vertex in  $V_1$ . Suppose that  $u$  is adjacent to a 1-vertex  $u'$  in  $V_2$ , and  $u'$  is adjacent to another  $k$ -vertex  $v'$  in  $V_1$ . Additionally,  $v'$  is adjacent to an  $n$ -vertex  $v$  in  $V_2$ , and  $v$  is adjacent to  $u$ . This forms a cycle  $u - u' - v' - v - u$  of length 4. Since bipartite graphs only contain cycles of even length, we conclude that  $\text{Girth}(G) = 4$ .  $\square$

**Remark 4.16.** Let  $G = H_B(n, k)$  with bipartition,  $V(G) = V_1 \cup V_2$ . We denote the set of all unordered pairs of vertices of  $V(G)$  by  $S = \{\{u, v\} | u, v \in V(G)\}$ . Then,  $S$  contains vertex pairs at distances 1, 2, 3, and 4. The subsets of  $S$  are of the form  $S(V(G), h) = \{\{u, v\} | d(u, v) = h, 1 \leq h \leq 4\}$ . Then,  $S = \bigcup_{h=1}^4 S(V(G), h)$ . Let  $d^{(h)}(u, v)$  be the cardinality of  $S(V(G), h)$  for  $h = 1, 2, 3$ , and 4. We denote by  $S(V_1, 2)$  and  $S(V_2, 2)$ , respectively, the sets of vertex pairs of  $V_1$  and  $V_2$  at distance 2. We have,  $S(V(G), 2) = S(V_1, 2) \cup S(V_2, 2)$ . Let  $d_{V_1}^{(2)}(u, v)$  and  $d_{V_2}^{(2)}(u, v)$ , respectively, denote the cardinalities of  $S(V_1, 2)$  and  $S(V_2, 2)$ . Consequently, we get,  $d^{(2)}(u, v) = d_{V_1}^{(2)}(u, v) + d_{V_2}^{(2)}(u, v)$ . As vertex pairs at distance 4 exists only in  $V_2$ , we denote the set containing them as  $S(V_2, 4)$ . We have,  $S(V(G), 4) = S(V_2, 4)$ . Let  $d_{V_2}^{(4)}(u, v)$  be the cardinality of  $S(V_2, 4)$ . Then,  $d^{(4)}(u, v) = d_{V_2}^{(4)}(u, v)$ .

The cardinalities of  $S(V(G), 1)$  and  $S(V(G), 3)$ , the total number of vertex pairs at distance 2 from  $V_2$  denoted by  $d_{V_2}^{(2)}(u, v)$ , and the total number of vertex pairs  $\{u, v\}$  at distances of 2 and 4, where  $u$  and  $v$  are from  $V_2$ , are determined in the next theorem

**Theorem 4.17.** *Consider the graph  $G = H_B(n, k)$ ,  $n > 2$ ,  $1 < k < n$ . For  $u, v \in V$ ,*

$$(i) d^{(3)}(u, v) = \binom{n}{k} \left( \frac{3^n-1}{2} - \binom{n}{k} - \left( \frac{3^k-3}{2} \right) - 2^{k-1}(3^{n-k} - 1) \right).$$

$$(ii) d_{V_1}^{(2)}(u, v) = \binom{n}{2}.$$

$$(iii) d_{V_2}^{(2)}(u, v) = \left( \frac{3^n-1}{2} - \binom{n}{k} \right) - d_{V_2}^{(4)}(u, v)$$

*Proof.* When  $u$  and  $v$  are in  $V_1$ , we have  $d(u, v) = 2$ . As there are  $\binom{n}{k}$   $k$ -vertices in  $V_1$ ,  $d_{V_1}^{(2)}(u, v) = \binom{n}{2}$ . When  $u \in V_1$  and  $v \in V_2$  are not adjacent, we have  $d(u, v) = 3$ .

The total number of unordered pairs of vertices such that one vertex is from  $V_1$  and the other from  $V_2$  is  $\binom{n}{k}(|V(G)| - \binom{n}{k})$ .

Thus,  $d^{(3)}(u, v) = \binom{n}{k}(|V(G)| - \binom{n}{k}) - |E(G)| = \binom{n}{k} \left( \frac{3^n-1}{2} - \binom{n}{k} - \left( \frac{3^k-3}{2} \right) - 2^{k-1}(3^{n-k} - 1) \right)$ .

Given that  $d_{V_2}^{(2)}(u, v)$  and  $d_{V_2}^{(4)}(u, v)$  are the counts of pairs at distances 2 and 4 respectively, it follows from theorem 4.11 that

$$\begin{aligned} d_{V_2}^{(2)}(u, v) + d_{V_2}^{(4)}(u, v) &= \binom{|V(G)|}{2} - \left( |E(G)| + \binom{n}{k} \left( |V(G)| - \binom{n}{k} \right) - |E(G)| \right) \\ &= \binom{\frac{3^n-1}{2}}{2} - \binom{n}{k} \end{aligned}$$

□

$d_{V_2}^{(4)}(u, v)$  is computed in the following theorem. As any  $i$ -vertex  $u$  and  $j$ -vertex  $v$  in  $V_2$  can have both positive and negative components, whenever we say common elements in an  $i$  vertex and a  $j$ -vertex, we mean  $|u^\dagger \cap v^\dagger|$ .

**Theorem 4.18.** Let  $P_{i,j}$  be the number of unordered pairs of  $i$ -vertices and  $j$ -vertices of  $V_2$  that are at distance 4.

$$\text{For } u, v \in V_2 \text{ in } H_B(n, k), d_{V_2}^{(4)}(u, v) = \sum_{\substack{i, j \in \{1, 2, \dots, n-1\} \\ i \leq j \\ k+1 \leq i+j \leq n+k-1}} P_{i,j}.$$

$$\text{Here, } P_{i,j} = \begin{cases} \sum_t \binom{n}{i} 2^{i-1} 2^{j-1} \binom{n-i}{j-t} \binom{i}{t} & \text{for } i \neq k \text{ and } j \neq k \\ \sum_t \binom{n}{k} (2^{k-1} - 1) 2^{j-1} \binom{n-k}{j-t} \binom{k}{t} & \text{for } i = k \text{ and } j \neq k \\ \sum_t \binom{n}{i} 2^{i-1} (2^{k-1} - 1) \binom{n-i}{k-t} \binom{i}{t} & \text{for } i \neq k \text{ and } j = k \\ \frac{1}{2} \left( \sum_t \binom{n}{k} (2^{k-1} - 1)^2 \binom{n-k}{k-t} \binom{k}{t} \right) & \text{for } i = j = k \\ \frac{1}{2} \left( \sum_t \binom{n}{i} 2^{2(i-1)} \binom{n-i}{i-t} \binom{i}{t} \right) & \text{for } i = j \text{ and } j \neq k \end{cases}$$

Here,  $|u^\dagger \cap v^\dagger| = t$  and  $t$  is a non-negative integer such that  $i + j - n \leq t < i + j - k$  and  $t < \min\{i, k\}$

*Proof.* Choose an  $i$ -vertex  $u = \{x_1, x_2, \dots, x_i\}$  and a  $j$ -vertex  $v = \{y_1, y_2, \dots, y_j\}$  in  $V_2$  such that  $d(u, v) = 4$ . Here  $i, j \in \{1, 2, \dots, n-1\}$ ,  $i \leq j$ , and  $k+1 \leq i+j \leq n+k-1$ . By the construction of  $H_B(n, k)$ , the vertices at distance 4 satisfy the conditions:  $|u^\dagger \cap v^\dagger| = t$ ,  $t \geq 0$ ,  $i + j - n \leq t < i + j - k$ , and  $t < \min\{i, k\}$ . Corresponding to an  $i$ -element subset of  $\mathcal{B}_n^+ = \{a_1, a_2, a_3, \dots, a_{n-1}, a_n\}$ ,  $2^{i-1}$ ,  $i$ -element subsets or  $i$ -vertices are seen in  $\phi(\mathcal{B}_n)$ . Similarly, corresponding to a  $j$ -element subset of  $\mathcal{B}_n^+$ ,  $2^{j-1}$ ,  $j$ -vertices are there in  $\phi(\mathcal{B}_n)$ .  $P_{i,j}$  is calculated in various cases.

**Case 1** For  $i, j$  such that  $i < j$  and  $i \neq k$  and  $j \neq k$ .

There are  $2^{j-1}$ ,  $j$ -vertices at distance 4 to  $u$ . As there are  $2^{i-1}$  vertices corresponding to  $u$ , the total number of 4 pairs between  $u$  and  $v$  and their corresponding vertices is  $2^{i-1} 2^{j-1}$ . As  $|u^\dagger \cap v^\dagger| = t$ , the remaining  $j-t$  elements in any other  $j$ -vertex at distance 4 can be selected from  $n-i$  elements of  $\mathcal{B}_n^+$  in  $\binom{n-i}{j-t}$  ways. Also,  $t$  elements can be selected from  $i$ -vertex in  $\binom{i}{t}$  ways. The total number of  $i$  element subsets of  $\mathcal{B}_n^+$  is  $\binom{n}{i}$ . Using the restrictions on  $t, i$  and  $j$ , we get  $P_{i,j} = \sum_t \binom{n}{i} 2^{i-1} 2^{j-1} \binom{n-i}{j-t} \binom{i}{t}$ .

**Case 2** For  $i, j$  such that  $i < j$ ,  $i = k$  and  $j \neq k$ .

Of the  $2^{k-1}$ ,  $k$ -vertices corresponding to  $\{x_1, x_2, \dots, x_k\}$  in  $\phi(\mathcal{B}_n)$ ,  $(2^{k-1} - 1)$  are in  $V_2$  and 1 in  $V_1$ . Therefore,  $P_{i,j} = \sum_t \binom{n}{i} (2^{k-1} - 1) 2^{j-1} \binom{n-k}{j-t} \binom{k}{t}$ .

**Case 3** For  $i, j$  such that  $i < j$ ,  $i \neq k$  and  $j = k$ .

Of the  $2^{k-1}$ ,  $k$ -vertices corresponding to  $\{y_1, y_2, \dots, y_k\}$  in  $\phi(\mathcal{B}_n)$ ,  $(2^{k-1} - 1)$  are in  $V_2$  and 1 in  $V_1$ . Therefore,  $P_{i,j} = \sum_t \binom{n}{i} 2^{i-1} (2^{k-1} - 1) \binom{n-i}{k-t} \binom{i}{t}$ .

**case 4** For  $i, j$  such that  $i = j = k$ .

Using similar arguments, we conclude that  $P_{i,j} = \frac{1}{2} \left( \sum_t \binom{n}{k} (2^{k-1} - 1)^2 \binom{n-k}{k-t} \binom{k}{t} \right)$

**Case 5** For  $i, j$  such that  $i = j$  and  $j \neq k$ .

Here,  $P_{i,j} = \frac{1}{2} \left( \sum_t \binom{n}{i} 2^{2(i-1)} \binom{n-i}{i-t} \binom{i}{t} \right)$ .

For  $i, j \in \{1, 2, \dots, n-1\}$ ,  $i \leq j$ , and  $k+1 \leq i+j \leq n+k-1$ , the total number of unordered pairs of vertices from  $V_2$  such that  $d(u, v) = 4$  is

$$\begin{aligned} d_{V_2}^{(4)}(u, v) &= \sum P_{i,j} \\ &= P_{1,k} + P_{1,k+1} + \dots + P_{1,n-1} + \\ &\quad P_{2,k-1} + P_{2,k} + \dots + P_{2,n-1} + \\ &\quad \dots + \\ &\quad P_{k,k} + \dots + P_{k,n-1} + P_{k+1,k+1} + \dots + P_{k+1,n-2} + \\ &\quad \dots + \\ &\quad P_{\frac{n+k-1}{2}, \frac{n+k-1}{2}}. \quad (\text{When } n+k-1 \text{ is even}) \end{aligned}$$

$$\begin{aligned} d_{V_2}^{(4)}(u, v) &= \sum P_{i,j} \\ &= P_{1,k} + P_{1,k+1} + \dots + P_{1,n-1} + \\ &\quad P_{2,k-1} + P_{2,k} + \dots + P_{2,n-1} + \\ &\quad \dots + \\ &\quad P_{k,k} + \dots + P_{k,n-1} + P_{k+1,k+1} + \dots + P_{k+1,n-2} + \\ &\quad \dots + \\ &\quad P_{\frac{n+k-2}{2}, \frac{n+k-2}{2}} + P_{\frac{n+k-2}{2}, \frac{n+k}{2}}. \quad (\text{When } n+k-1 \text{ is odd}) \end{aligned}$$

□

**Remark 4.19.** From theorem 4.17, we have got the cardinalities of  $S(V(G), 1)$  and  $S(V(G), 3)$ . Using theorems 4.17 and 4.18, the cardinalities of  $S(V(G), 2)$  and  $S(V(G), 4)$  are obtained as  $d^{(2)}(u, v) = d_{V_1}^{(2)}(u, v) + d_{V_2}^{(2)}(u, v)$  and  $d^{(4)}(u, v) = d_{V_2}^{(4)}(u, v)$ .

In the next example, we compute  $d^{(4)}(u, v)$  for the graph  $H_B(4, 2)$  using theorem 4.18.

**Example 4.20.** Consider the graph  $H_B(4, 2)$  with  $\mathcal{B}_4 = \{\pm 1, \pm 2, \pm 3, 4\}$ , and bipartition:

$$V_1 = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\},$$

$$\begin{aligned} V_2 = &\{\{1\}, \{2\}, \{3\}, \{4\}, \{-1, 2\}, \{-1, 3\}, \{-1, 4\} \\ &\{-2, 3\}, \{-2, 4\}, \{-3, 4\}, \{1, 2, 3\}, \{-1, 2, 3\}, \{1, -2, 3\}, \{-1, -2, 3\}, \\ &\{1, 2, 4\}, \{-1, 2, 4\}, \{1, -2, 4\}, \{-1, -2, 4\}, \{2, 3, 4\}, \{-2, 3, 4\}, \\ &\{2, -3, 4\}, \{-2, -3, 4\}, \{1, 3, 4\}, \{-1, 3, 4\}, \{1, -3, 4\}, \{-1, -3, 4\} \\ &\{1, 2, 3, 4\}, \{-1, 2, 3, 4\}, \{1, -2, 3, 4\}, \{1, 2, -3, 4\}, \{-1, -2, 3, 4\} \\ &\{1, -2, -3, 4\}, \{-1, 2, -3, 4\}, \{-1, -2, -3, 4\}\}. \end{aligned}$$

Then,  $d^{(4)}(u, v) = d_{V_2}^{(4)}(u, v) = \sum_{\substack{i,j \in \{1,2,3\} \\ i \leq j \\ 3 \leq i+j \leq 5}} P_{i,j} = P_{1,2} + P_{1,3} + P_{2,2} + P_{2,3}$ .

For  $t = 0$ ,  $P_{1,2} = \binom{4}{1}2^{1-1}(2^{2-1} - 1)\binom{4-1}{2-0}\binom{1}{0} = 12$ .  
 For  $t = 0$ ,  $P_{1,3} = \binom{4}{1}2^{1-1}2^{3-1}\binom{4-1}{3-0}\binom{1}{0} = 16$ .  
 For  $t = 0, 1$ ,  $P_{2,2} = \frac{1}{2} \left( \binom{4}{2}(2^{2-1} - 1)^2\binom{4-2}{2-0}\binom{2}{0} + \binom{4}{2}(2^{2-1} - 1)^2\binom{4-2}{2-1}\binom{2}{1} \right) = 15$ .  
 For  $t = 1$ ,  $P_{2,3} = \binom{4}{2}(2^{2-1} - 1)2^{3-1}\binom{4-2}{3-1}\binom{2}{1} = 48$ .  
 Therefore,  $d^{(4)}(u, v) = d_{V_2}^{(4)}(u, v) = 12 + 16 + 15 + 48 = 91$ .

**Table 1.** A table showing the values of  $d^{(h)}(u, v)$ ,  $1 \leq h \leq 4$ , and  $H_B(n, k)$  for some values of  $n$  and  $k$ .

|             | $d^{(1)}(u, v)$ | $d^{(2)}(u, v)$ | $d^{(3)}(u, v)$ | $d^{(4)}(u, v)$ |
|-------------|-----------------|-----------------|-----------------|-----------------|
| $H_B(4, 2)$ | 114             | 485             | 90              | 91              |
| $H_B(4, 3)$ | 80              | 486             | 64              | 150             |
| $H_B(5, 2)$ | 550             | 5275            | 560             | 875             |
| $H_B(5, 3)$ | 440             | 4125            | 670             | 2025            |
| $H_B(5, 4)$ | 275             | 4715            | 305             | 1965            |
| $H_B(6, 2)$ | 2445            | 54050           | 2790            | 6781            |

### 5 Conclusion

In this study, we have explored several invariants of the bipartite Kneser B-type graph  $H_B(n, k)$ . It extends bipartite Kneser graphs, which already appear in coding theory and cryptographic constructions. Since  $H_B(n, k)$  captures subset relations, it can model hierarchical influence networks (e.g., social media influence propagation). It can also be useful in ecological network analysis, where species interactions depend on hierarchical food chains. Since both the degree sequence and the pairwise vertex distances in  $H_B(n, k)$  have been explicitly determined, numerous molecular descriptors can be derived. Furthermore, distance-based properties facilitate the computation of various metric dimensions. Additionally, centrality measures, including degree centrality, closeness centrality, betweenness centrality, and eigenvector centrality, can be systematically analyzed for further insights into the graph’s structural significance.

### Conflict of Interest

The authors affirm that there are no conflicts of interest associated with this research work.

### Acknowledgements

The authors express their gratitude to the anonymous reviewers for their valuable critiques and suggestions. The first author is presently engaged in doctoral studies in mathematics at University College, Thiruvananthapuram. We also extend our appreciation to the University of Kerala for their support in conducting this research.

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