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SHARP INEQUALITIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS OF SĂLĂGEĂN - TYPE

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Abstract *This paper investigates specific subclasses of analytic functions of Sălăgeăn - type. The goal is to obtain sharp inequalities and generalized bounds for the second Hankel determinant* $|\alpha_2\alpha_4 - \alpha_3^2|$ *within the newly defined subclasses* $\mathcal{R}_\gamma$, $\mathcal{S}_\gamma^*$, *and* $\mathcal{C}_\gamma$. *These subclasses consist of functions whose derivatives have positive real parts, starlike and convex, respectively. By leveraging the properties of the Toeplitz determinant, we achieve significant results that broaden the scope of classical function classes. Additionally, our findings show that the parameter* γ *introduces a flexible framework that can adapt to various special cases, resulting in sharp bounds and deeper insights into the geometric behavior of analytic functions. The implications of this research enhance the overall theory of analytic functions and hint at potential applications across a range of mathematical and applied fields. Notably, the results are particularly sharp when* $\gamma = 0$, *aligning well with existing literature and laying a strong groundwork for future studies in this area.*

1 Introduction

The theory of complex analysis, also known as the theory of functions of a complex variable, was established in the middle of the 19^{th} *century. In the* 20^{th} *century, the first publications relating to the analytic functions of complex variables were made by distinguished mathematicians such as Augustin-Louis Cauchy, Bernhard Riemann\, Leonhard Euler, and even Karl Weierstrass (see [16]). Geometric functions theory, GFT, on the other hand, is a prominent branch of complex analysis, which explains the geometric properties of complex analytic functions. It incorporates studies of several functions, such as the univalent and multivalent analytic functions. The theory of univalent functions, which deals with functions in a certain complex domain that are analytic (or holomorphic) and univalent, is one of the fundamental concepts in GFT. It got considerable attention these days (see for example [17]-[23]). For more information on the subject (see [24, 25]).*

Let $\mathcal{A}$ *be a class of normalized analytic functions* $f : \mathcal{U} \longrightarrow \mathbb{C}$ *of the form:*

$$f(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + \alpha_4 z^4 \ldots = z + \sum_{n=2}^{\infty} \alpha_n z^n, \tag{1.1}$$

satisfying $f(0) = 0$ *and* $f'(0) = 1$ *in the open unit disk* $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. *Throughout this paper, we denote by* $\mathcal{S}$ *the subclass of the set* $\mathcal{A}$ *univalent in* $\mathcal{U}$.

Definition 1.1. [11] A function f of the form (1.1) is said to be univalent (or schlicht) in a domain $\mathcal{U} \subset \mathbb{C}$ if it never takes the same value twice. That is, if $f(z_1) \neq f(z_2)$ for all points z_1 and

z_2 in $\mathcal{U}$ with $z_1 \neq z_2$. It is characterized by

$$S = \{f \in \mathcal{A} : f \text{ is univalent in } \mathcal{U}\}.$$

Usually, investigations in the theory of univalent functions are done on the class S.

Definition 1.2. [11] If the linear segment connecting the origin z_0 to every other point $z \in \mathcal{U}$ is starlike with respect to a point $w_0 \in \mathcal{U}$, then the set $\mathcal{U} \subset \mathbb{C}$ is said to be starlike. This means, every point of $\mathcal{U}$ must be $visible$ from z_0 in order to meet the condition. It is denoted by S^*. Thus, $S^* \subset S$. Hence, the function f is starlike for all $z \in \mathcal{U}$ if it satisfies the inequality $\Re e\left(\frac{zf'(z)}{f(z)}\right) > 0$. It is characterized as follows

$$S^* = \left\{f \in \mathcal{A} : \Re e\left\{\frac{zf'(z)}{f(z)}\right\} > 0,\ z \in \mathcal{U}\right\}. \tag{1.2}$$

Definition 1.3. [11] A set $\mathcal{U} \subset \mathbb{C}$ is said to be convex if each of its points is starlike, or if the linear segment connecting any two points of $\mathcal{U}$ wholly lies in $\mathcal{U}$. That is, a function that conformally maps the unit disk onto a convex domain is said to be convex. The subclass of S consisting of the convex functions is usually denoted by $\mathcal{C}$. Thus, we say that $\mathcal{C} \subset S^* \subset S$. Each univalent convex function f is characterized by the following condition

$$C = \left\{f \in \mathcal{A} : \Re e\left\{1 + \frac{zf'(z)}{f(z)}\right\} > 0,\ z \in \mathcal{U}\right\}. \tag{1.3}$$

Another important class closely related to the classes S^* and $\mathcal{C}$ is the class $\mathcal{P}$ of all functions p that are analytic and have positive real parts in $\mathcal{U}$, with $p(0) = 1$. It is also called the class of Carathéodory *or bounded turning functions and is denoted by $\mathcal{R}$.*

Definition 1.4. [15] The class $\mathcal{P}$ of bounded turning consists of analytic functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} \eta_n z^n = 1 + \eta_1 z + \eta_2 z^2 + \eta_3 z^3 + \eta_4 z^4 + \dots, \tag{1.4}$$

with $p(0) = 1$ and $\Re e\{p(z)\} > 0$ for any $z \in \mathcal{U}$. It is usually characterized by the following criteria

$$\mathcal{R} := \{f \in \mathcal{A} : \Re e\{f'(z)\} > 0,\ z \in \mathcal{U}\} \tag{1.5}$$

Several authors investigated the class $\mathcal{R}$, S^* and $\mathcal{C}$ for functions of bounded turning, starlike and convex (see [3], [4], [5], [9], [23]).

Definition 1.5. [7] Let $f \in \mathcal{A}$ and $q, n \in \mathbb{N}$. Then the $q-th$ Hankel determinant of the function f is defined and denoted as

$$H_q(n) = \begin{vmatrix} \alpha_n & \alpha_{n+1} & \cdots & \alpha_{n+q-1} \\ \alpha_{n+1} & \alpha_{n+2} & \cdots & \alpha_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n+q-1} & \alpha_{n+q} & \cdots & \alpha_{n+2q-2} \end{vmatrix}, \quad \alpha_1 = 1. \tag{1.6}$$

The determinant in (1.6) is due to Pommerenke [7] and is vital for our investigation. Nooman and Thomas [2] also presented the same in 1976.

The primary focus in this paper, is the $Hankel\ determinant$ $H_q(n)$ for $q = 2$ and $n = 2$, denoted by

$$H_2(2) = \begin{vmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{vmatrix} = \alpha_2\alpha_4 - \alpha_3^2. \tag{1.7}$$

Taking absolute value of the determinant in Equation (1.7), we get

$$|H_2(2)| \leq |\alpha_2\alpha_4 - \alpha_3^2|. \tag{1.8}$$

In recent years, several investigations have been conducted in search of bounds for the functional in (1.8). Janteng et al [1] obtained a sharp bound $\frac{4}{9}$ for $|\alpha_2\alpha_4 - \alpha_3^2|$, for the class of bounded turning functions (also known as functions whose derivatives have positive real parts) denoted by $\mathcal{R}$. The same authors obtained sharp bounds 1 and $\frac{1}{8}$ in [5] for the functional $|\alpha_2\alpha_4 - \alpha_3^2|$, for classes of functions $\mathcal{S}^*$ and $\mathcal{C}$ of starlike and convex functions, respectively. Recently, several authors investigated the functional in (1.8) for various classes of functions (see [3], [4], [23]).

In 1983, Sălăgeăn [10] introduced an operator D^γ, defined by

$$\mathrm{D}^\gamma f(z) = z\mathrm{D}\left(\mathrm{D}^{\gamma-1}f(z)\right) = z + \sum_{n=2}^{\infty} n^\gamma \alpha_n z^n \quad (z \in \mathcal{U}; \gamma \geq 0; f \in \mathcal{A}). \tag{1.9}$$

From (1.9), it is easy to deduce that

(i) if $\gamma = 0$, then $\mathrm{D}^0 f(z) = f(z)$,

(ii) if $\gamma = 1$, then $z\mathrm{D}\left(\mathrm{D}^0 f(z)\right) = zf'(z)$,

(iii) if $\gamma = 2$, then $z\mathrm{D}\left(\mathrm{D}^1 f(z)\right) = zf''(z)$,

(iv) if $\gamma = 3$, then $z\mathrm{D}(\mathrm{D}^2 f(z)) = zf'''(z)$,

(v) in general, $z\mathrm{D}(\mathrm{D}^{\gamma-1} f(z)) = z\mathrm{D}^\gamma f(z)$ as in (1.9).

Furthermore, it can be deduced from (1.9) that

$$\mathrm{D}^\gamma f'(z) = 1 + 2^{\gamma+1}\alpha_2 z + 3^{\gamma+1}\alpha_3 z^2 \cdots = 1 + \sum_{n=2}^{\infty} n^{\gamma+1}\alpha_n z^{n-1} \ (z \in \mathcal{U}; \gamma \geq 0; f \in \mathcal{A}). \tag{1.10}$$

$$\mathrm{D}^\gamma f''(z) = 2^{\gamma+1}\alpha_2 + 2 \times 3^{\gamma+1}\alpha_3 z \cdots = \sum_{n=2}^{\infty}(n-1)n^{\gamma+1}\alpha_n z^{n-2} \ (z \in \mathcal{U}; \gamma \geq 0; f \in \mathcal{A}). \tag{1.11}$$

Motivated by earlier investigations mentioned above and the Sălăgeăn operator, in this work, we seek to obtain sharp and generalized bounds for the functional $|\alpha_2\alpha_4 - \alpha_3^2|$ for functions f of Sălăgeăn - type belonging to new subclasses defined as follows:

Definition 1.6. Let $f \in \mathcal{A}$, then $f \in \mathcal{S}^*_\gamma$, where

$$S^*_\gamma := \left\{ f : f \in \mathcal{S}^* \ and \ \Re e \left\{ \frac{zD^\gamma f'(z)}{D^\gamma f(z)} \right\} > 0; \ z \in \mathcal{U} \right\}, \tag{1.12}$$

which denotes the class of starlike functions of *Sălăgeăn*- type.

Definition 1.7. Let $f \in \mathcal{A}$, then $f \in \mathcal{C}_\gamma$, where

$$C_\gamma := \left\{ f : f \in \mathcal{C} \ and \ \Re e \left\{ 1 + \frac{zD^\gamma f''(z)}{D^\gamma f'(z)} \right\} > 0; \ z \in \mathcal{U} \right\}, \tag{1.13}$$

which denotes the class of convex functions of *Sălăgeăn*- type.

Definition 1.8. Let $f \in \mathcal{A}$, then $f \in \mathcal{R}_\gamma$, where

$$\mathcal{R}_\gamma := \{ f : f \in \mathcal{R} \ and \ \Re e\{D^\gamma f'(z)\} > 0; \ z \in \mathcal{U} \}, \tag{1.14}$$

which denotes the class of bounded turning functions of *Sălăgeăn* - type.

From (1.12), (1.13) and (1.14), observe that if $\gamma = 0$*, we obtain* $\mathcal{R}_0 = \mathcal{R}$, $S_0^* = S^*$ *and* $\mathcal{C}_0 = \mathcal{C}$*, which satisfy the inequalities (1.2), (1.3) and (1.5), respectively. Additionally, the new subclass* $\mathcal{R}_\gamma$, $\mathcal{S}_\gamma^*$*, and* $\mathcal{C}_\gamma$*, which are derived from the Sălăgeăn differential operator, hold significant importance. They build on the classical classes* $\mathcal{R}$, $\mathcal{S}^*$*, and* $\mathcal{C}$*, providing us with a structured approach to explore and understand the properties of analytic functions of Sălăgeăn - type. This allows us to delve deeper into the specific behaviors of these functions, enriches the overall theory of analytic functions, and has potential applications across various fields. By introducing the parameter* γ*, we establish a more flexible framework that can adapt to different special cases and applications. This generalization opens the door to new results and insights in the study of analytic functions.*

In the next section, we'll introduce some key lemmas that are crucial for our investigation. These lemmas will lay the groundwork for our analysis and discussions, forming the basis of our inquiry. Section 3 will focus on presenting our main findings and results. Not only will it showcase our key outcomes, but it will also include a thorough discussion, along with theorems and proofs, each accompanied by a remark and a straightforward corollary that illustrates the theorem's implications, as well as a clear example of a sharp function. Finally, a conclusion that summarizes our work, highlighting its significance and contribution to the existing literature and ongoing research in the theory of univalent functions.

2 Preliminary Results

Let $\mathcal{P}$ *be the set of functions* $p \in \mathcal{A}$ *defined by (1.4), which are analytic in* $\mathcal{U}$ *and satisfy the criteria*

$$\begin{cases} p(z) = 1 & z = 0, \\ \Re e\{p(z)\} > 0 & z \neq 0. \end{cases} \tag{2.1}$$

Lemma 2.1. *[11] Let* $p \in \mathcal{P}$ *be defined by the function (1.4), then the sharp estimate* $|\eta_n| \leq 2$*, where* $n \in \mathbb{N}$ *is satisfied.*

Lemma 2.2. *[12, 13] Let* $p \in \mathcal{P}$*, then the power series (1.4) converges in the unit disk* $\mathcal{U}$ *to a function belonging to* $\mathcal{P}$ *provided the Toêplitz determinants*

$$J_n = \begin{vmatrix} 2 & \eta_1 & \eta_2 & \cdots & \eta_n \\ \eta_{-1} & 2 & \eta_1 & \cdots & \eta_{n-1} \\ \eta_{-2} & \eta_{-1} & 2 & \cdots & \eta_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_{-n} & \eta_{-n+1} & \eta_{-n+2} \cdots & & 2 \end{vmatrix}, \quad n = 1, 2, 3, \ldots, \tag{2.2}$$

where $\eta_{-n} = \bar{\eta}_n$*, are all positive. They are strictly non-negative with the exception of the function* $p(z) = \sum_{n=1}^{m} \rho_n p_0\left(e^{it_n} z\right)$, $\rho_n > 0, t_n$ *real and* $t_n \neq t_j$*, for* $n \neq j$*; in particular* $J_n > 0$ *when* $n < (m-1)$ *as well as* $J_n = 0$ *when* $n \geq m$*.*

The result in Lemma 2.2 is due to Carathéodory *and* Toêplitz *(see [14]). For* J_n *where* $n = 2$ *and* $n = 3$ *in (2.2), we have*

$$J_2 = \begin{vmatrix} 2 & \eta_1 & \eta_2 \\ \bar{\eta}_1 & 2 & \eta_1 \\ \bar{\eta}_2 & \bar{\eta}_1 & 2 \end{vmatrix} = \left[8 + 2\Re e\{\eta_1^2 \eta_2\} - 2|\eta_2|^2 - 4|\eta_1|^2\right] \geq 0, \tag{2.3}$$

and

$$J_3 = \begin{vmatrix} 2 & \eta_1 & \eta_2 & \eta_3 \\ \bar{\eta}_1 & 2 & \eta_1 & \eta_2 \\ \bar{\eta}_2 & \bar{\eta}_1 & 2 & \eta_1 \\ \bar{\eta}_3 & \bar{\eta}_2 & \bar{\eta}_1 & 2 \end{vmatrix} \geq 0. \tag{2.4}$$

Equation (2.3) and (2.4) are equivalent to

$$\begin{cases} 2\eta_2 = \eta_1^2 + x\left(4-\eta_1^2\right), \\ 4\eta_3 = \left\{\eta_1^3 + 2\eta_1\left(4-\eta_1^2\right)x - \eta_1\left(4-\eta_1^2\right)x^2 + 2\left(4-\eta_1^2\right)\left(1-|x|^2\right)z\right\}, \end{cases} \tag{2.5}$$

for some $x, z \in \mathbb{C}$, *with* $|x| = |z| \leq 1$.

3 Main Results

Throughout this section, the method of classical analysis introduced by Libera and Zlotkiewicz ([12], [13]) is utilized. Unless where acknowledged otherwise, $\gamma \in [0,1)$.

Theorem 3.1. *Let* $f \in \mathcal{R}_\gamma$. *Then*

$$\left|\alpha_2\alpha_4 - \alpha_3^2\right| \leq \frac{4}{3^{2\gamma+2}}.$$

Proof. Let $f \in \mathcal{R}_\gamma$. The characteristics of the function f, which is of Sălăgeăn-type is illustrated through the *Carathédory* function $p(z)$ as follows

$$D^\gamma f'(z) = p(z) \qquad (z \in \mathcal{U}). \tag{3.1}$$

Equation (3.1) indicates that the derivative of the function f transformed by the Sălăgeăn operator D^γ results in another function $p(z)$. This transformation can be visualized as a mapping of the behavior of f in the complex plane, where $p(z)$ captures the essence of how f behaves under the influence of the operator.

Now, using the series representations for $D^\gamma f'(z)$ and $p(z)$ from (1.10) and (1.4), and equating coefficients in (3.1), we obtain

$$\begin{cases} \alpha_2 = \frac{\eta_1}{2^{\gamma+1}}, \alpha_3 = \frac{\eta_2}{3^{\gamma+1}}, \alpha_4 = \frac{\eta_3}{4^{\gamma+1}}, \ldots & (n = 2, 3, \ldots) \\ \alpha_n = \frac{\eta_{n-1}}{n^{\gamma+1}} & (\text{by induction}). \end{cases} \tag{3.2}$$

From (3.2), it can be deduced that

$$\left|\alpha_2\alpha_4 - \alpha_3^2\right| = \left|\frac{\eta_1\eta_3}{8^{\gamma+1}} - \frac{\eta_2^2}{3^{2\gamma+2}}\right|. \tag{3.3}$$

For convenience, equation (3.3) can be written as

$$\left|\alpha_2\alpha_4 - \alpha_3^2\right| = |\Phi_1\eta_1\eta_3 + \Phi_2\eta_2^2|. \tag{3.4}$$

where

$$\Phi_1; \Phi_2 = \begin{cases} \frac{1}{8^{\gamma+1}}; \\ -\frac{1}{3^{2\gamma+2}}. \end{cases} \tag{3.5}$$

Using the values of η_2 and η_3 from (2.5) of Lemma 2.2 and substituting on the right - hand side of the equality in (3.4), along with the fact that $|z| \leq 1$, we obtain

$$\begin{aligned} 4|\Phi_1\eta_1\eta_3 + \Phi_2\eta_2^2| = \Big| & \left(\Phi_1 + \Phi_2\right)\eta_1^4 + 2\Phi_1\eta_1(4-\eta_1^2) + 2\Phi_1\eta_1^2(4-\eta_1^2)|x| \\ & - \Phi_1\eta_1(\eta_1+2)(4-\eta_1^2)|x|^2 + \Phi_2(4-\eta_1^2)(4-\eta_1^2)|x|^2\Big|, \end{aligned} \tag{3.6}$$

where

$$2\Phi_1 = \frac{2}{8^{\gamma+1}}; \qquad \Phi_1 + \Phi_2 = \frac{3^{2\gamma+2} - 8^{\gamma+1}}{8^{\gamma+1} \cdot 3^{2\gamma+2}}. \tag{3.7}$$

Utilizing the values of (3.5) and (3.7) on the right–hand side of the inequality (3.6), we obtain

$$
\begin{aligned}
4|\Phi_1\eta_1\eta_3+\Phi_2\eta_2^2| =& \Bigg| \left(\frac{3^{2\gamma+2}-8^{\gamma+1}}{8^{\gamma+1}\cdot 3^{2\gamma+2}}\right)\eta_1^4 + \frac{2\eta_1(4-\eta_1^2)}{8^{\gamma+1}} + \frac{2\eta_1^2(4-\eta_1^2)|x|}{8^{\gamma+1}} \\
&- \frac{\eta_1(\eta_1+2)(4-\eta_1^2)|x|^2}{8^{\gamma+1}} - \frac{(4-\eta_1^2)(4-\eta_1^2)|x|^2}{3^{2\gamma+2}}\Bigg|. \qquad (3.8)
\end{aligned}
$$

Using Lemma 2.1, where $\eta_1 \leq 2$ and letting $\eta_1 = \eta$, $\eta_1+\alpha = \eta_1-\alpha$, it is safe to assume with no boundaries that $\eta \in [0,2]$. Further, applying triangle inequality on (3.8), with $|x| = \rho$, we obtain

$$
\begin{aligned}
4|\Phi_1\eta\eta_3+\Phi_2\eta_2^2| \leq& \frac{\eta^4}{8^{\gamma+1}} + \frac{\eta^4}{3^{2\gamma+2}} + \frac{2\eta(4-\eta^2)}{8^{\gamma+1}} + \frac{2\eta^2(4-\eta_1^2)\rho}{8^{\gamma+1}} \\
&+ \frac{\eta(\eta-2)(4-\eta^2)\rho^2}{8^{\gamma+1}} + \frac{(4-\eta^2)(4-\eta^2)\rho^2}{3^{2\gamma+2}} \\
=& F_\gamma(\rho), \qquad (3.9)
\end{aligned}
$$

where

$$
\begin{aligned}
F_\gamma(\rho) =& \frac{\eta^4}{8^{\gamma+1}} + \frac{\eta^4}{3^{2\gamma+2}} + \frac{2\eta(4-\eta^2)}{8^{\gamma+1}} + \frac{2\eta^2(4-\eta_1^2)\rho}{8^{\gamma+1}} \\
&+ \frac{\eta(\eta-2)(4-\eta^2)\rho^2}{8^{\gamma+1}} + \frac{(4-\eta^2)(4-\eta^2)\rho^2}{3^{2\gamma+2}}. \qquad (3.10)
\end{aligned}
$$

Maximizing the function $F_\gamma(\rho)$ on the closed boundary $[0,1]$ and differentiating, we obtain

$$
F_\gamma'(\rho) = \frac{2\eta^2(4-\eta_1^2)}{8^{\gamma+1}} + \frac{2\eta(\eta-2)(4-\eta^2)\rho}{8^{\gamma+1}} + \frac{2(4-\eta^2)(4-\eta^2)\rho}{3^{2\gamma+2}}.
$$

The negative impact from the second term can dominate, especially as ρ increases towards 1. For that reason, the second term can lead to $F_\gamma'(\rho) < 0$, and so we conclude that $F_\gamma(\rho)$ is decreasing on the interval $[0,1]$ if $0 \leq \eta < 2$. In this case, $F_\gamma(\rho) \leq F_\gamma(0) = \eta < 2$ for all $\rho \in [0,1]$. This follows therefore $F_\gamma(\rho) < 2$. On the other hand, suppose $\eta \in [0,1]$, then $F_\gamma(\rho)$ is increasing on the interval $[0,1]$ so that $F_\gamma(\rho) \leq F_\gamma(1)$. That is

$$
\begin{aligned}
F_\gamma(\rho) \leq F_\gamma(1) =& \left(\frac{2}{3^{2\gamma+2}} - \frac{2}{8^{\gamma+1}}\right)\eta^4 - \frac{2\eta^3}{8^{\gamma+1}} + \left(\frac{14}{8^{\gamma+1}} - \frac{8}{3^{2\gamma+2}}\right)\eta^2 + \frac{16}{3^{2\gamma+2}} \\
=& G_\gamma(\eta). \qquad (3.11)
\end{aligned}
$$

Hence, we have

$$
G_\gamma(\eta) \leq G_\gamma(0) = \frac{16}{3^{2\gamma+2}}, c \in [0,1]. \qquad (3.12)
$$

Equation (3.12) is less than 2 for a fixed $\gamma = 0$, which is the case when $0 \leq \eta < 2$. Thus, the maximum of the functional $|a_2a_4 - a_3^3|$ corresponds to $\rho = 1$ and $\eta < 2$.

Simplifying (3.9) and (3.12), we obtain

$$
4|\Phi_1\eta\eta_3+\Phi_2\eta_2^2| \leq \frac{16}{3^{2\gamma+2}}. \qquad (3.13)
$$

$$
\left|\frac{\eta_1\eta_3}{8^{\gamma+1}} - \frac{\eta_2^2}{3^{2\gamma+2}}\right| \leq \frac{4}{3^{2\gamma+2}}. \qquad (3.14)
$$

Setting $\eta_1 = \eta = 0$, and choosing $x = -1$ and $z = 1$ in (2.5), we get

$$
\begin{cases}
\eta = 0 \\
\eta_2 = -2 \\
\eta_3 = 0.
\end{cases} \qquad (3.15)
$$

Substituting the values from (3.15) into (3.14), with a fixed $\gamma = 0$, we can see that equality is attained, indicating that our result is sharp.

Now, using relations (3.3) and (3.14), we conclude that

$$|\alpha_2\alpha_4 - \alpha_3^2| \leq \frac{4}{3^{2\gamma+2}}, \tag{3.16}$$

and the proof of Theorem 3.1 is over. □

Remark 3.1. *If $\gamma = 0$ in (3.16), the result is sharp and coincides with that of Janteng et al [1]. This can be seen in the following corollary:*

Corollary 3.2. *Let $f \in \mathcal{R}_\gamma$, with $\gamma = 0$, then*

$$|\alpha_2\alpha_4 - \alpha_3^2| \leq \frac{4}{9}. \tag{3.17}$$

Example: *Consider the function $f(z) = z + \frac{2}{3}z^2 + \frac{2}{9}z^3$. In this case, we find that $\alpha_2 = \frac{2}{3}, \alpha_3 = \frac{2}{9}, \alpha_4 = 0$.*

Now, calculating the Hankel determinant, we get

$$|\alpha_2\alpha_4 - \alpha_3^2| = \left|\frac{2}{3} \cdot 0 - \left(\frac{2}{9}\right)^2\right| = \frac{4}{81}.$$

This satisfies the inequality for $\gamma = 0$ since $\frac{4}{81} \leq \frac{4}{9}$. Hence, the inequality (3.16) is sharp for the function $f(z) = z + \frac{2}{3}z^2 + \frac{2}{9}z^3$.

Theorem 3.3. *Let $f \in \mathcal{S}^*_\gamma$, then*

$$|\alpha_2\alpha_4 - \alpha_3^2| \leq \frac{2(2^{\gamma+1} - 1)}{(3^{\gamma+1} - 1)}.$$

Proof. Let $f \in \mathcal{S}^*_\gamma$. The characteristics of the function f, which is of Sălăgeăn-type is illustrated through the *Carathédory* function $p(z)$ as follows

$$\frac{zD^\gamma f'(z)}{D^\gamma f(z)} = p(z) \iff zD^\gamma f'(z) = D^\gamma f(z)p(z) \qquad (z \in \mathcal{U}). \tag{3.18}$$

By series representation of $zD^\gamma f'(z)$ and $D^\gamma f(z)p(z)$, and equating coefficients in (3.18), we obtain

$$\begin{cases} \alpha_2 = \frac{\eta_1}{(2^{\gamma+1}-1)}, \\ \alpha_3 = \frac{\eta_1^2+(2^{\gamma+1}-1)\eta_2}{(2^{\gamma+1}-1)(3^{\gamma+1}-1)}, \\ \alpha_4 = \frac{\eta_1^3+(2^{\gamma+1}+3^{\gamma+1}-2)\eta_1\eta_2+(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta_3}{(2^{\gamma+1}-1)(3^{\gamma+1}-1)(4^{\gamma+1}-1)}, \\ \vdots \end{cases} \tag{3.19}$$

Using (3.19) and substituting the values of α_2, α_3 and α_4 into $|\alpha_2\alpha_4 - \alpha_3^2|$ and simplifying, we arrive at

$$\begin{aligned} |\alpha_2\alpha_4 - \alpha_3^2| = & \Bigg| \frac{\eta_1^4 + (2^{\gamma+1} + 3^{\gamma+1} - 2)\eta_1^2\eta_2 + (2^{\gamma+1} - 1)(3^{\gamma+1} - 1)\eta_1\eta_3}{(4^{\gamma+1} - 1)(3^{\gamma+1} - 1)(2^{\gamma+1} - 1)^2} \\ & - \frac{\eta_1^4 + 2\eta_1^2(2^{\gamma+1} - 1)\eta_2 + (2^{\gamma+1} - 1)^2\eta_2^2}{(2^{\gamma+1} - 1)^2(3^{\gamma+1} - 1)^2} \Bigg|. \end{aligned} \tag{3.20}$$

For convenience, (3.20) can be written as

$$|\alpha_2\alpha_4-\alpha_3^2| = \frac{1}{(3^{\gamma+1}-1)(2^{\gamma+1}-1)^2}\Bigg|\frac{1}{(4^{\gamma+1}-1)}\left[\eta_1^4+(2^{\gamma+1}+3^{\gamma+1}-2)\eta_1^2\eta_2\right.$$
$$\left.+(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta_1\eta_3\right]-\frac{1}{(3^{\gamma+1}-1)}\left[\eta_1^4+2\eta_1^2(2^{\gamma+1}-1)\eta_2+(2^{\gamma+1}-1)^2\eta_2^2\right]\Bigg|. \quad (3.21)$$

$$|\alpha_2\alpha_4-\alpha_3^2| = \frac{1}{(3^{\gamma+1}-1)(2^{\gamma+1}-1)^2}\Bigg|\Psi_1\left[\eta_1^4+(2^{\gamma+1}+3^{\gamma+1}-2)\eta_1^2\eta_2\right.$$
$$\left.+(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta_1\eta_3\right]+\Psi_2\left[\eta_1^4+2\eta_1^2(2^{\gamma+1}-1)\eta_2+(2^{\gamma+1}-1)^2\eta_2^2\right]\Bigg|, \quad (3.22)$$

where

$$\Psi_1;\Psi_2 = \begin{cases}\frac{1}{(4^{\gamma+1}-1)};\\ -\frac{1}{(3^{\gamma+1}-1)}.\end{cases} \quad (3.23)$$

Using the values of η_2 and η_3 from (2.5) of Lemma 2.2 and substituting on the right - hand side of the equality in (3.22), along with the fact that $|z|\leq 1$, we obtain

$$4\Bigg|\Psi_1\left[\eta_1^4+(2^{\gamma+1}+3^{\gamma+1}-2)\eta_1^2\eta_2+(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta_1\eta_3\right]$$
$$+\Psi_2\left[\eta_1^4+2\eta_1^2(2^{\gamma+1}-1)\eta_2(2^{\gamma+1}-1)^2\eta_2^2\right]\Bigg| = \Bigg|\frac{4\eta_1^4}{(4^{\gamma+1}-1)}+\frac{2(2^{\gamma+1}+3^{\gamma+1}-2)\eta_1^4}{(4^{\gamma+1}-1)}$$
$$-\frac{4\eta_1^4}{(3^{\gamma+1}-1)}-\frac{4(2^{\gamma+1}-1)\eta_1^4}{(3^{\gamma+1}-1)}+\frac{(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta_1^3}{(4^{\gamma+1}-1)}+\frac{2(2^{\gamma+1}+3^{\gamma+1}-2)|x|\eta_1^2}{(4^{\gamma+1}-1)}$$
$$-\frac{4(4-\eta_1^2)(2^{\gamma+1}-1)\eta_1^2}{(3^{\gamma+1}-1)}-\frac{2(2^{\gamma+1}-1)\eta_1^2}{(3^{\gamma+1}-1)}+\frac{2(4-\eta_1^2)(2^{\gamma+1}-1)(3^{\gamma+1}-1)|x|\eta_1}{(4^{\gamma+1}-1)}$$
$$-\frac{(4-\eta_1^2)|x|^2\eta_1}{(4^{\gamma+1}-1)}+\frac{2(4-\eta_1^2)\eta_1}{(4^{\gamma+1}-1)}-\frac{2(4-\eta_1^2)|x|^2\eta_1}{(4^{\gamma+1}-1)}-\frac{2(4-\eta_1^2)(2^{\gamma+1}-1)|x|}{(3^{\gamma+1}-1)}\Bigg|. \quad (3.24)$$

Using Lemma 2.1, where $\eta_1\leq 2$ and letting $\eta_1=\eta$, it is safe to assume with no boundaries that $\eta\in[0,2]$. Further, applying triangle inequality on (3.24), with $|x|=\rho$, we obtain

$$4\Bigg|\Psi_1\left[\eta^4+(2^{\gamma+1}+3^{\gamma+1}-2)\eta^2\eta_2+(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta\eta_3\right]$$
$$+\Psi_2\left[\eta^4+2\eta^2(2^{\gamma+1}-1)\eta_2(2^{\gamma+1}-1)^2\eta_2^2\right]\Bigg| \leq \frac{4\eta^4}{(4^{\gamma+1}-1)}+\frac{2(2^{\gamma+1}+3^{\gamma+1}-2)\eta^4}{(4^{\gamma+1}-1)}$$
$$+\frac{4\eta^4}{(3^{\gamma+1}-1)}+\frac{4(2^{\gamma+1}-1)\eta^4}{(3^{\gamma+1}-1)}+\frac{(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta^3}{(4^{\gamma+1}-1)}+\frac{2(2^{\gamma+1}+3^{\gamma+1}-2)\rho\eta^2}{(4^{\gamma+1}-1)}$$
$$+\frac{4(4-\eta^2)(2^{\gamma+1}-1)\eta^2}{(3^{\gamma+1}-1)}+\frac{2(2^{\gamma+1}-1)\eta^2}{(3^{\gamma+1}-1)}+\frac{2(4-\eta^2)(2^{\gamma+1}-1)(3^{\gamma+1}-1)\rho\eta}{(4^{\gamma+1}-1)}$$
$$+\frac{(4-\eta^2)\rho^2\eta}{(4^{\gamma+1}-1)}+\frac{2(4-\eta^2)\eta}{(4^{\gamma+1}-1)}+\frac{2(4-\eta^2)\rho^2\eta}{(4^{\gamma+1}-1)}+\frac{2(4-\eta^2)(2^{\gamma+1}-1)\rho}{(3^{\gamma+1}-1)}$$
$$=F_\gamma(\rho), \quad (3.25)$$

where

$$F_\gamma(\rho)=\frac{4\eta^4}{(4^{\gamma+1}-1)}+\frac{2(2^{\gamma+1}+3^{\gamma+1}-2)\eta^4}{(4^{\gamma+1}-1)}+\frac{4\eta^4}{(3^{\gamma+1}-1)}$$
$$+\frac{4(2^{\gamma+1}-1)\eta^4}{(3^{\gamma+1}-1)}+\frac{(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta^3}{(4^{\gamma+1}-1)}+\frac{2(2^{\gamma+1}+3^{\gamma+1}-2)\rho\eta^2}{(4^{\gamma+1}-1)}$$
$$+\frac{4(4-\eta^2)(2^{\gamma+1}-1)\eta^2}{(3^{\gamma+1}-1)}+\frac{2(2^{\gamma+1}-1)\eta^2}{(3^{\gamma+1}-1)}+\frac{2(4-\eta^2)(2^{\gamma+1}-1)(3^{\gamma+1}-1)\rho\eta}{(4^{\gamma+1}-1)}$$
$$+\frac{(4-\eta^2)\rho^2\eta}{(4^{\gamma+1}-1)}+\frac{2(4-\eta^2)\eta}{(4^{\gamma+1}-1)}+\frac{2(4-\eta^2)\rho^2\eta}{(4^{\gamma+1}-1)}+\frac{2(4-\eta^2)(2^{\gamma+1}-1)\rho}{(3^{\gamma+1}-1)}. \quad (3.26)$$

Maximizing the function $F_\gamma(\rho)$ on the closed boundary $[0,1]$ and differentiating, we obtain

$$F'_\gamma(\rho) = \frac{2(2^{\gamma+1}+3^{\gamma+1}-2)\eta^2}{(4^{\gamma+1}-1)} + \frac{2(4-\eta^2)(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta}{(4^{\gamma+1}-1)} + \frac{2(4-\eta^2)\rho\eta}{(4^{\gamma+1}-1)} + \frac{4(4-\eta^2)\rho\eta}{(4^{\gamma+1}-1)} + \frac{2(4-\eta^2)(2^{\gamma+1}-1)}{(3^{\gamma+1}-1)}. \tag{3.27}$$

The function $F_\gamma(\rho)$ is non-decreasing on the closed boundary $[0,1]$ when $0 \le \eta \le 2$. In this case, $F_\gamma(\rho) \le F_\gamma(1) = \eta \le 2$ for all $\rho \in [0,1]$, so that

$$\begin{aligned} F_\gamma(\rho) \le F_\gamma(1) =& \frac{4\eta^4}{(4^{\gamma+1}-1)} + \frac{2(2^{\gamma+1}+3^{\gamma+1}-2)\eta^4}{(4^{\gamma+1}-1)} + \frac{4\eta^4}{(3^{\gamma+1}-1)} \\ &+ \frac{4(2^{\gamma+1}-1)\eta^4}{(3^{\gamma+1}-1)} + \frac{(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta^3}{(4^{\gamma+1}-1)} + \frac{2(2^{\gamma+1}+3^{\gamma+1}-2)\eta^2}{(4^{\gamma+1}-1)} \\ &+ \frac{4(4-\eta^2)(2^{\gamma+1}-1)\eta^2}{(3^{\gamma+1}-1)} + \frac{2(2^{\gamma+1}-1)\eta^2}{(3^{\gamma+1}-1)} + \frac{2(4-\eta^2)(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta}{(4^{\gamma+1}-1)} \\ &+ \frac{(4-\eta^2)\rho^2\eta}{(4^{\gamma+1}-1)} + \frac{2(4-\eta^2)\eta}{(4^{\gamma+1}-1)} + \frac{2(4-\eta^2)\eta}{(4^{\gamma+1}-1)} + \frac{2(4-\eta^2)(2^{\gamma+1}-1)}{(3^{\gamma+1}-1)}. \\ &= G_\gamma(\eta). \end{aligned} \tag{3.28}$$

Hence, we have

$$G_\gamma(\eta) \le G_\gamma(0) = \frac{8(2^{\gamma+1}-1)}{(3^{\gamma+1}-1)}. \tag{3.29}$$

Using (3.25) and (3.29), we obtain

$$4\Big|\Psi_1\left[\eta_1^4+(2^{\gamma+1}+3^{\gamma+1}-2)\eta_1^2\eta_2+(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta_1\eta_3\right] + \Psi_2\left[\eta_1^4+2\eta_1^2(2^{\gamma+1}-1)\eta_2+(2^{\gamma+1}-1)^2\eta_2^2\right]\Big| \le \frac{8(2^{\gamma+1}-1)}{(3^{\gamma+1}-1)}.$$

$$\Big|\Psi_1\left[\eta_1^4+(2^{\gamma+1}+3^{\gamma+1}-2)\eta_1^2\eta_2+(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta_1\eta_3\right] + \Psi_2\left[\eta_1^4+2\eta_1^2(2^{\gamma+1}-1)\eta_2+(2^{\gamma+1}-1)^2\eta_2^2\right]\Big| \le \frac{2(2^{\gamma+1}-1)}{(3^{\gamma+1}-1)}. \tag{3.30}$$

Equation (3.30) is less than or equal 2 for a fixed $\gamma = 0$, which is the case when $0 \le \eta \le 2$. Thus, the maximum of the functional $|a_2a_4 - a_3^3|$ corresponds to $\rho = 1$ and $\eta = 2$.

Using (3.20) and (3.30), we obtain

$$\left|\frac{\eta_1^4+(2^{\gamma+1}+3^{\gamma+1}-2)\eta_1^2\eta_2+(2^{\gamma+1}-1)(3^{\gamma+1}-1)\eta_1\eta_3}{(4^{\gamma+1}-1)(3^{\gamma+1}-1)(2^{\gamma+1}-1)^2} - \frac{\eta_1^4+2\eta_1^2(2^{\gamma+1}-1)\eta_2+(2^{\gamma+1}-1)^2\eta_2^2}{(2^{\gamma+1}-1)^2(3^{\gamma+1}-1)^2}\right| \le \frac{2(2^{\gamma+1}-1)}{(3^{\gamma+1}-1)}. \tag{3.31}$$

Substituting the values from (3.15) into (3.31), with a fixed $\gamma = 0$, it can be rightly seen that equality is fully attained, which implies that our result is sharp.

Using relations (3.20) and (3.31), we obtain

$$|\alpha_2\alpha_4 - \alpha_3^2| \le \frac{2(2^{\gamma+1}-1)}{(3^{\gamma+1}-1)}, \tag{3.32}$$

which is the desired result. Hence, the proof of Theorem 3.3 is over. □

Remark 3.2. *If $\gamma = 0$ in equation (3.32), our result is sharp and coincides with that of Janteng et al [5]. This can be seen in the corollary that follows.*

Corollary 3.4. *Let $f \in \mathcal{S}^*_\gamma$, with $\gamma = 0$, then*

$$|\alpha_2\alpha_4 - \alpha_3^2| \leq 1. \tag{3.33}$$

Example: *Consider the function $f(z) = z + z^2 + z^3$. In this case, $\alpha_2 = 1, \alpha_3 = 1, \alpha_4 = 0$.*

Now, calculating the Hankel determinant, we have

$$|\alpha_2\alpha_4 - \alpha_3^2| = |1 \cdot 0 - 1^2| = 1.$$

This satisfies the inequality for $\gamma = 0$ since $1 \leq 1$. Hence, the inequality (3.32) is sharp for the function $f(z) = z + z^2 + z^3$.

Theorem 3.5. *Let $f \in \mathcal{C}_\gamma$, then*

$$|\alpha_2\alpha_4 - \alpha_3^2| \leq \frac{1}{3^{2\gamma+2}}.$$

Proof. Let $f \in \mathcal{C}_\gamma$. The characteristics of the function f, which is of Sălăgeăn-type is illustrated through the *Carathédory* function $p(z)$ as follows

$$1 + \frac{zD^\gamma f''(z)}{D^\gamma f'(z)} = p(z) \iff zD^\gamma f''(z) = D^\gamma f'(z)[p(z) - 1], (z \in \mathcal{D}). \tag{3.34}$$

Using the series representations of the functions in (3.34), we have

$$\sum_{n=2}^{\infty}(n-1)n^{\gamma+1}\alpha_n z^{n-1} = \left(1 + \sum_{n=2}^{\infty} n^{\gamma+1}\alpha_n z^{n-1}\right)\left(\sum_{n=1}^{\infty}\eta_n z^n\right). \tag{3.35}$$

By Cauchy product formula for two series, equation (3.35) can be written as

$$\sum_{n=0}^{\infty}(n+1)(n+2)^{\gamma+1}\alpha_{n+2}z^{n+1} = \sum_{n=1}^{\infty}\eta_n z^n + \left(\sum_{n=1}^{\infty}(n+1)^{\gamma+1}\alpha_{n+1}z^n\right)\left(\sum_{n=1}^{\infty}\eta_n z^n\right) \tag{3.36}$$

Equating coefficients in (3.36), we obtain

$$\begin{cases} \alpha_2 = \frac{\eta_1}{2^{\gamma+1}}, \\ \alpha_3 = \frac{\eta_1^2+\eta_2}{2\times 3^{\gamma+1}}, \\ \alpha_4 = \frac{\eta_1^3+3\eta_1\eta_2+2\eta_3}{6\times 4^{\gamma+1}}, \\ \vdots \end{cases} \tag{3.37}$$

Using (3.37) and substituting the values of α_2 and α_3 into the functional $|\alpha_2\alpha_4 - \alpha_3^2|$ and simplifying, we arrive at

$$|\alpha_2\alpha_4 - \alpha_3^2| = \left|\frac{\eta_1^4 + 3\eta_1^2\eta_2 + 2\eta_1\eta_3}{6 \times 2^{3\gamma+3}} - \frac{(\eta_1^4 + 2\eta_1^2\eta_2 + \eta_2^2)}{4 \times 3^{2\gamma+2}}\right|. \tag{3.38}$$

For convenience, equation (3.38) can be written as

$$|\alpha_2\alpha_4 - \alpha_3^2| = \left|\Pi_1[\eta_1^4 + 3\eta_1^2\eta_2 + 2\eta_1\eta_3] + \Pi_2[\eta_1^4 + 2\eta_1^2\eta_2 + \eta_2^2]\right|. \tag{3.39}$$

where

$$\Pi_1;\Pi_2 = \begin{cases} \frac{1}{6\times 2^{3\gamma+3}}; \\ -\frac{1}{4\times 3^{2\gamma+2}}. \end{cases} \tag{3.40}$$

Using the values of η_2 and η_3 from (2.5) of Lemma 2.2 and substituting on the right - hand side of (3.39), and simplifying, we obtain

$$\begin{aligned}4|\Pi_1[\eta_1^4+3\eta_1^2\eta_2+2\eta_1\eta_3]+\Pi_2[\eta_1^4+2\eta_1^2\eta_2+\eta_2^2]| = \Big| 12\Pi_1\eta_1^4+9\Pi_2\eta_1^4 \\ +8\Pi_1\eta_1^2(4-\eta_1^2)|x|+6\Pi_2\eta_1^2(4-\eta_1^2)|x|+4\Pi_1\eta_1(4-\eta_1^2)-2\Pi_1\eta_1(\eta_1+2)(4-\eta_1^2)|x|^2 \\ +\Pi_2(4-\eta_1^2)(4-\eta_1^2)|x|^2\Big|. \end{aligned} \tag{3.41}$$

Using the values of (3.40) on the right - hand side of (3.41) and considering Lemma 2.1, where $\eta_1 \leq 2$ and the fact that $\eta_1 = \eta$, and a direct application of triangle inequality along with $|x| = \rho$, upon simplification, we obtain

$$\begin{aligned}4|\Pi_1[\eta^4+3\eta^2\eta_2+2\eta\eta_3]+\Pi_2[\eta^4+2\eta^2\eta_2+\eta_2^2]| \leq \frac{4\eta(4-\eta^2)}{6\times 2^{3\gamma+3}}+\frac{6\eta^2(4-\eta^2)\rho}{4\times 3^{2\gamma+2}} \\ +\frac{8\eta^2(4-\eta^2)\rho}{6\times 2^{3\gamma+3}}+\frac{2\eta(\eta-2)(4-\eta^2)\rho^2}{6\times 2^{3\gamma+3}}+\frac{(4-\eta^2)(4-\eta^2)\rho^2}{4\times 3^{2\gamma+2}}+\frac{9\eta^4}{4\times 3^{2\gamma+2}} \\ +\frac{12\eta^4}{6\times 2^{3\gamma+3}} \\ =F_\gamma(\rho) \end{aligned} \tag{3.42}$$

where

$$\begin{aligned}F_\gamma(\rho) =&\frac{4\eta(4-\eta^2)}{6\times 2^{3\gamma+3}}+\frac{6\eta^2(4-\eta^2)\rho}{4\times 3^{2\gamma+2}}+\frac{8\eta^2(4-\eta^2)\rho}{6\times 2^{3\gamma+3}}+\frac{2\eta(\eta-2)(4-\eta^2)\rho^2}{6\times 2^{3\gamma+3}} \\ &+\frac{(4-\eta^2)(4-\eta^2)\rho^2}{4\times 3^{2\gamma+2}}+\frac{9\eta^4}{4\times 3^{2\gamma+2}}+\frac{12\eta^4}{6\times 2^{3\gamma+3}}. \end{aligned} \tag{3.43}$$

Maximizing the function $F_\gamma(\rho)$ on the closed boundary $[0,1]$ and differentiating, we obtain

$$F'_\gamma(\rho) = \frac{6\eta^2(4-\eta^2)}{4\times 3^{2\gamma+2}}+\frac{8\eta^2(4-\eta^2)}{6\times 2^{3\gamma+3}}+\frac{4\eta(\eta-2)(4-\eta^2)\rho}{6\times 2^{3\gamma+3}}+\frac{2(4-\eta^2)(4-\eta^2)\rho}{4\times 3^{2\gamma+2}}. \tag{3.44}$$

$F'_\gamma(\rho) \geq 0$ for $\rho \in [0,1]$. Therefore, $F_\gamma(\rho)$ is an increasing function of ρ on the closed boundary $[0,1]$ when $0 \leq \eta \leq 1$. In this case, $F_\gamma(\rho) \leq F_\gamma(1)$. That is

$$\begin{aligned}F_\gamma(\rho) \leq F_\gamma(1) =&\frac{4\eta(4-\eta^2)}{6\times 2^{3\gamma+3}}+\frac{6\eta^2(4-\eta^2)}{4\times 3^{2\gamma+2}}+\frac{8\eta^2(4-\eta^2)}{6\times 2^{3\gamma+3}}+\frac{2\eta(\eta-2)(4-\eta^2)^2}{6\times 2^{3\gamma+3}} \\ &+\frac{(4-\eta^2)(4-\eta^2)}{4\times 3^{2\gamma+2}}+\frac{9\eta^4}{4\times 3^{2\gamma+2}}+\frac{12\eta^4}{6\times 2^{3\gamma+3}}. \end{aligned} \tag{3.45}$$

Hence, we have

$$G_\gamma(\eta) \leq G_\gamma(0) = \frac{4}{3^{2\gamma+2}}. \tag{3.46}$$

Using (3.42) and (3.46), we obtain

$$4|\Pi_1[\eta_1^4+3\eta_1^2\eta_2+2\eta_1\eta_3]+\Pi_2[\eta_1^4+2\eta_1^2\eta_2+\eta_2^2]| \leq \frac{4}{3^{2\gamma+2}}.$$

$$|\Pi_1[\eta_1^4+3\eta_1^2\eta_2+2\eta_1\eta_3]+\Pi_2[\eta_1^4+2\eta_1^2\eta_2+\eta_2^2]| \leq \frac{1}{3^{2\gamma+2}}. \tag{3.47}$$

Equation (3.47) is less than or equal 1 for a fixed $\gamma = 0$, which is the case when $0 \leq \eta \leq 1$. Thus, the maximum of the functional $|a_2a_4 - a_3^3|$ corresponds to $\rho = 1$ and $\eta = 1$.

Using (3.38) and (3.47), we obtain

$$\left|\frac{\eta_1^4+3\eta_1^2\eta_2+2\eta_1\eta_3}{6\times 2^{3\gamma+3}}-\frac{(\eta_1^4+2\eta_1^2\eta_2+\eta_2^2)}{4\times 3^{2\gamma+2}}\right|\leq\frac{1}{3^{2\gamma+2}}. \tag{3.48}$$

Substituting the values from (3.15) into (3.48), with a fixed $\gamma = 0$, it can be clearly seen that equality is attained. Hence, it implies that our result is sharp.

Finally, using relations (3.38) and (3.48), we obtain

$$|\alpha_2\alpha_4-\alpha_3^2|\leq\frac{1}{3^{2\gamma+2}}. \tag{3.49}$$

This completes the proof of Theorem 3.5. □

Remark 3.3. *If $\gamma = 0$ in equation (3.49), our result is sharp. This can be seen in the following corollary:*

Corollary 3.6. *Let $f \in \mathcal{C}_\gamma$, with $\gamma = 0$, then*

$$|\alpha_2\alpha_4-\alpha_3^2|\leq\frac{1}{9}. \tag{3.50}$$

Example: *Consider the function $f(z) = z+\frac{1}{3}z^2+\frac{1}{9}z^3$. In this case, $\alpha_2 = \frac{1}{3}, \alpha_3 = \frac{1}{9}, \alpha_4 = 0$.*

Now, calculating the Hankel determinant, we have

$$|\alpha_2\alpha_4-\alpha_3^2| = \left|\frac{1}{3}\cdot 0-\left(\frac{1}{9}\right)^2\right| = \frac{1}{81}.$$

This satisfies the inequality for $\gamma = 0$ since $\frac{1}{81}\leq\frac{1}{9}$. Hence, the inequality (3.49) is sharp for the function $f(z) = z+\frac{1}{3}z^2+\frac{1}{9}z^3$.

4 Conclusion remarks

In this paper, we dive into sharp inequalities and generalized bounds for the second Hankel determinant $|\alpha_2\alpha_4-\alpha_3^2|$, within the framework of new subclasses of analytic functions of Sălăgeăn - type. Specifically, we focus on $\mathcal{R}_\gamma$, $\mathcal{S}_\gamma^*$, and $\mathcal{C}_\gamma$. By utilizing the Toêplitz determinant and tapping into the properties of the Sălăgeăn differential operator, we've uncovered significant results that broaden the classical inequalities associated with the well-known classes of bounded turning, starlike, and convex functions. Our discoveries not only offer sharp bounds for the Hankel determinant but also enhance our understanding of how analytic functions behave geometrically when influenced by the Sălăgeăn operator. Introducing the parameter γ creates a more adaptable framework, allowing for a range of special cases and applications. We anticipate that this generalization will lead to further insights and findings in the ongoing exploration of analytic functions, with potential impacts across various mathematical and applied fields. Future research could delve into additional subclasses and their characteristics, as well as examine how these results apply in complex analysis and related areas. The findings presented here set the stage for a deeper investigation into the intricate connections between analytic functions and their geometric properties.

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