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An Improved Class of Newton Raphson Technique for Numerical Simulation of Nonlinear Equations

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Abstract *Numerous branches of research and engineering deal with nonlinear equations, which frequently require for numerical approximation methods to solve. Analytical solutions to nonlinear equations are often not obtained in simple way, so we move towards numerical techniques to solve these equations to obtained approximate solutions. In this article, we present a newly iterative technique based on Newton – Obadah technique to evaluate second derivative Both numerical results and theoretical analysis demonstrate robustness and effectiveness of the proposed technique for solving wide range of nonlinear equations. The proposed technique has eight order of convergence which is higher than the existing techniques like KT, CTV, and CLM and, also proposed method required lesser number of iterations to converges the root of the equations in comparison to the existing techniques like KT, CTV, and CLM. This provided grounds for confidence in modified method and provides for further refinement in future work.*

1 Introduction

Nonlinear equations are ubiquitous in various branches of engineering and science, arising in problems related to fluid mechanics, heat transfer, structural analysis, and many other fields [1]. Iterative techniques have gained preference among many numerical methods due to their capacity to handle broad spectrum of nonlinear equations [2].

The Newton-Raphson method is one of the most frequently used iterative method for solving nonlinear equations [3]. However, the standard Newton-Raphson method has certain limitations, such as the requirement of evaluating the derivative of the function at each iteration which may be not possible in some cases, and convergence of the method depends on initial value [4].

To overcome these limitations, various enhancement to Newton-Raphson technique have been proposed in the literature which aims to improve convergence rate and reduce number of iterations [5, 6]. Other iterative techniques, such as Secant technique and Fixed-Point iteration technique, have also been frequently used for solving nonlinear equations [7, 8].

A well-known technique for solving the equation $f(x) = 0$ *is the Newton's technique, which has second order convergence and an efficiency index of 1.412 specified by:*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots \tag{1.1}$$

Following that, the third order convergence iteration was shown by Halley's with efficiency index of 1.4422 [5]:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}, \quad n = 0, 1, 2, \dots \tag{1.2}$$

We are particularly interested in the Obadah [9] concept, which is considered an advancement over the Halley method.

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{2[f(x_0)]^2 f'(x_0) f''(x_0)}{4[f(x_0)]^4 - 4f(x_0)[f'(x_0)]^2 f''(x_0) + [f(x_0)]^2[f''(x_0)]^2} \tag{1.3}$$

The third-order convergence iteration of Chebyshev's algorithm [6] was demonstrated with an efficiency index of 1.4422, as indicated by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f'(x_n)}{2[f'(x_n)]^3}, \quad n = 0, 1, 2, \ldots \tag{1.4}$$

In this research we present the modified approaches each with three step iterations. Taylor's series is used to reduce functions in order to estimate second order derivative with each iteration. Equation (1.3), which we use to represent the concept of Obadah [9], is then rewritten using Newton's approach. Using Newton's approach, we raise the iteration step by one for the modified method. Subsequently, the improved method's order of convergence was analyzed. Other iteration methods and numerical examples were used to test the effectiveness.

Inderjeet and Rashmi Bhardwaj [10, 11, 12, 13] presented modified Newton Raphson approaches to tackle the nonlinear equations with improved accuracy and robustness of the technique, and the proposed approaches are better than the existing techniques like Secant technique, Fixed – Point iteration etc.

2 Proposed Methodology

Third order convergence and efficiency index at 1.4422 could be expressed using Newton's method and the Obadah [9] notion as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.1}$$

$$x_{n+1} = y_n - \frac{2[f(x_n)]^2 f'(x_n) f''(x_n)}{4[f'(x_n)]^4 - 4f(x_n)[f'(x_n)]^2 f''(x_n) + [f(x_n)]^2[f''(x_n)]^2} \tag{2.2}$$

Taylor series expansion of $f(y_n)$ around $x = x_n$ up-to second order term, we get

$$f(y_n) \approx f(x_n) + f'(x_n)(y_n - x_n) + \frac{f''(x_n)(y_n - x_n)^2}{2!}, \quad \textit{where } y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

Putting the value of $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$ and simply we get

$$f(y_n) \approx \frac{f''(x_n)\,[f(x_n)]^2}{2\,[f'(x_n)]^2} \tag{2.3}$$

Put the value of equation (2.3) in equation (2.2), we obtained
$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = y_n - \frac{[f(x_n)]^2 f(y_n)}{[f'(x_n)]^2 f'(x_n) - 2f(x_n)f'(x_n)f(y_n) + f'(x_n)[f(y_n)]^2} \tag{2.4}$$

By using Newton's approach, equation (2.4) was expanded step by step, and algorithm 2.1 provides the new technique.
Algorithm 2.1
We use iterative approaches to calculate the approximate solution x_{n+1} for a given x_0
$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$

$$z_n = y_n - \frac{[f(x_n)]^2 f(y_n)}{[f(x_n)]^2 f'(x_n) - 2f(x_n)f'(x_n)f(y_n) + f'(x_n)[f(y_n)]^2} \tag{2.5}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \tag{2.6}$$

3 Convergence Analysis

Theorem 3.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function on an open interval I, and let $\alpha \in I$ be a simple zero of $f(x)$. If an initial point x_0 is sufficiently close to α, then the algorithm 2.1 has eight orders of convergence, and satisfies the following error equation:*
$e_{n+1} = \left(c_3^2 c_2^3 - 4c_3 c_2^5 + 4c_2^7\right) e_n^8 + o(e_n^9)$
Where $e_n = x_n - \alpha$ *and* $c_n = \frac{f^n(\alpha)}{n! f'(\alpha)}, \quad n = 2, 3, 4, \ldots$

Proof. Let α be a simple zero of $f(x)$, so $f(\alpha) = 0$.
By using Taylor series expansion of $f(x_n)$ around $x_n = \alpha$ is given by:
$f(x_n) = f(\alpha + e_n) = f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2!} f''(\alpha) + \frac{e_n^3}{3!} f'''(\alpha) + \frac{e_n^4}{4!} f^{(4)}(\alpha) + \frac{e_n^5}{5!} f^{(5)}(\alpha) + \frac{e_n^6}{6!} f^{(6)}(\alpha) + \frac{e_n^7}{7!} f^{(7)}(\alpha) + \frac{e_n^8}{8!} f^{(8)}(\alpha) + \ldots$

$$f(x_n) = f(\alpha + e_n) = f'(\alpha)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + \ldots\right] \tag{3.1}$$

$$f'(x_n) = f'(\alpha)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + 8c_8 e_n^7 + \ldots\right] \tag{3.2}$$

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 - (2c_3 - 2c_2^2)e_n^3 - (3c_4 - 7c_2 c_3 + 4c_2^3)e_n^4 + \ldots \tag{3.3}$$

$$y_n = \alpha + c_2 e_n^2 + \left(2c_3 - 2c_2^2\right) e_n^3 + \left(3c_4 - 7c_2 c_3 + 4c_2^3\right) e_n^4 + \ldots \tag{3.4}$$

$$f(y_n) = f'(\alpha)\left[c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (5c_2^3 - 7c_2 c_3 + 3c_4)e_n^4 + \ldots\right] \tag{3.5}$$

Put the value of the equations (3.1), (3.2), and (3.5) in the equation (2.5), we get

$$z_n = \alpha + \left(2c_2^3 - c_2 c_3\right) e_n^4 + \left(14c_3 c_2^2 - 10c_2^4 - 2c_3^2 - 2c_2 c_4\right) e_n^5 + \ldots \tag{3.6}$$

$$f(z_n) = f'(\alpha)\left[\left(2c_2^3 - c_2 c_3\right) e_n^4 + \left(14c_3 c_2^2 - 10c_2^4 - 2c_3^2 - 2c_2 c_4\right) e_n^5 + \ldots\right] \tag{3.7}$$

and

$$f'(z_n) = f'(\alpha)\left[1 + 2c_2^2(2c_2^2 - c_3)e_n^4 - 4c_2(5c_2^4 - 7c_3 c_2^2 + c_3^2 + c_2 c_4)e_n^5\right] + \ldots \tag{3.8}$$

$$\frac{f(z_n)}{f'(z_n)} = \left[(2c_2^3 - c_2 c_3)e_n^4 + (14c_3 c_2^2 - 10c_2^4 - 2c_3^2 - 2c_2 c_4)e_n^5\right] + \ldots \tag{3.9}$$

Put the value of (3.9) in (2.6) we get
$x_{n+1} = \alpha + \left(c_3^2 c_2^3 - 4c_3 c_2^5 + 4c_2^7\right) e_n^8 + o(e_n^9)$
Using $e_{n+1} = x_{n+1} - \alpha$, we get
$e_{n+1} = \left(c_3^2 c_2^3 - 4c_3 c_2^5 + 4c_2^7\right) e_n^8 + o(e_n^9)$
Hence, the algorithm (2.1) has eighth order of convergence. □

4 Numerical Experiments and Comparison of Results

To evaluate the performance of the modified Newton-Raphson approach, numerical simulations were conducted on several benchmark problems involving nonlinear equations. The proposed method was compared with others existing techniques Cordero, and Torregrosa and Vassileva (CTV), Lotfi et al (KT), and Chebyshev-Lagrange method (CLM).

All the computational experiments are performed on Intel(R) Core (TM) i3, 2.1 GHz, 8GB RAM.

The stopping criterion are used in the proposed algorithm is $|x_{n+1} - x_n| < \epsilon = 10^{-15}$

The difference between successive approximation δ and number of iterations (IT) all are shown below in tables.

Benchmark Problems

Problem1: $f_1(x) = 4x^2(x^2 - 1)$ and initial point is $x_0 = 0.8$

Problem 2: $f_2(x) = \ln(x^2 + 1) + e^x \sin x$ and initial point is $x_0 = 2.9$

Problem 3: $f_3(x) = \sin^2(x) - x^2 + 1$ and initial point is $x_0 = -2.5$

Problem 4: $f_4(x) = (x - 2)^{23} - 1$ and initial point is $x_0 = 2.9$

Problem 5: $f_5(x) = 5x^3 - 0.4x^2 - 1$ and initial point is $x_0 = 2$

Table 1. Comparison between the newly created approach and the existing approach in terms of iterations

Problem	**Method**	**Number of Iterations**	x_n	δ
1	KT	5	1	8884929×10^{-37}
	CTV	9	1	2820598×10^{-23}
	CLM	10	1	2459319×10^{-34}
	PM	3	1	4973714×10^{-79}
2	KT	5	$3237562984023921 \times 10^{-15}$	2898834×10^{-24}
	CTV	7	$323756298402392 \times 10^{-15}$	8429563×10^{-29}
	CLM	7	$3237562984023921 \times 10^{-15}$	1535332×10^{-32}
	PM	3	3237562984023921^{-15}	2044221×10^{-27}
3	KT	7	$-1404491648215341 \times 10^{-15}$	2432583×10^{-24}
	CTV	11	$-1404491648215341 \times 10^{-15}$	1505241×10^{-29}
	CLM	9	$-1404491648215341 \times 10^{-15}$	4083291×10^{-30}
	PM	4	$-1404491648215341 \times 10^{-15}$	2229438×10^{26}
4	KT	7	3	3935067×10^{-39}
	CTV	27	3	5681689×10^{-47}
	CLM	divergent	-	-
	PM	6	3	6396552×10^{-84}
5	KT	6	$2690647448028614 \times 10^{-15}$	1987149×10^{-88}
	CTV	divergent	-	-
	CLM	9	$2690647448028614 \times 10^{-15}$	2990186×10^{-81}
	PM	4	$2690647448028614 \times 10^{-15}$	1413885×10^{-114}

5 Result discussions and conclusion

In this article, we presented a modified Newton-Raphson technique for numerically simulating and approximation the roots of nonlinear equations. The proposed method enhances efficiency and convergence rate of traditional Newton-Raphson technique. Theoretical analysis and convergence results were established, showing that the modified Newton-Raphson method achieves a convergence rate of order 8, which is faster than the traditional Newton-Raphson method. Numerical experiments on benchmark problems demonstrated the superior performance of the proposed method compared to others methods like KT, CLV, and CTM. The modified technique exhibited faster convergence, higher accuracy, and required fewer number of iterations.

In conclusion, the modified approach presented in this paper offers a powerful and efficient numerical tool for simulating and solving nonlinear equations. Its enhanced convergence properties, fewer number of iterations, and robustness make it a promising approach for tackling a wide range of nonlinear problems in engineering and science.

Conflict of interest

The authors declare no conflict of interest.

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