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Applications of measure of non-compactness for the solvability of an infinite system of integral equations of Hammerstein type of three variables in tempered sequence spaces

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Abstract *In this paper, we examine the solvability of an infinite system of integral equations of Hammerstein type in three variables in two newly constructed tempered sequence spaces* bv_0^ρ *and* cs^ρ*, with the help of measure of non-compactness and Meir-Keeler condensing operator. In order to demonstrate and establish the significance of our results, we additionally offer suitable examples.*

1 Introduction

Integral equations (IE) are a significant extension of nonlinear functional analysis. Nonlinear integral equations often arise to explain a broad range of real-world issues. They occur in tackling various engineering, population dynamics, feedback systems, stability of nuclear reactors, and economics challenges [2]. One of the most important and thoroughly researched nonlinear integral equation is the Hammerstein one. In recent times, many researchers have been interested in analyzing the solutions of various functional equations. Measure of noncompactness (MNC) is a useful tool in fixed point theory, which plays an important role in figuring out the solvability of functional equations.

Kuratowski [1] was the first who described the concept of MNC as a function that provides a quantitative measure of how far a given set is from being compact. Later on, many prominent mathematicians and researchers further generalized the concept. In 1955, Darbo [3] introduced the famous Darbo's fixed point theorem using Kuratowski measure of non-compactness. Over the period of time, a number of researchers have used MNC to demonstrate the existence or solvability of several linear and non-linear equations in various Banach sequence spaces. We suggest the readers to visit [7], [9], [12], [14], [17] and the references therein for such results.

Finding the solutions of distinct differential or integral equations on different classical spaces can occasionally be challenging , as the solutions may not be present in classical sequence spaces. Therefore, it is necessary to extend the classical sequence spaces. Banaś and Krajewska filled the void in [15] by proposing the concept of a positive and non-increasing sequence named as tempered sequence. For example, if we take the classical sequence space bv_0 *and the tempered sequence* $(\rho_m) = \frac{1}{m^2}$, $(m \in \mathbb{N})$*, then it gives a new sequence space* bv_0^ρ*, allowing* $\sum_{m=1}^{\infty} \rho_m |u_m - u_{m+1}| < \infty$*. We go into great depth on the tempered sequence in section 4.*

Using the concept of tempered sequence, Ghasemi and Khanehgir [16] studied the solution of n-variable integral equations in c_0^β *and* ℓ_1^β*, in [18], Mehravaran et al. introduced the tempered sequence* $m^\beta(\phi)$*, in [19], Rao et al. checked the solvability of infinite systems of nonlinear* *tempered fractional order BVPs in $C(Q, c_0^l)$ and $C(Q, c^l)$. In [20], Simbeye et al. checked the solvability of an infinite system of integral equations of Hammerstein type in three variables in tempering sequence spaces c_0^β and ℓ_1^β.*

Motivated by all these works, in this article, we extend the results of [20] and check the solvability of the following infinite system of IE with a Hammerstein structure in three variables in two newly generated tempered sequence spaces bv_0^ρ and cs^ρ.

$$\begin{aligned} x_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) &= r_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) + \int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}} K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) h_n(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, \\ &\quad x(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})) d\overline{\psi}_{11} d\overline{\psi}_{22} d\overline{\psi}_{33} \end{aligned} \tag{1.1}$$

where $n \in \mathbb{N}$ and $\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33} \in I^3$ for $I = [\hat{\mu}, \hat{\nu}]$.

The significance of this work lies in the novel application of measure of non-compactness to infinite-dimensional sequence spaces tailored by tempering sequences, which opens new avenues in the study of integral equations. These results not only enrich the theoretical understanding but also lay the groundwork for future studies involving more complex or generalized systems.

For convenience, we use $\mathcal{NBS}$ for nonempty bounded subset and $\mathcal{BCS}$ for a bounded closed set.

This work is organized as follows: in section 2, we present some preliminary findings and analysis of MNC. We imply the Hausdorff measure of non-compactness (H-MNC) on sequence spaces in section 3. We construct two new tempered sequence spaces bv_0^ρ and cs^ρ and define H-MNC on these spaces in section 4. The solvability of the system (1.1) in bv_0^ρ is examined in section 5, and our conclusion is demonstrated with an example. Section 6 examines the solvability of (1.1) in cs^ρ followed by an example to support our findings.

2 Preliminaries

Assume $\mathfrak{E}$ as a complete normed linear space (Banach space) over the real numbers with the norm $\|\cdot\|$. Suppose $\mathfrak{B}(x_0, d_0)$ represents a closed ball with radius d_0 and center at x_0 in $\mathfrak{E}$. For any subset Y of $\mathfrak{E}$, let $\overline{Y}$ and $Conv(Y)$ be the closure and convex closure of Y respectively. Additionaly, let $\mathfrak{R}_{\mathfrak{E}}$ be a subfamily of all relatively compact sets, and $\mathfrak{P}_{\mathfrak{E}}$ is set of all $\mathcal{NBS}$ of $\mathfrak{E}$. Moreover, $\mathfrak{P}_{\mathfrak{E}}^C$ be the subfamily of all $\mathcal{BCS}$.

Definition 2.1. [5] A mapping

$$\tilde{f} : \mathfrak{P}_{\mathfrak{E}} \to \mathbb{R}_+$$

is called MNC in $\mathfrak{E}$ if

(i) $\ker \tilde{f} = \{\mathfrak{S} \in \mathfrak{P}_{\mathfrak{E}} : \tilde{f}(\mathfrak{S}) = 0\}$ is non-empty and $\ker \tilde{f} \subset \mathfrak{R}_{\mathfrak{E}}$.

(ii) $\mathfrak{S}_1 \subset \mathfrak{S}_2 \Rightarrow \tilde{f}(\mathfrak{S}_1) \leq \tilde{f}(\mathfrak{S}_2)$.

(iii) $\tilde{f}(\overline{\mathfrak{S}}) = \tilde{f}(\mathfrak{S})$.

(iv) $\tilde{f}(Conv(\mathfrak{S})) = \tilde{f}(\mathfrak{S})$.

(v) $\tilde{f}(s\mathfrak{S}_1 + (1-s)\mathfrak{S}_2) \leq s\tilde{f}(\mathfrak{S}_1) + (1-s)\tilde{f}(\mathfrak{S}_2), \forall s \in (0,1)$.

(vi) If $(\mathcal{I}_n)$ is a non-increasing sequence in $\mathfrak{P}_{\mathfrak{E}}^C$ and if $(\tilde{f}(\mathcal{I}_n))$ is in c_0, then there exists at least one $j \in \mathfrak{E}$ such that $j \in \mathcal{I}_n, \forall\, n \in \mathbb{N}$.

Note that, the MNC $\tilde{f}$ is called sublinear when

(a) $\tilde{f}(s\mathfrak{S}) = |s|\tilde{f}(\mathfrak{S})$ for $s \in \mathbb{R}$ and

(b) $\tilde{f}(\mathfrak{S}_1 + \mathfrak{S}_2) \leq \tilde{f}(\mathfrak{S}_1) + \tilde{f}(\mathfrak{S}_2)$.

Definition 2.2. **[6]** Suppose $\tilde{f}_1$ and $\tilde{f}_2$ be two MNC's on the Banach spaces $\mathfrak{E}_1$ and $\mathfrak{E}_2$ respectively. An operator $\mathcal{H} : \mathfrak{E}_1 \to \mathfrak{E}_2$ is $(\tilde{f}_1, \tilde{f}_2)$-condensing operator when

i. $\mathcal{H}$ is continuous,

ii. $\tilde{f}_2(\mathcal{H}(C)) < \tilde{f}_1(C)$, where C is bounded as well as non compact subset of $\mathfrak{E}_1$. A $(\tilde{f}_1, \tilde{f}_2)$-condensing operator becomes $\tilde{f}$-condensing operator when $\tilde{f}_1 = \tilde{f}_2 = \tilde{f}$ and $\mathfrak{E}_1 = \mathfrak{E}_2$.

Definition 2.3. **[4]** Consider $\mathfrak{M}$ as a metric space with metric ϱ. A mapping

$$\mathfrak{I} : \mathfrak{M} \to \mathfrak{M}$$

is a Meir-Keeler contraction or simply, a M-K contraction if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$ with $\varrho(\mathfrak{I}u, \mathfrak{I}v) < \epsilon$ whenever $\epsilon \le \varrho(u, v) < \epsilon + \delta$, $\forall\, u, v \in \mathfrak{M}$.

Theorem 2.4. *[4] Whenever $(\mathfrak{M}, \varrho)$ is complete, M-K contraction $\mathfrak{I}$ on $\mathfrak{M}$ always has a unique fixed point.*

Definition 2.5. **[13]** Let $\mathfrak{E}$ be a Banach space with MNC η, and $\mathfrak{D} \subset \mathfrak{E}$. An operator $\mathfrak{J} : \mathfrak{D} \to \mathfrak{D}$ is called Meir-Keeler condensing operator or simply M-K condensing operator if $\forall\, \epsilon > 0$, $\exists$ $\delta > 0$ s.t. $\eta(\mathfrak{J}(\mathfrak{D}_1)) < \epsilon$ whenever $\epsilon \le \eta(\mathfrak{D}_1) < \epsilon + \delta$ $\forall$ bounded subset $\mathfrak{D}_1$ of $\mathfrak{D}$.

Theorem 2.6. *[13] For a subset $\mathfrak{D}$ of $\mathfrak{E}$ and arbitrary MNC η, if $\mathfrak{J}$, the Meir-Keeler condensing operator on $\mathfrak{D}$, is continuous then $\mathfrak{J}$ has at least one fixed point and the set of all fixed points of $\mathfrak{J}$ in $\mathfrak{D}$ is a compact set.*

Suppose $C(\mathfrak{J}, \mathfrak{E})$ denote a Banach space of all continuous functions defined on $\mathfrak{J}$ with values in $\mathfrak{E}$ equipped with the norm

$$\|y\|_c = \sup\{\|y(z)\| : z \in \mathfrak{J}\}.$$

3 The H-MNC on sequence spaces

Suppose $\mathcal{S}$ be a bounded subset of a metric space $(\mathfrak{X}, \mathfrak{d})$. Then the Hausdorff measure of non-compactness (H-MNC) on $\mathcal{S}$, denoted by $\chi(\mathcal{S})$, is defined as

$$\chi(\mathcal{S}) = \inf\left\{\varsigma > 0 : \mathcal{S} \subset \bigcup_{\mathfrak{i}=1}^{n} \mathscr{B}(\mathfrak{x}_\mathfrak{i}, \mathfrak{r}_\mathfrak{i}),\ \mathfrak{x}_\mathfrak{i} \in \mathfrak{X},\ \mathfrak{r}_\mathfrak{i} < \varsigma, 1 \le \mathfrak{i} \le n, n \in \mathbb{N}\right\},$$

where $\mathscr{B}(\mathfrak{x}_\mathfrak{i}, \mathfrak{r}_\mathfrak{i})$ is an open ball in $\mathfrak{X}$.

Consider $e^{(n)} = \{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}, \dots\}$ as a sequence such that

$$e_i^{(n)} = \begin{cases} 1, & \text{if } i = n \\ 0, & \text{if } i \ne n \end{cases} \qquad \forall n \in \mathbb{N}.$$

For $x = (x_i)$, its n-section is formulated as

$$x^{[n]} = \sum_{i=1}^{n} x_i e^{(i)}.$$

Definition 3.1. Let X be a sequence space of real or complex valued sequences. Then,

i) X is known as FK-space when

- X is a complete linear metric space, and
- $\forall\, n \in \mathbb{N}$, the map $p_n : X \to \mathbb{C}$ formulated by $p_n(x) = (x_n)$ is continuous.

ii) A FK-space together with a norm is called a BK-space.

iii) A FK-space X has AK if $\forall$ sequence $x = (x_n) \in X$ possesses a distinctive form $x = \lim_{n\to\infty} x^{[n]}$.

Lemma 3.2. *[11] Assume $\mathfrak{X}$ as a BK space with Schauder basis b_n, $\mathcal{Q} \in \mathfrak{P}_{\mathfrak{X}}$, $\mathfrak{S}_n : \mathfrak{X} \to \mathfrak{X}$, is the projection onto the linear span of $\{e_1, e_2, \cdots, e_n\}$ and $\mathcal{I}$ is the identity operator. Then*

$$\frac{1}{\mathfrak{a}} \limsup_{n\to\infty} \left(\sup_{x\in Q} ||(\mathcal{I} - \mathfrak{S}_n)(x)|| \right) \leq \tilde{f}(\mathcal{Q}) \leq \inf_n \left(\sup_{x\in\mathcal{Q}} ||(\mathcal{I} - \mathfrak{S}_n)(x)|| \right) \leq \limsup_{n\to\infty} \left(\sup_{x\in\mathcal{Q}} ||(\mathcal{I} - \mathfrak{S}_n)(x)|| \right),$$

where, $\mathfrak{a} = \limsup_{n\to\infty} ||\mathcal{I} - \mathfrak{S}_n||$.
When

$$\mathfrak{S}_n : \mathfrak{X} \to \mathfrak{X}$$

s.t.

$$\mathfrak{S}_n(x_1, x_2, \cdots) = x^{[n]} = (x_1, x_2, \cdots, x_n, 0, 0, \cdots) \quad \forall x = (x_1, x_2, \cdots) \in \mathfrak{X}$$

then

$$\tilde{f}(\mathcal{Q}) = \lim_{n\to\infty} \left(\sup_{x\in Q} ||(\mathcal{I} - \mathfrak{S}_n)(x)|| \right).$$

4 H-MNC on bv_0^ρ and cs^ρ

The sequence space bv_0 is defined by

$$bv_0 = \left\{ u = (u_m) : \sum_{m=1}^{\infty} |u_m - u_{m+1}| < \infty \right\}$$

is a BK-space with $\|u\|_{bv_0} = \sum_{m=1}^{\infty} |u_m - u_{m+1}|$. The space bv_0 is denoted by $bv \cap c_0$ and $bv_0 \subset bv$ (details in [10]). Therefore,

$$bv_0 = \left\{ u = (u_m) : \sum_{m=1}^{\infty} |u_m - u_{m+1}| < \infty \right\}.$$

Theorem 4.1. *The H-MNC on bv_0 is given by*

$$\chi(H) = \lim_{n\to\infty} \sup_{u\in H} \left\{ \sum_{m=n}^{\infty} |u_m - u_{m+1}| \right\}, \text{where } H \text{ be } \mathcal{NBS} \text{ of } bv_0.$$

Applying a fixed, non-increasing positive sequence $\rho = (\rho_n)_{n=1}^{\infty}$ known as tempering sequence, Banaś and Krajewska [15] first instituted the idea of tempered sequence space.

Suppose $\mathfrak{L}$ consists the sequences (real or complex) $z = (z_n)_{n=1}^{\infty}$ so that $\rho_n z_n \to 0$ as $n \to \infty$. Let $\mathfrak{L} := c_0^\rho$ is a Banach space with the norm

$$\|z\|_{c_0^\rho} = \sup_{n\in\mathbb{N}} \{\rho_n |z_n|\}.$$

Inspired by this construction, we have established new tempered sequence spaces bv_0^ρ and cs^ρ and discussed H-MNC on it.

Consider a set $\mathfrak{H} = bv_0^\rho$ with real or complex sequences $u = (u_m)_{m=1}^{\infty}$ so that

$$\sum_{m=1}^{\infty} \rho_m |u_m - u_{m+1}| < \infty.$$

Clearly $\mathfrak{H}$ forms a Banach space with the norm

$$\|u\|_{bv_0^\rho} = \sum_{m=1}^{\infty} \rho_m |u_m - u_{m+1}|.$$

When $\rho_m = 1 \; \forall \; m \in \mathbb{N}$ then $bv_0^\rho = bv_0$.

Proposition 4.2. *The spaces bv_0^ρ and bv_0 are isometric.*

Proof. Consider a mapping $\mathfrak{V} : bv_0^\rho \to bv_0$ defined as

$$\mathfrak{V}(u) = \mathfrak{V}(\mathfrak{u}_i)_{i=1}^{\infty} = (\rho_i \mathfrak{u}_i)_{i=1}^{\infty},$$

where $\mathfrak{u} = (\mathfrak{u}_i) \in bv_0^\rho$. For fix $\mathfrak{v} = (\mathfrak{v}_i), \mathfrak{w} = (\mathfrak{w}_i) \in bv_0^\rho$ we get

$$\begin{aligned}
\|\mathfrak{V}(\mathfrak{v}) - \mathfrak{V}(\mathfrak{w})\|_{bv_0} &= \|(\rho_i \mathfrak{v}_i)_{i=1}^{\infty} - (\rho_i \mathfrak{w}_i)_{i=1}^{\infty}\|_{bv_0} \\
&= \|\{\rho_i(\mathfrak{v}_i - \mathfrak{w}_i)\}\|_{bv_0} \\
&= \sum_{i=1}^{\infty} |\rho_i\{(\mathfrak{v}_i - \mathfrak{w}_i) - (\mathfrak{v}_{i+1} - \mathfrak{w}_{i+1})\}| \\
&= \sum_{i=1}^{\infty} \rho_i |(\mathfrak{v}_i - \mathfrak{w}_i) - (\mathfrak{v}_{i+1} - \mathfrak{w}_{i+1})| \\
&= \|\mathfrak{v} - \mathfrak{w}\|_{bv_0^\rho},
\end{aligned}$$

□

which implies that $\mathfrak{V}$ is an isometry between bv_0^ρ and bv_0.

Observing Theorem (4.1) and Proposition (4.2) $\mathcal{NBS}$ H_1 of bv_0^ρ, we have concluded that

$$\chi_{bv_0^\rho}(H_1) = \lim_{n\to\infty} \sup_{\mathfrak{u}\in H_1} \left\{ \sum_{m=n}^{\infty} \rho_m |\mathfrak{u}_m - \mathfrak{u}_{m+1}| \right\}.$$

Again, the sequence space cs is defined as

$$cs = \left\{ z = (z_i)_{i=0}^{\infty} : \lim_{n\to\infty} \sum_{i=0}^{n} z_i < \infty \right\}$$

is a sequence space with BK , AK properties and normed by $\|z\| = \sup_n |\sum_{i=0}^{n} z_i|$.

Theorem 4.3. *For any bounded subset $T \subset cs$, the H-MNC is defined as*

$$\chi_{cs}(T) = \lim_{n\to\infty} \left(\sup_{z\in T} \sup_{m} \left| \sum_{i=n}^{m} z_i \right| \right).$$

Let $\mathfrak{Z}$ contains the sequences (real or complex) $z = (z_i)_{i=1}^{\infty}$ such that $\sup_n \rho_n |\sum_{i=0}^{n} z_i| < \infty$. Suppose $\mathfrak{Z} := cs^\rho$ is a Banach space with the norm

$$\|z\|_{cs^\rho} = \sup_n \rho_n |\sum_{i=0}^{n} z_i|.$$

Proposition 4.4. *The spaces cs^ρ and cs are isometric.*

Proof. Consider a mapping $\tilde{f} : cs^{\rho} \to cs$ defined as

$$\tilde{f}(z) = \tilde{f}(z_i)_{i=1}^{\infty} = (\rho_i z_i)_{i=1}^{\infty},$$

where $z = (z_i) \in cs^{\rho}$. Let $\mathfrak{x} = (\mathfrak{x}_i), \mathfrak{y} = (\mathfrak{y}_i) \in cs^{\rho}$ we get

$$\begin{aligned}\|\tilde{f}(\mathfrak{x}) - \tilde{f}(\mathfrak{y})\|_{cs} &= \|\rho_i \mathfrak{x}_i - \rho_i \mathfrak{y}_i\|_{cs} \\ &= \|\{\rho_i(\mathfrak{x}_i - \mathfrak{y}_i)\}\|_{cs} \\ &= \sup_n \left| \sum_{i=0}^{n} \rho_i(\mathfrak{x}_i - \mathfrak{y}_i) \right| \\ &= \|\mathfrak{x} - \mathfrak{y}\|_{cs^{\rho}}.\end{aligned}$$

□

∴ *From Theorem (4.3) and Proposition (4.4), we have*

$$\chi_{cs^{\rho}}(T_1) = \lim_{n\to\infty} \left(\sup_{z\in T_1} \sup_{m\geq n} \rho_m \left| \sum_{i=n}^{m} z_i \right| \right),$$

where T_1 be the $\mathcal{NBS}$ of cs^{ρ}.

5 Solution in bv_0^{ρ}

Our assumptions are as below:

(A.) The functions $(h_k)_{k=1}^{\infty}$ on $I^3 \times \mathbb{R}^{\infty}$ are continuous and real-valued. Define the operator $Y : I^3 \times bv_0^{\rho} \to bv_0^{\rho}$ as

$$(Yx)(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) = (h_1(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, x), h_2(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, x), \dots)$$

maps $I^3 \times bv_0^{\rho}$ to bv_0^{ρ}. The family of functions $(Yx)(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})$ hold equicontinuity at all points of the space bv_0^{ρ}.

(B.) $c_i(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})$ and $d_i(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})$ are continuous functions on I^6 such that

$$\begin{aligned}|K_n h_n - K_{n+1} h_{n+1}| \quad \leq \quad & c_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) + d_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\ & |x_n(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) - x_{n+1}(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})|.\end{aligned}$$

Here the function

$$\sum_{n=1}^{\infty} \rho_n c_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})$$

is uniformly convergent on I^6. $d_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})_{n\in\mathbb{N}}$ is equibounded on I^6. The function $c(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})$ is given by

$$c(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) = \sum_{n=1}^{\infty} \rho_n c_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}).$$

Consider

$$\mathcal{D} = \sup\{d_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) : (\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \in I^6, n \in \mathbb{N}\}.$$

$$\mathcal{C} = \sup\{c(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) : (\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \in I^6, n \in \mathbb{N}\}.$$

(C.) The functions $K_n : I^6 \to \mathbb{N}$ are continuous throughout the entire domain I^6 where $n \in \mathbb{N}$. Again, $K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})$ exhibit equicontinuity w.r.t. the variables $(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})$, also $K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})$s are equibounded on I^6. The constant $\mathcal{K}$ is defined as

$$\mathcal{K} = \sup\left\{|K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})|\right\}$$

(D.) The functions $r_n : I^3 \to \mathbb{R}$ *are continuous. Again,* $\mathcal{R}$ *is given by*

$$\mathcal{R} = \sum_{n=1}^{\infty} |r_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) - r_{n+1}(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})|.$$

Theorem 5.1. *On account of assumptions (A)-(D), the system (1.1) has a solution* $x(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) = x_k(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})_{k=1}^{\infty}\ bv_0^\rho$ *for a fixed* $(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) \in I^3$*, whenever* $(\hat{\nu} - \hat{\mu})^3 \mathcal{D} < 1$.

Proof. First, we consider an operator $\mathfrak{Y}$ on $C(I^3, bv_0^\rho)$ as -

$$\begin{aligned}
(\mathfrak{Y}x)(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) &= ((\mathfrak{Y}x)_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})) \\
&= r_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) + \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\
&\quad h_n(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, x(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}))\, d\overline{\psi}_{11}\, d\overline{\psi}_{22}\, d\overline{\psi}_{33} \\
&= \Bigg(r_1(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) + \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} K_1(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\
&\quad h_1(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, x(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}))\, d\overline{\psi}_{11}\, d\overline{\psi}_{22}\, d\overline{\psi}_{33} + \dots \Bigg).
\end{aligned} \tag{5.1}$$

Now, we will show that $\mathfrak{Y}$ self-maps on the space $C(I^3, bv_0^\rho)$. Let $n \in \mathbb{N}$ and $(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) \in I^3$. Then by using assumptions (A) and (C), we get

$$\begin{aligned}
&\|(\mathfrak{Y}x)_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})\|_{bv_0^\rho} \\
&= \sum_{n=1}^{\infty} \rho_n |(\mathfrak{Y}x)_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) - (\mathfrak{Y}x)_{n+1}(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})| \\
&= \sum_{n=1}^{\infty} \rho_n \Bigg| r_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) + \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\
&\quad h_n(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, x(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}))\, d\overline{\psi}_{11}\, d\overline{\psi}_{22}\, d\overline{\psi}_{33} \\
&\quad - r_{n+1}(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) - \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} K_{n+1}(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\
&\quad h_{n+1}(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, x(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}))\, d\overline{\psi}_{11}\, d\overline{\psi}_{22}\, d\overline{\psi}_{33} \Bigg| \\
&\leq \mathcal{R} + \sum_{n=1}^{\infty} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} \rho_n \Bigg| K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\
&\quad h_n(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, x(\overline{\psi}_{11}, \overline{\psi}_{22}, \xi_3)) - K_{n+1}(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\
&\quad h_{n+1}(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, x(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})) \Bigg| \, d\overline{\psi}_{11}\, d\overline{\psi}_{22}\, d\overline{\psi}_{33} \\
&\leq \mathcal{R} + \sum_{n=1}^{\infty} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} \rho_n \{ |c_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})| + d_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\
&\quad |x_n(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) - x_{n+1}(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})| \} d\overline{\psi}_{11} d\overline{\psi}_{22} d\overline{\psi}_{33}.
\end{aligned}$$

Using Lebesgue monotone convergence theorem **[8]**, we get-

$$
\begin{aligned}
&|(\mathfrak{Y}x)_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})|_{bv_0^\rho}\\
&\leq \mathcal{R}+\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\sum_{n=1}^{\infty}\rho_n\{|c_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})|+d_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})\\
&\quad |x_n(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})-x_{n+1}(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})|\}\\
&\leq \mathcal{R}+\mathcal{C}(\hat{\nu}-\hat{\mu})^3+\mathcal{D}(\hat{\nu}-\hat{\mu})^3\sup\{||x||_{bv_0^\rho}\}<\infty. \qquad (5.2)
\end{aligned}
$$

Thus $(\mathfrak{Y}x)(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})\in bv_0^\rho\ \forall\ (\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})\in I^3$.
Again,

$$\|x(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})\|_{bv_0^\rho}\leq \mathcal{R}+\mathcal{C}(\hat{\nu}-\hat{\mu})^3+\mathcal{D}(\hat{\nu}-\hat{\mu})^3||x||_{bv_0^\rho}.$$

As $\mathcal{D}(\hat{\nu}-\hat{\mu})^3<1$, so, we have -

$$\|x(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})\|_{bv_0^\rho}\leq \frac{\mathcal{R}+(\hat{\nu}-\hat{\mu})^3\mathcal{C}}{\{1-(\hat{\nu}-\hat{\mu})^3\mathcal{D}\}}=\hat{r}.$$

Therefore, using (5.2), we conclude that $\mathfrak{Y}$ is self-map on $C(I^3,bv_0^\rho)$.

Also,

$$||(\mathfrak{Y}x)(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})-0||\leq\hat{r},$$

which implies $\mathfrak{Y}$ maps to a ball with radius $\hat{r}$ and center at the origin, $\mathfrak{B}_{\hat{r}}$ which belongs to $C(I^3,bv_0^\rho)$.

Next, we will check continuity of $\mathfrak{Y}$ on $\mathfrak{B}_{\hat{r}}$. Consider a fix $\epsilon_1>0$ and $y\in\mathfrak{B}_{\hat{r}}$. Choose $\bar{y}\in\mathfrak{B}_{\hat{r}}$ s.t. $|y-\bar{y}|<\epsilon_1$ for arbitrary fix $(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})\in I^3$ and $n\in\mathbb{N}$, we get

$$
\begin{aligned}
&\|(\mathfrak{Y}y)(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})-(\mathfrak{Y}\bar{y})(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})\|\\
&\leq\sum_{n=1}^{\infty}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\rho_n|K_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})|\\
&\quad|h_n(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},y(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}))-h_n(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},\bar{y}(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}))|d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33}\\
&\leq\mathcal{K}\sum_{n=1}^{\infty}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\rho_n\{|h_n(\xi_1,\overline{\psi}_{22},\overline{\psi}_{33},y(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}))\\
&-h_n(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},\bar{y}(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}))|\}d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33}. \qquad (5.3)
\end{aligned}
$$

Now, by condition (A), define $\delta_1(\epsilon_1)$ as -

$$\delta_1(\epsilon_1)=\sup\left\{|h_n(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},y)-h_n(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},\bar{y})|:\bar{y},y\in\mathfrak{B}_{\hat{r}},||y-\bar{y}|<\epsilon_1\right\},$$

so that $\delta_1(\epsilon_1)\to 0$ whenever $\epsilon_1\to 0$.

$\therefore$ Using the Lebesgue monotone convergence theorem **[8]**, equation (5.3) and assumption (C), we obtain

$$\|(\mathfrak{Y}y)-(\mathfrak{Y}\bar{y})\| \quad\leq\quad \mathcal{K}(\hat{\nu}-\hat{\mu})^3\delta_1(\epsilon_1),$$

which prove the continuity of $\mathfrak{Y}$.

Lastly, we will show that that $\mathfrak{Y}$ is a M-K Condensing Operator, i.e. for $\epsilon>0$ and $\delta>0$, we have

$$\chi(\mathfrak{B}_{\hat{r}})\in[\epsilon,\epsilon+\delta)\Rightarrow\chi(\mathfrak{Y}(\mathfrak{B}_{\hat{r}}))<\epsilon.$$

Using H-MNC on bv_0^ρ and the assumptions (B) - (D)-

$$\begin{aligned}
\chi(\mathfrak{Y}(\mathfrak{B}_{\hat{r}})) &= \lim_{n\to\infty} \sup_{v\in\mathfrak{B}_{\hat{r}}} \left(\sum_{k\geq n} \rho_k |x_k - x_{k+1}| \right) \\
&= \lim_{n\to\infty} \sup_{v\in\mathfrak{B}_{\hat{r}}} \left(\sum_{k\geq n} \rho_k \Bigg| r_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) + \int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}} K_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) \right. \\
&\quad h_k(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},x(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}))\, d\overline{\psi}_{11}\, d\overline{\psi}_{22}\, d\overline{\psi}_{33} \\
&\quad - r_{k+1}(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) - \int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}} K_{k+1}(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) \\
&\quad \left. h_{k+1}(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},x(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}))\, d\overline{\psi}_{11}\, d\overline{\psi}_{22}\, d\overline{\psi}_{33} \Bigg| \right) \\
&\leq \lim_{n\to\infty} \sup_{v\in\mathfrak{B}_{\hat{r}}} \left(\sum_{k\geq n} \int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}} \rho_k |c_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) + d_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) \right. \\
&\quad \left. |x_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) - x_{k+1}(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})|\, d\overline{\psi}_{11}\, d\overline{\psi}_{22}\, d\overline{\psi}_{33} \right) \\
&\leq \lim_{n\to\infty} \sup_{v\in\mathfrak{B}_{\hat{r}}} \left(\sum_{k\geq n} \int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}} \rho_k c_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) + d_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) \right. \\
&\quad \left. \rho_k |x_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) - x_{k+1}(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})|\, d\overline{\psi}_{11}\, d\overline{\psi}_{22}\, d\overline{\psi}_{33} \right) \\
&\leq (\hat{\nu}-\hat{\mu})^3 \mathcal{D} \chi(\mathfrak{B}_{\hat{r}}).
\end{aligned}$$

Thus,

$$\chi(\mathfrak{Y}(\mathfrak{B}_{\hat{r}})) \leq \mathcal{D}(\hat{\nu}-\hat{\mu})^3 \chi(\mathfrak{B}_{\hat{r}}) < \epsilon \Rightarrow \chi(\mathfrak{B}_{\hat{r}}) < \frac{\epsilon}{(\hat{\nu}-\hat{\mu})^3\mathcal{D}}.$$

Taking $\delta = \dfrac{\epsilon[1-(\hat{\nu}-\hat{\mu})^3\mathcal{D}]}{(\hat{\nu}-\hat{\mu})^3\mathcal{D}}$, we get $\epsilon \leq \chi(\mathfrak{B}_{\hat{r}}) < \epsilon + \delta$.

Hence $\mathfrak{Y}$ is an M-K condensing operator on $\mathfrak{B}_{\hat{r}} \subset bv_0^\rho$.

Therefore, we notice that $\mathfrak{Y}$ fulfills all the conditions of theorem (2.6). Thus $\mathfrak{Y}$ has a fixed point on $\mathfrak{B}_{\hat{r}}$ which behaves as a solution of the system of equations (1.1). Therefore, integral equation (1.1) has a solution in bv_0^ρ. □

Example 5.2. To illustrate our result, we examine the following Hammerstein type integral equation

$$\begin{aligned}
x_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) = \frac{1}{n}\sin(\overline{\phi}_{11}\overline{\phi}_{22}\overline{\phi}_{33}) + \int_1^2\int_1^2\int_1^2 \frac{(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33}+\overline{\phi}_{11}+\overline{\phi}_{22}+\overline{\phi}_{33})}{3n} \\
\left(\frac{\cos(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})}{4n} \sum_{k=n}^{\infty} (x_k - x_{k+1}) \right) d\overline{\psi}_{11}\, d\overline{\psi}_{22}\, d\overline{\psi}_{33}, \quad (5.4)
\end{aligned}$$

for $(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) \in [1,2]\times[1,2]\times[1,2]$, $n\in\mathbb{N}$ and let $I_3 = [1,2]$.

Comparing (5.4) to (1.1) we have

$$\begin{aligned} r_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) &= \frac{1}{n}\sin(\overline{\phi}_{11}\overline{\phi}_{22}\overline{\phi}_{33}), \\ K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) &= \frac{(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33}+\overline{\phi}_{11}+\overline{\phi}_{22}+\overline{\phi}_{33})}{3n}, \\ h_n(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, x(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})) &= \left(\frac{\cos(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})}{4n} + \sum_{k=n}^{\infty}(x_k - x_{k+1})\right), \\ \rho_n &= \frac{1}{n}. \end{aligned}$$

By observing Theorem (5.1), we see that all the assumptions are satisfied. Define the operator F_1 as

$$(F_1x)(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) = h_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, x(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}))$$

that maps $I_3^3 \times bv_0^\rho$ into bv_0^ρ. Now, we will show that $\{(F_1x)(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})\}$ is equicontinuous at arbitrary $x \in bv_0^\rho$.

Fix $\epsilon > 0$, $n \in \mathbb{N}$, and $(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) \in I_3^3$, suppose $y \in bv_0^\rho$ so that $||x-y||_{bv_0^\rho} < \epsilon$.

$$\begin{aligned} |h_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, x) - h_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, y)| &= \left|\sum_{m=n}^{\infty}(x_m - x_{m+1}) - \sum_{m=n}^{\infty}(y_m - y_{m+1})\right| \\ &\leq \sum_{m=n}^{\infty}|(x_m - x_{m+1}) - (y_m - y_{m+1})| \\ &\leq ||x-y|| \\ &< \epsilon. \end{aligned}$$

Hence, $\{(F_1x)(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})\}$ is equicontinuous.

The functions $K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})$ are continuous on I_3^3 and sequence of functions $K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})$ are equicontinuous on I_3^3. Also,

$$\mathcal{K} = \sup|K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})| = 0.$$

Again, consider fix $\epsilon > 0$, $(x, y) \in I_3^3$, $n \in \mathbb{N}$ then for any arbitrary $(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})$, $(\bar{\bar{\phi}}_1, \bar{\bar{\phi}}_2, \bar{\bar{\phi}}_3) \in I_3^3$ with

$$|\bar{\bar{\phi}}_1 - \overline{\phi}_{11}| < \epsilon, |\bar{\bar{\phi}}_2 - \overline{\phi}_{22}| < \epsilon, |\bar{\bar{\phi}}_3 - \overline{\phi}_{33}| < \epsilon.$$

We have,

$$\begin{aligned} |K_n(\bar{\bar{\phi}}_1, \bar{\bar{\phi}}_2, \bar{\bar{\phi}}_3, x, y, z) - K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, x, y, z)| &\leq \left|\frac{\bar{\bar{\phi}}_1+\bar{\bar{\phi}}_2+\bar{\bar{\phi}}_3+x+y+z}{3n}\right. \\ &- \left.\frac{\overline{\phi}_{11}+\overline{\phi}_{22}+\overline{\phi}_{33}+x+y+z}{3n}\right| \\ &\leq \frac{1}{3n}|(\bar{\bar{\phi}}_1 - \overline{\phi}_{11}) + (\bar{\bar{\phi}}_2 - \overline{\phi}_{22}) + (\bar{\bar{\phi}}_3 - \overline{\phi}_{33})| \\ &\leq \frac{1}{3n}(|\bar{\bar{\phi}}_1 - \overline{\phi}_{11}| + |\bar{\bar{\phi}}_2 - \overline{\phi}_{22}| + |\bar{\bar{\phi}}_3 - \overline{\phi}_{33}|) \\ &\leq \epsilon. \end{aligned}$$

Therefore, $K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, x, y, z)$ is equicontinuous.

Now,

$$|k_k h_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},x) - k_{k+1}h_{k+1}(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},x)|$$
$$\leq \left|\frac{(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33}+\overline{\phi}_{11}+\overline{\phi}_{22}+\overline{\phi}_{33})\cos(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})(2k+1)}{4k^2(k+1)^2}\right|$$
$$+\frac{\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33}+\overline{\phi}_{11}+\overline{\phi}_{22}+\overline{\phi}_{33}}{3k(k+1)}\sum_{m=n}^{\infty}|x_m - x_{m+1}|$$
$$= c_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) + d_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})|x_k - x_{k+1}|.$$

Assuming

$$c_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) = \frac{(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33}+\overline{\phi}_{11}+\overline{\phi}_{22}+\overline{\phi}_{33})\cos(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})(2k+1)}{4k^2(k+1)^2}$$
$$d_k(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) = \frac{\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33}+\overline{\phi}_{11}+\overline{\phi}_{22}+\overline{\phi}_{33}}{3k(k+1)},$$

it is clear that c_n and d_n are real-valued functions. In addition, $\mathcal{D} = \sup\{d_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})\} = 0$, $\forall n \in \mathbb{N}$, and $\sum_{n=1}^{\infty}\rho_n c_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})$ converges uniformly to $\frac{6\zeta(3)-12+\pi^2}{24} \approx 0.21175$ on I_3^6, where $\zeta(x)$ is a zeta function. Again we have $(\hat{\nu}-\hat{\mu})^3\mathcal{D} < 1$,therefore,e by the theorem (5.1) we can conclude that the above example (5.2) contains a solution in bv_0^ρ.

6 Solution in cs^ρ

Assume the following suppositions:

i. *$(h_k)_{k=1}^{\infty}$ on $I^3 \times \mathbb{R}^\infty$ are continuous and real-valued functions. Define the operator $Z : I^3 \times cs^\rho \to cs^\rho$ as*

$$(Zx)(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) = h_1(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},x), h_2(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},x),\ldots$$

   maps $I^3 \times cs^\rho$ to cs^ρ. The family of functions $(Zx)(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})$ hold equicontinuity at all points of the space cs^ρ.

ii. *Suppose $j_i(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})$ and $l_i(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})$ are continuous functions on I^3 so that*

$$|h_n(\overline{\phi}_{11}\overline{\phi}_{22},\overline{\phi}_{33},x(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}))| \leq j_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) + l_n(\overline{\phi}_{11}\overline{\phi}_{22},\overline{\phi}_{33})|x_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})|.$$

   Here, $(j_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}))_{n\in\mathbb{N}}$, $(l_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}))_{n\in\mathbb{N}}$ are real-valued continuous functions on I^3. The function

$$\sum_{n=1}^{\infty}\rho_n j_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})$$

   is uniformly convergent on I^3. $l_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})_{n\in\mathbb{N}}$ is equibounded on I^3. The function $j(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})$ is given by

$$j(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) = \sum_{n=1}^{\infty}\rho_n j_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}).$$

   Consider

$$\mathcal{L} = \sup\{l_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) : (\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) \in I^3, n \in \mathbb{N}\}.$$
$$\mathcal{J} = \max\{j_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) : (\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) \in I^3, n \in \mathbb{N}\}.$$

iii. *The functions* $K_n : I^6 \to \mathbb{N}$ *are continuous on* I^6. *Again,* $K_n(\overline{\phi}_{11}\overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})$ *exhibit equicontinuity with respect to the variables* $(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})$, *and equibounded on* I^6. *Consider* $\mathfrak{K}$ *as*

$$\mathfrak{K} = \sup\left\{|K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})| : \overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33} \in I^3\right\} < \infty.$$

iv. *The functions* $r_n : I^3 \to \mathbb{R}$ *are continuous. Also, the sequence of functions* (r_n) *converge uniformly to 0 . Again,*

$$\mathfrak{R} = \sup\{|r_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})| : (\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})\}.$$

Theorem 6.1. *On account of assumptions (i) - (iv), the system of equations (1.1) has a solution* $x(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) = (x_k(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}))_{k=1}^{\infty}$ *in* $C(I^3, cs^\rho)$ *for a fixed* $(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) \in I^3$ *, whenever* $(\hat{\nu} - \hat{\mu})^3 \mathfrak{K}\mathcal{L} < 1$.

Proof. First, we consider an operator $\mathfrak{U}$ on the space $C(I^3, cs^\rho)$ by -

$$\begin{aligned}
(\mathfrak{U}x)(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) &= (\mathfrak{U}x)_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) \\
&= r_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) + \int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}} K_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\
&\quad h_n(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, x(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}))d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33} \\
&= \Bigg\{ r_1(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) + \int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}} K_1(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\
&\quad h_1(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, x(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}))d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33} + \cdots \Bigg\}. \qquad (6.1)
\end{aligned}$$

Now we will show that $\mathfrak{U}$ is self mapped on $C(I^3, cs^\rho)$. For $n \in \mathbb{N}$, $(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) \in I^3$, from assumptions (i) and (iii) we have-

$$\begin{aligned}
|(\mathfrak{U}x)_n(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})|_{cs^\rho} &= \sup_n \rho_n \left|\sum_{i=0}^{n} (\mathfrak{U}x)_i\right| \\
&\leq \sup_n \rho_n \Bigg| \sum_{i=0}^{n} r_i(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) + \sum_{i=0}^{n} \int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}} K_i(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\
&\quad h_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, x(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}))d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33} \Bigg| \\
&\leq \mathfrak{R} + \sup_n \rho_n \int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}} \Bigg| \sum_{i=0}^{n} K_i(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\
&\quad h_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, x(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}))d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33} \Bigg| \\
&\leq \mathfrak{R} + \mathfrak{K} \sup_n \int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}} \Bigg| \sum_{i=0}^{n} \Bigg(\rho_i j_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) + l_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\
&\quad \rho_i |x_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})| \Bigg) d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33} \Bigg| \\
&\leq \mathfrak{R} + \mathfrak{K} \sup_n \int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}} |\sum_{i=0}^{n} \rho_i j_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})| d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33} \\
&+ \mathfrak{K}\mathfrak{L} \sup_n \int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}}\int_{\hat{\mu}}^{\hat{\nu}} \rho_i |\sum_{i=0}^{n} x_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})| d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33},
\end{aligned}$$

thus

$$\sum_{i=0}^{n} |(\mathfrak{U}x)_i(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})| \leq \mathfrak{R} + \mathfrak{K}(\hat{\nu} - \hat{\mu})^3\{\mathfrak{J} + \mathfrak{L}\|x\|_{cs^\rho}\} < \infty,$$

and hence $\lim_{n\to 0} \sum_{i=0}^{n} |(\mathfrak{U}x)_i(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})|$ exists. Thus the operator $(\mathfrak{U}x)(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) \in cs^\rho$ for arbitrary $(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) \in I^3$.

Further,

$$\begin{aligned} \|x(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})\|_{cs^\rho} &= \sup_n \rho_n \Bigg| \sum_{i=1}^{n} r_i(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) + \sum_{i=0}^{n} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} K_i(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}) \\ &\quad h_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, x(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}))d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33}\Bigg| \\ &\leq \mathfrak{R} + \mathfrak{K}(\hat{\nu} - \hat{\mu})^3\{\mathfrak{J} + \mathfrak{L}\|x\|_{cs^\rho}\}. \end{aligned}$$

Since $\mathfrak{K}\mathfrak{L}(\hat{\nu} - \hat{\mu})^3 < 1$ we have -

$$\|x(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})\|_{cs^\rho} \leq \frac{\mathfrak{R} + \mathfrak{K}(\hat{\nu} - \hat{\mu})^3\mathfrak{J}}{1 - \mathfrak{K}(\hat{\nu} - \hat{\mu})^3\mathfrak{L}} = \acute{r}.$$

Thus we can say that $\mathfrak{U}$ is a self-map on $C(I^3, cs^\rho)$. Again $|(\mathfrak{U}x)(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})| < \bar{r}$, which shows that the operator $\mathfrak{U}$ is self mapped into the ball $\mathcal{B}_{\acute{r}}$ with centered at the origin and radius of $\acute{r}$.

Next, we will investigate continuity of $\mathfrak{U}$ on $\mathcal{B}_{\acute{r}}$. Fix $\epsilon_2 > 0$ and $b \in \mathcal{B}_{\acute{r}}$. Choose $a \in \mathcal{B}_{\acute{r}}$ such that $|a - b| < \epsilon_2$. Then for arbitrary fixed $(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) \in I^3$ and $n \in \mathbb{N}$, we get -

$$\begin{aligned} &\|(\mathfrak{U}b)(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) - (\mathfrak{U}a)(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33})\|_{cs^\rho} \\ &\leq \sup_n \rho_n \Bigg| \sum_{i=1}^{n} \Bigg\{ \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} K_i(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}, \overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})\{h_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, b(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})) \\ &- h_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, a(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}))\}d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33}\Bigg\}\Bigg| \\ &\leq \mathfrak{K} \sup_n \rho_n \Bigg| \sum_{i=1}^{n} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} \int_{\hat{\mu}}^{\hat{\nu}} h_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, b(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33})) \\ &- h_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, a(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}))d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33}\Bigg|. \end{aligned}$$

Now, by assumption (i), define the set $\delta_2(\epsilon_2)$ as -

$$\delta_2(\epsilon_2) = \sup\Bigg\{|h_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, b) - h_i(\overline{\psi}_{11}, \overline{\psi}_{22}, \overline{\psi}_{33}, a)| : a, b \in \mathfrak{B}_{\acute{r}}, ||b - a||_{cs^\rho} \in \epsilon_2\Bigg\},$$

whenever $\epsilon_2 \to 0$, $\delta_2(\epsilon_2) \to 0$.

Applying (iii) and the Lebesgue monotone convergence theorem **[8]**, we deduce -

$$\begin{aligned}\|(\mathfrak{U}b)(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) &- (\mathfrak{U}a)(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})\|_{cs^\rho}\\ &\leq \mathfrak{K}\sup_n \rho_n\left|\sum_{i=1}^{n}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\{h_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},b(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}))\right.\\ &- \left.h_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},a(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}))\}d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33}\right|\\ &\leq \mathfrak{K}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\int_{\hat{\nu}}^{\hat{\hat{\mu}}}|h_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},b)-h_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},a)|_{cs^\rho}d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33}\\ &\leq \mathfrak{K}(\hat{\nu}-\hat{\hat{\mu}})^3\delta_2(\epsilon_2).\end{aligned}$$

Taking the supremum $\forall\ (\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})\in I^3$,

$$\|(\mathfrak{U}b)(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})-(\mathfrak{U}a)(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})\|_{C(I^3,cs^\rho)}\leq\mathfrak{K}(\hat{\nu}-\hat{\hat{\mu}})^3\delta_2(\epsilon_2). \tag{6.2}$$

Since , the equation (6.2) holds for arbitrary $(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})$, hence, we can conclude that $\mathfrak{U}$ is a continuous operator on $\mathcal{B}_{\acute{r}}$.

To prove $\mathfrak{U}$ as a M-K condensing operator, we will show that $\exists\ \epsilon>0$ and $\delta>0$ so that

$$\chi(\mathcal{B}_{\acute{r}})\in[\epsilon,\epsilon+\delta)\Rightarrow\chi(\mathfrak{U}(\mathfrak{B}_{\acute{r}}))<\epsilon.$$

Considering assumtions (ii), (iii), and (iv), we get

$$\begin{aligned}\chi(\mathfrak{U}(\mathfrak{B}_{\acute{r}})) &= \lim_{n\to\infty}\left[\sup_{x\in\mathfrak{B}_{\acute{r}}}\left(\rho_n\left|\sum_{i\geq n}x_i\right|\right)\right]\\ &\leq \lim_{n\to\infty}\left[\sup_{z\in\mathfrak{B}_{\acute{r}}}\left(\rho_n\left|\sum_{i\geq n}r_i(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})+\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}K_i(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})\right.\right.\right.\\ &\quad\left.\left.\left. h_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},x(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}))d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33}\right|\right)\right]\\ &\leq \lim_{n\to\infty}\left[\sup_{z\in\mathfrak{B}_{\acute{r}}}\left(\rho_n\left|\sum_{i\geq n}r_i(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})\right|+\rho_n\left|\sum_{i\geq n}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}K_i(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})\right.\right.\right.\\ &\quad\left.\left.\left. h_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},x(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}))d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33}\right|\right)\right]\\ &\leq \lim_{n\to\infty}\left[\sup_{z\in\mathfrak{B}_{\acute{r}}}\left(\mathfrak{K}\rho_n\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\left|\sum_{i\geq n}h_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},x(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})\right|d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33}\right)\right]\\ &\leq \lim_{n\to\infty}\left[\sup_{z\in\mathfrak{B}_{\acute{r}}}\left(\mathfrak{K}\rho_n\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}\int_{\hat{\hat{\mu}}}^{\hat{\nu}}|\sum_{i\geq n}j_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})|\right.\right.\\ &+ \left.\left. l_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})|x_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})|d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33}\right)\right]\\ &\leq \mathfrak{K}\mathfrak{L}(\hat{\nu}-\hat{\hat{\mu}})^3\chi(\mathfrak{B}_{\acute{r}}).\end{aligned}$$

Therefore,

$$\chi(\mathfrak{U}(\mathfrak{B}_{\acute{r}}))\leq\mathfrak{K}\mathfrak{L}(\hat{\nu}-\hat{\hat{\mu}})^3\chi(\mathfrak{B}_{\acute{r}})<\epsilon\Rightarrow\chi(\mathfrak{B}_{\acute{r}})<\frac{\epsilon}{\mathfrak{K}\mathfrak{L}(\hat{\nu}-\hat{\hat{\mu}})^3}.$$

By assuming

$$\delta = \frac{\epsilon[1-(\hat{\nu}-\hat{\hat{\mu}})^3\mathfrak{K}\mathfrak{L}]}{(\hat{\nu}-\hat{\hat{\mu}})^3\mathfrak{K}\mathfrak{L}},$$

we get

$$\epsilon \leq \chi(\mathfrak{B}_{\acute{r}}) < \epsilon + \delta.$$

Hence, $\mathfrak{U}$ is a M-K condensing operator on $\mathfrak{B}_{\acute{r}} \in cs^\rho$. Thus, by theorem (2.6), $\mathfrak{U}$ holds a fixed point on $\mathfrak{B}_{\acute{r}}$, which performs as a solution of (1.1) in cs^ρ. □

Example 6.2. To illustrate our result we examine the below Hammerstein IE

$$x_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) = \frac{1}{n}\arctan(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) + \int_2^3\int_2^3\int_2^3 \cos\left(\frac{\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33}+\overline{\phi}_{11}+\overline{\phi}_{22}+\overline{\phi}_{33}}{n}\right) \left(\frac{1+(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})^2\sum_{i=n}^{\infty} x_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]}\right) d\overline{\psi}_{11}d\overline{\psi}_{22}d\overline{\psi}_{33}, \tag{6.3}$$

for $(\psi_1,\psi_2,\psi_3) \in [2,3]\times[2,3]\times[2,3]$, $n \in \mathbb{N}$ and let $I_4 = [2,3]$.

Comparing (6.3) to (1.1) we have

$$\begin{aligned} r_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) &= \frac{1}{n}\arctan(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}),\\ K_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) &= \cos\left(\frac{\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33}+\overline{\phi}_{11}+\overline{\phi}_{22}+\overline{\phi}_{33}}{n}\right),\\ h_n(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},x(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})) &= \left(\frac{1+(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})^2\sum_{i=n}^{\infty} x_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]}\right). \end{aligned}$$

By observing Theorem (6.1), we see that all the assumptions are satisfied. Again, the operator F_2 defined by

$$(F_2x)(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) = (h_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},x(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})))$$

which maps $I_3^4 \times cs^\rho$ into cs^ρ. Now, we will show that $\{(F_2x)(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})\}$ is equicontinuous at arbitrary $x \in cs^\rho$. Fix $\epsilon > 0$, $n \in \mathbb{N}$ and $(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33}) \in I_3^4$. Suppose $y \in cs^\rho$ so that $\|x-y\|_{cs^\rho} < \epsilon$. Then

$$\begin{aligned} |h_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},x) - h_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},y)| &= \left|\frac{(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})^2}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]}\sum_{i=n}^{\infty}\{x_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})\right.\\ &\quad \left. - y_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})\}\right|\\ &\leq \left|\frac{(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})^2}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]}\right|\left|\sum_{i=n}^{\infty}\{x_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})\right.\\ &\quad \left. - y_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})\}\right|\\ &\leq \frac{1}{3}\sup_n\left|\sum_{i=n}^{\infty} x_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) - y_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})\right|\\ &\leq \frac{1}{3}\|x-y\|_{cs^\rho}\\ &\leq \frac{\epsilon}{3}. \end{aligned}$$

Hence, $\{(F_2x)(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})\}$ *is equicontinuous.*

The functions $K_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})$ *are continuous on* $I_4^3=[2,3]\times[2,3]\times[2,3]$ *and the sequence of functions* $K_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})$ *are equicontinuous on* I_4^3*. Also,*

$$\mathfrak{K}=\sup\left\{|K_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}|:\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}\in I_4^3\right\}=1.$$

Again, consider a fix $\epsilon>0$, $(x,y)\in I_4^3$, $n\in\mathbb{N}$*. Then for any arbitrary* $(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})$ *and* $(\bar{\bar{\phi}}_1,\bar{\bar{\phi}}_2,\bar{\bar{\phi}}_3)\in I_4^3$ *with*

$$|\bar{\bar{\phi}}_1-\overline{\phi}_{11}|<\frac{\epsilon}{3},|\bar{\bar{\phi}}_2-\overline{\phi}_{22}|<\frac{\epsilon}{3},|\bar{\bar{\phi}}_3-\overline{\phi}_{33}|<\frac{\epsilon}{3},$$

we have,

$$\begin{aligned}|K_n(\bar{\bar{\phi}}_1,\bar{\bar{\phi}}_2,\bar{\bar{\phi}}_3,p,q,r)-K_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},p,q,r)| &\le \left|\frac{\bar{\bar{\phi}}_1+\bar{\bar{\phi}}_2+\bar{\bar{\phi}}_3+p+q+r}{3n}\right.\\ &- \left.\frac{\overline{\phi}_{11}+\overline{\phi}_{22}+\overline{\phi}_{33}+p+q+r}{3n}\right|\\ &\le \frac{1}{3n}|\bar{\bar{\phi}}_1-\overline{\phi}_{11}+\bar{\bar{\phi}}_2-\overline{\phi}_{22}+\bar{\bar{\phi}}_3-\overline{\phi}_{33}|\\ &\le \frac{1}{3n}(|\bar{\bar{\phi}}_1-\overline{\phi}_{11}|+|\bar{\bar{\phi}}_2-\overline{\phi}_{22}|+|\bar{\bar{\phi}}_3-\overline{\phi}_{33}|)\\ &\le \epsilon.\end{aligned}$$

Therefore, $K_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33},p,q,r)$ *are equicontinous.*

Fixing $(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})\in I_4^3$, $x\in cs^\rho$*, and* $n\in\mathbb{N}$*, we get*

$$\begin{aligned}|h_n(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33},x)| &= \left|\frac{1+(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})^2\sum_{i=n}^{\infty}x_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]}\right|\\ &= \left|\frac{1}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]}+\frac{(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})^2\sum_{i=n}^{\infty}x_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]}\right|\\ &\le \frac{1}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]}+\frac{(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})^2|\sum_{i=n}^{\infty}x_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})|}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]}\\ &\le \frac{1}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]}+\frac{(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})^2}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]}|\sum_{i=n}^{\infty}x_i(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33})|\end{aligned}$$

Here,

$$\begin{aligned}j_n(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) &= \frac{1}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]},\\ l_n(\overline{\psi}_{11},\overline{\psi}_{22},\overline{\psi}_{33}) &= \frac{(\overline{\psi}_{11}\overline{\psi}_{22}\overline{\psi}_{33})^2}{3[n^2+(\overline{\psi}_{11}+\overline{\psi}_{22}+\overline{\psi}_{33})^4]}.\end{aligned}$$

Clearly both j_n *and* l_n *are real-valued continuous functions on* I_4^3. $\sum_{n=1}^{\infty}\rho_n j_n$ *is uniformly convergent on* I_4^3*.*
Moreover,

$$|l_n(\overline{\phi}_{11},\overline{\phi}_{22},\overline{\phi}_{33})|\le\frac{1}{3},\quad\forall n\in\mathbb{N},$$

and

$$\mathfrak{L} = \sup_n \frac{(\overline{\phi}_{11}\overline{\phi}_{22}\overline{\phi}_{33})^2}{3[n^2 + (\overline{\phi}_{11} + \overline{\phi}_{22} + \overline{\phi}_{33})^4]} = \frac{1}{3}.$$

Further, the function $r_n(\overline{\phi}_{11},, \overline{\phi}_{22}, \overline{\phi}_{33})$ *is continuous for all* $(\overline{\phi}_{11}, \overline{\phi}_{22}, \overline{\phi}_{33}) \in I_4^3$ *and it converges uniformly to 0.*

Again,

$$(\hat{\nu} - \hat{\hat{\mu}})^3 \mathfrak{K}\mathfrak{L} < 1.$$

Thus, the theorem (6.1) is satisfied. Hence we can imply that the example (6.3) holds a solution on cs^ρ.

7 Conclusion

In this paper, we successfully investigated the solvability of an infinite system of Hammerstein-type integral equations involving three variables within the framework of two newly constructed tempered sequence spaces, bv_0^ρ *and* cs^ρ*.By employing the concepts of Hausdorff measure of non-compactness and Meir-Keeler condensing operators, we established the existence of solutions under specific conditions. Our results extend and generalize earlier work in the area by introducing new sequence spaces that provide a more flexible analytical setting, especially when classical spaces prove insufficient.*

Potential directions for future research include exploring the stability, uniqueness, and approximation of solutions in these and other generalized sequence spaces. Additionally, the framework could be extended to fractional or stochastic integral equations or applied to practical models in physics, biology, or engineering where non-compact behaviors arise naturally. Further development of numerical methods tailored to tempered sequence spaces may also enable practical computation of such solutions.

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