On the application of the $(G'/G, 1/G)$-expansion method to the time fractional Chaffee-Infante equation

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Abstract The Time-fractional Chaffee Infante equation is used in variety of field including diffusion of gases in a homogeneous medium, which are useful in physics of combustion, transportation of masses, and also the dynamics of solitons. In this paper, we find the soliton solutions of the time-fractional equation using $(G'/G, 1/G)$-expansion method. This method is useful in finding the solution of a nonlinear evolution equation in form of hyperbolic-type, trigonometric-type and rational type functions. We shall check the effect of the time fractional variable on the solution using 3D graphical representations, density plots and line graph.

1 Introduction

In the 19th century, Scottish Engineer John Scott Russell observed an interesting pattern during his study of water waves [1], which he later described as the "waves of translation". Later, Zabusky and Kruskal coined the term 'soliton' to introduce these waves. Solitons are defined as the localised solitary waves that doesn't change its shape and structure while propagating through a medium. Since then, solitons are used in variety of fields like plasma physics [2, 5], optical fiber mechanisms [3, 4], medical imaging, long-distance communications [6], ocean engineering [7], fluid dynamics [8], etc. These are phenomena are better described with the help of nonlinear partial differential equations (NLPDEs) [9] and the fractional version of NLDEs [10].

One significant example of a fractional NLEE in the context of solitons is the TF-CI equation, that is used in homogeneous medium to model and study the behaviour of gas diffusion. There are several field like combustion physics, mass transportation, soliton dynamics, etc. The equation of classical CI equation is given by [11]

$$u_{xt} + (-u_{xx} + \alpha u^3 - \alpha u)_x + \sigma u_{yy} = 0, \tag{1}$$

and the time-fraction version of the TF-CI equation [12] is given by

$$\left(\frac{\partial^\beta \mathcal{U}}{\partial t^\beta}\right)_x - \left(\frac{\partial^2 \mathcal{U}}{\partial x^2} - \alpha \mathcal{U}^3 + \alpha \mathcal{U}\right)_x + \theta \frac{\partial^2 \mathcal{U}}{\partial y^2} = 0, \tag{2}$$

where α represents the coefficient of diffusion and θ represent degradation coefficient. The diffusion of a gas in a homogeneous medium is an important phenomenon in a physical context and the CI model provides a useful model to study such phenomena. The TF-CI equation, in porous medium, polymers or any disordered sytems, helps in modelling the subdiffusion behaviour. These disordered systems are the systems where the diffusion is slower than the classical Brownian motion. The mentioned equation is also useful in examining and predicting the biological systems, such as neutral circuits and tomour development. Due to the introduction of fractional derivatives, with their intrinsic memory properties, the prediction of growth or reduction in tomours can be explained by TF-CI equation.

In the 17th century, Leibnitz et. al.[13] proposed the concept of fractional calculus to generalize the classical calculus operations of differentiation and integration to non-integer orders.

This generalization allows for the consideration of rates of change and integrals of non-integer order, providing a powerful tool to describe and analyze various complex systems and phenomena. Various phenomena utilize fractional differential equations to describe them perfectly. In particular, time fractional differential equations represent a type of differential equation in which time is represented as a fractional variable [14]. This approach is particularly useful for modeling and understanding phenomena with memory effects, hereditary properties, and anomalous diffusion.

There are numerous examples of different time fractional differential equations (TFDEs), like TF Korteweg-de Vries (KdV) equation [15], TF Burgers' equation [16], TF-Fisher equation [17], TF-Zakharov-Kuznetsov (ZK) equation [18], TF-FitzHugh-Nagumo equation, TF-Zakharov equations, TF modified Korteweg-de Vries (mKdV) equation [19], TF-Cahn-Hilliard equation, TF-Ginzburg-Landau equation, etc., for which its soliton solutions have been found. To study the soliton solutions of nonlinear TFDEs, a variety of reliable and efficient methods are available. These include the sine-cosine method [20, 21], the extended direct algebraic method [22], the first integral method [23], modified Kudryashov method [24], the homogeneous balance method [25], simplified Hirota's method [26], modified simple equation method [27, 28], the direct algebraic method [29], generalized Kudryashov method [30, 31], Exp-function method [32, 33], Jacobi elliptic function expansion method [34, 35], the Hirota bilinear method [36], the (G'/G)-expansion method [37, 38], and the (G'/G^2)-expansion method [39, 40].

Here, we shall employ the $(G'/G, 1/G)$-expansion method to find the soliton solutions of the TF-CI equation and also show the effect of change in fractional variable β, using 3D plot, density plot and line graph. But before understanding the methodology of the above expansion method, we first discuss the conformable fractional derivative that we have used in (2).

2 Conformable fractional derivatives (CFDs)

There are different types of fractional derivatives like Riemann-Liouville Derivative [41], Caputo Derivative [42], etc. One of the following derivatives is CFD, which is defined as "For the function $g(t)$, a CFD [43, 44] of order β is defined as,

$$D_t^\beta g(t) = \lim_{\epsilon \to 0} \frac{g(t + \epsilon t^{1-\beta} - g(t)}{\epsilon},$$

where $(0 < \beta \le 1)$".

Theorem 1 [43]: Suppose $\alpha \in (0, 1]$, and f and g be α-differentiable at $t > 0$. Then

(i) $D_t^\beta (0 < \beta \le 1)(af + bg) = a D_t^\beta f + b D_t^\beta g, \qquad \forall a, b \in R,$

(ii) $D_t^\beta(t^\Omega) = \Omega t^{\Omega - \beta}, \qquad \forall \Omega \in R,$

(iii) $D_t^\beta(fg) = f D_t^\beta g + g D_t^\beta f,$

(iv) $D_t^\beta \left(\frac{f}{g}\right) = \frac{g D_t^\beta f - f D_t^\beta g}{g^2},$

(v) If f is differentiable, then $D_t^\beta(f(t)) = t^{1-\beta} \frac{df}{dt}(t)$.

Theorem 2 [43]: Suppose $f : (0, \infty) \to R$ be a function such that f is differentiable and also α-differentiable. Let g be a function defined in the range of f and also differentiable. Then

$$D_t^\beta(f \circ g)(t) = t^{1-\beta} g'(t) f'(g(t)).$$

3 Methodology

Before discussing the main steps of the method mentioned above, we consider the following second-order linear ordinary differential equation as given in [45, 46]:

$$G''(\eta) + \Lambda G(\eta) = \Omega, \tag{3}$$

where Λ, Ω are constants, and the equation (3) follows the conditions given below:

$$(\omega)' = -\omega^2 + \Omega\Upsilon - \Lambda,$$
$$(\Upsilon)' = -\omega\Upsilon,$$

where $\omega = \dfrac{G'}{G}$, and $\Upsilon = \dfrac{1}{G}$.

The solutions of equation (3) can be classified into the following three cases:

Case 1: (Hyperbolic type functions) If $\Lambda < 0$, the general solution is given by

$$G(\eta) = A_1 \sinh\left(\eta\sqrt{-\Lambda}\right) + A_2 \cosh\left(\eta\sqrt{-\Lambda}\right) + \frac{\Omega}{\Lambda}, \tag{4}$$

with

$$\Upsilon^2 = \frac{-\Lambda}{\Lambda^2(A_1^2 - A_2^2) + \Omega^2}(\omega^2 - 2\Omega\Upsilon + \Lambda), \tag{5}$$

Case 2: (Trignometric type functions) If $\Lambda > 0$, the general solution is given by

$$G(\eta) = A_1 \sin\left(\eta\sqrt{\Lambda}\right) + A_2 \cos\left(\eta\sqrt{\Lambda}\right) + \frac{\Omega}{\Lambda}, \tag{6}$$

with

$$\Upsilon^2 = \frac{\Lambda}{\Lambda^2(A_1^2 + A_2^2) - \Omega^2}(\omega^2 - 2\Omega\Upsilon + \Lambda), \tag{7}$$

Case 3: (Rational type functions) If $\Lambda = 0$, the general solution is given by

$$G(\eta) = \frac{\Omega}{2}\eta^2 + A_1\eta + A_2, \tag{8}$$

with

$$\Upsilon^2 = \frac{1}{(A_1^2 - 2\Omega A_2)}(\omega^2 - 2\Omega\Upsilon), \tag{9}$$

such that A_1 and A_2 are arbitrary constants.

Using these solutions, we begin the main steps of the methods, which are as follows:
Consider the general nonlinear fractional differential equation in three independent variables, x, y, and t:

$$K(\mathcal{U}, D_t^\beta\mathcal{U}, U_x, U_y, U_{xx}, \ldots) = 0, \qquad 0 < \beta \leq 1, \tag{10}$$

where $D_t^\beta\mathcal{U}$ is the conformable derivative of $\mathcal{U}$ with respect to time t, and $\mathcal{U}_x, \mathcal{U}_y$ are the partial derivatives of spatial dimensions x, y respectively, giving K, a polynomial of unknown function $\mathcal{U} = \mathcal{U}(x, y, t)$ and its derivatives.

Using the transformation,

$$\mathcal{U}(x, y, t) = \mathcal{W}(\eta), \text{ and } \eta = x + y - c\frac{t^\beta}{\beta}, \tag{11}$$

where c is a constant that is to be determined later, we reduce (10) to an ODE in $\mathcal{W} = \mathcal{W}(\eta)$ as

$$\mathcal{N}(\mathcal{W}, \mathcal{W}', \mathcal{W}'', \ldots) = 0, \tag{12}$$

where $\mathcal{N}$ is a polynomial of $\mathcal{W}$ and its ordinary derivatives, such that the notation $(')$ in equation (12) denotes the ordinary derivative with respect to η. Now, the steps are as follows:

Step 1: We assume that the equation (12) can be represented as the polynomial in two variables ω and Υ, as shows below:

$$\mathcal{W}(\eta) = \sum_{i=0}^{N} p_i\omega^i + \sum_{j=1}^{N} q_j\omega^{j-1}\Upsilon, \tag{13}$$

where p_i (for $i = 0, 1, 2, \ldots, N$) and q_j (for $j = 1, 2, \ldots, N$) are constants to be determined, with the condition $p_N^2 + q_N^2 \neq 0$.

Step 2: Using the homogeneous balancing principle, we determine the positive integer N in (13). In this, we equate the highest-order derivatives with the highest nonlinear term in (12), considering the following rules, such that the degree of $\mathcal{W}(\eta)$ is given by $\text{Deg}[\mathcal{W}(\eta)] = N$,

$$\text{Deg}\left[\frac{d^q \mathcal{W}(\eta)}{d\eta^q}\right] = N + q,$$

$$\text{Deg}\left[(\mathcal{W}(\eta))^p \left(\frac{d^q \mathcal{W}(\eta)}{d\eta^q}\right)^s\right] = Np + s(N + q). \tag{14}$$

In case, the balanced number N for certain equations, after reduced using equation (11), isn't a positive integer, specialized transformations are employed to reduce the NLEEs.

Step 3: After substituting the value of N in (13), we employ the resultant expansion into (12), giving us the polynomial of ω and Υ, where the degree of Υ doesn't exceed one, with the help of (4) and (5). Then we equate the coefficient of each power of the resultant polynomial to zero, we formulate a system of equations. The system of equations can be solved by using mathematical software like Maple or Mathematica to determine the unknowns: $p_i, q_j, c, \Omega, \Lambda < 0, A_1$, and A_2. The resulting solution for $\Lambda < 0$ is expressed using hyperbolic type functions.

Step 4: We apply the same procedure using (6) and (7) and solve the equations to determine the unknowns: $p_i, q_j, c, \Omega, \Lambda > 0, A_1$, and A_2. This gives the solutions in the form of trigonometric type functions. Then we employ (8) and (9) in (12) to get the solutions when $\Lambda = 0$, which are in the form of rational type functions.

Now, we use the steps above to find the soliton solutions of the TF-CI equation in the next section.

4 Application

To find the solutions of the TF-CI equation in (2), we reduce it to an ODE using the transformation in (11), which gives us the follwing equation:

$$\mathcal{W}''' + (k - \theta)\mathcal{W}'' - 3\alpha \mathcal{W}^2 \mathcal{W}' + \alpha \mathcal{W}' = 0.$$

We integrate the above equation once which gives us

$$\mathcal{W}'' + (c - \theta)\mathcal{W}' + \alpha \mathcal{W} - \alpha \mathcal{W}^3 = 0. \tag{15}$$

For the next step, we found $N = 1$ for (15). Therefore, we suppose that the solution of the above euation is:

$$\mathcal{W} = p_0 + p_1 \omega^1 + q_1 \Upsilon^1. \tag{16}$$

Putting the value of $\mathcal{W}$ from (16) in (15) and solve the system of equations, that is, by the coefficients of ω and Υ for each case of the mentioned method.

4.1 Hyperbolic type function

We get the following system of equations after using the equation (5) for the substitution of Υ^2 as follows:

$$\omega^0 = -ap_0^3 + ap_0 - p_1 c\Lambda + p_1 \Lambda \theta + \frac{3ap_0 q_1^2 \Lambda^2 + q_1 \Lambda^2 \Omega}{\Lambda^2 \left(A_1^2 - A_2^2\right) + \Omega^2} + \frac{2aq_1^3 \Lambda^3 \Omega}{\left(\Lambda^2 \left(A_1^2 - A_2^2\right) + \Omega^2\right)^2};$$

$$\omega^1 = -3ap_0^2 p_1 + ap_1 + 2p_1 \Lambda + \frac{3ap_1 q_1^2 \Lambda^2}{\Lambda^2 \left(A_1^2 - A_2^2\right) + \Omega^2};$$

$$\omega^2 = -3ap_0 p_1^2 - p_1 c + p_1 \theta + \frac{3ap_0 q_1^2 \Lambda + q_1 \Lambda \Omega}{\Lambda^2 \left(A_1^2 - A_2^2\right) + \Omega^2} + \frac{2aq_1^3 \Lambda^2 \Omega}{\left(\Lambda^2 \left(A_1^2 - A_2^2\right) + \Omega^2\right)^2};$$

$$\omega^3 = p_1(2 - ap_1^2) + \frac{3ap_1 q_1^2 \Lambda}{\Lambda^2 \left(A_1^2 - A_2^2\right) + \Omega^2};$$

$$\Upsilon\omega^0 = -3ap_0^2q_1 + aq_1 + p_1c\Omega - p_1\Omega\theta + q_1\Lambda + \frac{-6ap_0q_1^2\Lambda\Omega + aq_1^3\Lambda^2 - 2q_1\Lambda\Omega^2}{\Lambda^2\left(A_1^2 - A_2^2\right) + \Omega^2}$$

$$- \frac{4aq_1^3\Lambda^2\Omega^2}{\left(\Lambda^2\left(A_1^2 - A_2^2\right) + \Omega^2\right)^2};$$

$$\Upsilon\omega^1 = -6ap_0p_1q_1 - 3p_1\Omega - q_1c + q_1\theta - \frac{6ap_1q_1^2\Lambda\Omega}{\Lambda^2\left(A_1^2 - A_2^2\right) + \Omega^2};$$

$$\Upsilon\omega^2 = -3ap_1^2q_1 + 2q_1 + \frac{aq_1^3\Lambda}{\Lambda^2\left(A_1^2 - A_2^2\right) + \Omega^2}.$$

Here, we solved the above system of equations using Mathematica 14 to find the values of $p_i, q_j, c, \Lambda, \Omega, A_1$, and A_2. Given below are the results as follows:

(i) $\left\{p_0 \to -\frac{1}{2}, p_1 \to -\frac{i}{2\sqrt{\Lambda}}, q_1 \to -\frac{\sqrt{A_1^2\Lambda^2 - A_2^2\Lambda^2 + \Omega^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta + 3i\sqrt{\Lambda}\right\}$

(ii) $\left\{p_0 \to -\frac{1}{2}, p_1 \to -\frac{i}{2\sqrt{\Lambda}}, q_1 \to \frac{\sqrt{A_1^2\Lambda^2 - A_2^2\Lambda^2 + \Omega^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta + 3i\sqrt{\Lambda}\right\}$

(iii) $\left\{p_0 \to \frac{1}{2}, p_1 \to -\frac{i}{2\sqrt{\Lambda}}, q_1 \to -\frac{\sqrt{A_1^2\Lambda^2 - A_2^2\Lambda^2 + \Omega^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta - 3i\sqrt{\Lambda}\right\}$

(iv) $\left\{p_0 \to \frac{1}{2}, p_1 \to -\frac{i}{2\sqrt{\Lambda}}, q_1 \to \frac{\sqrt{A_1^2\Lambda^2 - A_2^2\Lambda^2 + \Omega^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta - 3i\sqrt{\Lambda}\right\}$

(v) $\left\{p_0 \to -\frac{1}{2}, p_1 \to \frac{i}{2\sqrt{\Lambda}}, q_1 \to -\frac{\sqrt{A_1^2\Lambda^2 - A_2^2\Lambda^2 + \Omega^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta - 3i\sqrt{\Lambda}\right\}$

(vi) $\left\{p_0 \to -\frac{1}{2}, p_1 \to \frac{i}{2\sqrt{\Lambda}}, q_1 \to \frac{\sqrt{A_1^2\Lambda^2 - A_2^2\Lambda^2 + \Omega^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta - 3i\sqrt{\Lambda}\right\}$

(vii) $\left\{p_0 \to \frac{1}{2}, p_1 \to \frac{i}{2\sqrt{\Lambda}}, q_1 \to -\frac{\sqrt{A_1^2\Lambda^2 - A_2^2\Lambda^2 + \Omega^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta + 3i\sqrt{\Lambda}\right\}$

(viii) $\left\{p_0 \to \frac{1}{2}, p_1 \to \frac{i}{2\sqrt{\Lambda}}, q_1 \to \frac{\sqrt{A_1^2\Lambda^2 - A_2^2\Lambda^2 + \Omega^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta + 3i\sqrt{\Lambda}\right\}$

(ix) $\left\{p_0 \to 0, p_1 \to -\frac{i}{\sqrt{\Lambda}}, q_1 \to -\frac{\sqrt{A_1^2\Lambda^2 - A_2^2\Lambda^2 + \Omega^2}}{\Lambda}, a \to -\frac{\Lambda}{2}, c \to \theta\right\}$

These results gives us the value of $\mathcal{U}_{1i}$'s, in the form of hyperbolic type function with the help of (4) when we put all the coefficients in equation (13).

$$\mathcal{U}_{11}(x,y,t) = -\frac{1}{2} - \frac{\sqrt{\Omega^2 + (A_1^2 - A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)}$$
$$+ \frac{\Lambda\left(A_1\cosh\left(\sqrt{-\Lambda}\eta\right) + A_2\sinh\left(\sqrt{-\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)}, \tag{15}$$

where $\eta = \left(x + y - \dfrac{(\theta + 3i\sqrt{\Lambda})t^{\beta}}{\beta} \right)$

$$
\begin{aligned}
\mathcal{U}_{12}(x,y,t) = &-\frac{1}{2} + \frac{\sqrt{\Omega^2 + (A_1^2 - A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)} \\
&+ \frac{\Lambda\left(A_1\cosh\left(\sqrt{-\Lambda}\eta\right) + A_2\sinh\left(\sqrt{-\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)},
\end{aligned}
\tag{16}
$$

where $\eta = \left(x + y - \dfrac{(\theta + 3i\sqrt{\Lambda})t^{\beta}}{\beta} \right)$

$$
\begin{aligned}
\mathcal{U}_{13}(x,y,t) = &\frac{1}{2} - \frac{\sqrt{\Omega^2 + (A_1^2 - A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)} \\
&+ \frac{\Lambda\left(A_1\cosh\left(\sqrt{-\Lambda}\eta\right) + A_2\sinh\left(\sqrt{-\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)},
\end{aligned}
\tag{17}
$$

where $\eta = \left(x + y - \dfrac{(\theta - 3i\sqrt{\Lambda})t^{\beta}}{\beta} \right)$

$$
\begin{aligned}
\mathcal{U}_{14}(x,y,t) = &\frac{1}{2} + \frac{\sqrt{\Omega^2 + (A_1^2 - A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)} \\
&+ \frac{\Lambda\left(A_1\cosh\left(\sqrt{-\Lambda}\eta\right) + A_2\sinh\left(\sqrt{-\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)},
\end{aligned}
\tag{18}
$$

where $\eta = \left(x + y - \dfrac{(\theta - 3i\sqrt{\Lambda})t^{\beta}}{\beta} \right)$

$$
\begin{aligned}
\mathcal{U}_{15}(x,y,t) = &-\frac{1}{2} - \frac{\sqrt{\Omega^2 + (A_1^2 - A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)} \\
&- \frac{\Lambda\left(A_1\cosh\left(\sqrt{-\Lambda}\eta\right) + A_2\sinh\left(\sqrt{-\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)},
\end{aligned}
\tag{19}
$$

where $\eta = \left(x + y - \dfrac{(\theta - 3i\sqrt{\Lambda})t^{\beta}}{\beta} \right)$

$$
\begin{aligned}
\mathcal{U}_{16}(x,y,t) = &-\frac{1}{2} + \frac{\sqrt{\Omega^2 + (A_1^2 - A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)} \\
&- \frac{\Lambda\left(A_1\cosh\left(\sqrt{-\Lambda}\eta\right) + A_2\sinh\left(\sqrt{-\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)},
\end{aligned}
\tag{20}
$$

where $\eta = \left(x + y - \dfrac{(\theta - 3i\sqrt{\Lambda})t^{\beta}}{\beta} \right)$

$$
\begin{aligned}
\mathcal{U}_{17}(x,y,t) = &\frac{1}{2} - \frac{\sqrt{\Omega^2 + (A_1^2 - A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)} \\
&- \frac{\Lambda\left(A_1\cosh\left(\sqrt{-\Lambda}\eta\right) + A_2\sinh\left(\sqrt{-\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)},
\end{aligned}
\tag{21}
$$

where $\eta = \left(x + y - \dfrac{(\theta + 3i\sqrt{\Lambda})t^{\beta}}{\beta} \right)$

$$
\begin{aligned}
\mathcal{U}_{18}(x,y,t) = &\frac{1}{2} + \frac{\sqrt{\Omega^2 + (A_1^2 - A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)} \\
&- \frac{\Lambda\left(A_1\cosh\left(\sqrt{-\Lambda}\eta\right) + A_2\sinh\left(\sqrt{-\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)},
\end{aligned}
\tag{22}
$$

where $\eta = \left(x + y - \frac{(\theta + 3i\sqrt{\Lambda})t^\beta}{\beta}\right)$

$$\mathcal{U}_{19}(x,y,t) = \frac{1}{2} - \frac{\sqrt{\Omega^2 + (A_1^2 - A_2^2)\Lambda^2}}{\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)} \\ - \frac{\Lambda\left(A_1\cosh\left(\sqrt{-\Lambda}\eta\right) + A_2\sinh\left(\sqrt{-\Lambda}\eta\right)\right)}{\left(A_1\Lambda\sinh\left(\sqrt{-\Lambda}\eta\right) + A_2\Lambda\cosh\left(\sqrt{-\Lambda}\eta\right) + \Omega\right)}, \tag{23}$$

where $\eta = \left(x + y - \frac{\theta t^\beta}{\beta}\right)$

4.2 Trigonometric type functions

Here, we employ the equation (7) for the substitution of Υ^2, giving us the system of equations as follows:

$$\omega^0 = -ap_0^3 + ap_0 - p_1 c\Lambda + p_1\Lambda\theta - \frac{3ap_0 q_1^2\Lambda^2 + q_1\Lambda^2\Omega}{\Lambda^2\left(A_1^2 + A_2^2\right) - \Omega^2} + \frac{2aq_1^3\Lambda^3\Omega}{\left(\Lambda^2\left(A_1^2 + A_2^2\right) - \Omega^2\right)^2};$$

$$\omega^1 = -3ap_0^2 p_1 + ap_1 + 2p_1\Lambda - \frac{3ap_1 q_1^2\Lambda^2}{\Lambda^2\left(A_1^2 + A_2^2\right) - \Omega^2};$$

$$\omega^2 = -3ap_0 p_1^2 - p_1 c + p_1\theta - \frac{3ap_0 q_1^2\Lambda + q_1\Lambda\Omega}{\Lambda^2\left(A_1^2 + A_2^2\right) - \Omega^2} + \frac{2aq_1^3\Lambda^2\Omega}{\left(\Lambda^2\left(A_1^2 + A_2^2\right) - \Omega^2\right)^2};$$

$$\omega^3 = p_1(2 - ap_1^2) - \frac{3ap_1 q_1^2\Lambda}{\Lambda^2\left(A_1^2 + A_2^2\right) - \Omega^2};$$

$$\Upsilon\omega^0 = -3ap_0^2 q_1 + aq_1 + p_1 c\Omega - p_1\Omega\theta + q_1\Lambda + \frac{6ap_0 q_1^2\Lambda\Omega - aq_1^3\Lambda^2 + 2q_1\Lambda\Omega^2}{\Lambda^2\left(A_1^2 + A_2^2\right) - \Omega^2} - \frac{4aq_1^3\Lambda^2\Omega^2}{\left(\Lambda^2\left(A_1^2 + A_2^2\right) - \Omega^2\right)^2};$$

$$\Upsilon\omega^1 = -6ap_0 p_1 q_1 - 3p_1\Omega - q_1 c + q_1\theta + \frac{6ap_1 q_1^2\Lambda\Omega}{\Lambda^2\left(A_1^2 + A_2^2\right) - \Omega^2};$$

$$\Upsilon\omega^2 = -3ap_1^2 q_1 + 2q_1 - \frac{aq_1^3\Lambda}{\Lambda^2\left(A_1^2 + A_2^2\right) - \Omega^2};$$

We proceed with solving the above system using Mathematica 14 to find all the coefficients along with c, Λ, Ω, A_1, and A_2, whose results are given below:

(i) $\left\{p_0 \to -\frac{1}{2}, p_1 \to -\frac{i}{2\sqrt{\Lambda}}, q_1 \to -\frac{\sqrt{\Omega^2 - (A_1^2 + A_2^2)\Lambda^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta + 3i\sqrt{\Lambda}\right\},$

(ii) $\left\{p_0 \to -\frac{1}{2}, p_1 \to -\frac{i}{2\sqrt{\Lambda}}, q_1 \to \frac{\sqrt{\Omega^2 - (A_1^2 + A_2^2)\Lambda^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta + 3i\sqrt{\Lambda}\right\},$

(iii) $\left\{p_0 \to \frac{1}{2}, p_1 \to -\frac{i}{2\sqrt{\Lambda}}, q_1 \to -\frac{\sqrt{\Omega^2 - (A_1^2 + A_2^2)\Lambda^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta - 3i\sqrt{\Lambda}\right\},$

(iv) $\left\{p_0 \to \frac{1}{2}, p_1 \to -\frac{i}{2\sqrt{\Lambda}}, q_1 \to \frac{\sqrt{\Omega^2 - (A_1^2 + A_2^2)\Lambda^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta - 3i\sqrt{\Lambda}\right\},$

(v) $\left\{p_0 \to -\frac{1}{2}, p_1 \to \frac{i}{2\sqrt{\Lambda}}, q_1 \to -\frac{\sqrt{\Omega^2 - (A_1^2 + A_2^2)\Lambda^2}}{2\Lambda}, a \to -2\Lambda, c \to \theta - 3i\sqrt{\Lambda}\right\},$

(vi) $\left\{ p_0 \to 0, p_1 \to -\dfrac{i}{\sqrt{\Lambda}}, q_1 \to -\dfrac{\sqrt{\Omega^2 - (A_1^2 + A_2^2)\Lambda^2}}{\Lambda}, a \to -\dfrac{\Lambda}{2}, c \to \theta \right\}$

These results gives us the value of $\mathcal{U}_{2i}$'s, in the form of trigonometric type function with the help of (6) when we put all the coefficients in equation (13).

$$\mathcal{U}_{21}(x,y,t) = -\frac{1}{2} - \frac{\sqrt{\Omega^2 - (A_1^2 + A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sin\left(\sqrt{\Lambda}\eta\right) + A_2\Lambda\cos\left(\sqrt{\Lambda}\eta\right) + \Omega\right)}$$
$$- \frac{i\Lambda\left(A_1\cos\left(\sqrt{\Lambda}\eta\right) - A_2\sin\left(\sqrt{\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sin\left(\sqrt{\Lambda}\eta\right) + A_2\Lambda\cos\left(\sqrt{\Lambda}\eta\right) + \Omega\right)}, \tag{24}$$

where $\eta = \left(x + y - \frac{(\theta + 3i\sqrt{\Lambda})t^\beta}{\beta}\right)$

$$\mathcal{U}_{22}(x,y,t) = -\frac{1}{2} + \frac{\sqrt{\Omega^2 - (A_1^2 + A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sin\left(\sqrt{\Lambda}\eta\right) + A_2\Lambda\cos\left(\sqrt{\Lambda}\eta\right) + \Omega\right)}$$
$$- \frac{i\Lambda\left(A_1\cos\left(\sqrt{\Lambda}\eta\right) - A_2\sin\left(\sqrt{\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sin\left(\sqrt{\Lambda}\eta\right) + A_2\Lambda\cos\left(\sqrt{\Lambda}\eta\right) + \Omega\right)}, \tag{25}$$

where $\eta = \left(x + y - \frac{(\theta + 3i\sqrt{\Lambda})t^\beta}{\beta}\right)$

$$\mathcal{U}_{23}(x,y,t) = \frac{1}{2} - \frac{\sqrt{\Omega^2 - (A_1^2 + A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sin\left(\sqrt{\Lambda}\eta\right) + A_2\Lambda\cos\left(\sqrt{\Lambda}\eta\right) + \Omega\right)}$$
$$- \frac{i\Lambda\left(A_1\cos\left(\sqrt{\Lambda}\eta\right) - A_2\sin\left(\sqrt{\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sin\left(\sqrt{\Lambda}\eta\right) + A_2\Lambda\cos\left(\sqrt{\Lambda}\eta\right) + \Omega\right)}, \tag{26}$$

where $\eta = \left(x + y - \frac{(\theta - 3i\sqrt{\Lambda})t^\beta}{\beta}\right)$

$$\mathcal{U}_{24}(x,y,t) = \frac{1}{2} + \frac{\sqrt{\Omega^2 - (A_1^2 + A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sin\left(\sqrt{\Lambda}\eta\right) + A_2\Lambda\cos\left(\sqrt{\Lambda}\eta\right) + \Omega\right)}$$
$$- \frac{i\Lambda\left(A_1\cos\left(\sqrt{\Lambda}\eta\right) - A_2\sin\left(\sqrt{\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sin\left(\sqrt{\Lambda}\eta\right) + A_2\Lambda\cos\left(\sqrt{\Lambda}\eta\right) + \Omega\right)}, \tag{27}$$

where $\eta = \left(x + y - \frac{(\theta - 3i\sqrt{\Lambda})t^\beta}{\beta}\right)$

$$\mathcal{U}_{25}(x,y,t) = -\frac{1}{2} - \frac{\sqrt{\Omega^2 - (A_1^2 + A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sin\left(\sqrt{\Lambda}\eta\right) + A_2\Lambda\cos\left(\sqrt{\Lambda}\eta\right) + \Omega\right)}$$
$$+ \frac{i\Lambda\left(A_1\cos\left(\sqrt{\Lambda}\eta\right) - A_2\sin\left(\sqrt{\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sin\left(\sqrt{\Lambda}\eta\right) + A_2\Lambda\cos\left(\sqrt{\Lambda}\eta\right) + \Omega\right)}, \tag{28}$$

where $\eta = \left(x + y - \frac{(\theta - 3i\sqrt{\Lambda})t^\beta}{\beta}\right)$

$$\mathcal{U}_{26}(x,y,t) = -\frac{\sqrt{\Omega^2 - (A_1^2 + A_2^2)\Lambda^2}}{2\left(A_1\Lambda\sin\left(\sqrt{\Lambda}\eta\right) + A_2\Lambda\cos\left(\sqrt{\Lambda}\eta\right) + \Omega\right)}$$
$$+ \frac{i\Lambda\left(A_1\cos\left(\sqrt{\Lambda}\eta\right) - A_2\sin\left(\sqrt{\Lambda}\eta\right)\right)}{2\left(A_1\Lambda\sin\left(\sqrt{\Lambda}\eta\right) + A_2\Lambda\cos\left(\sqrt{\Lambda}\eta\right) + \Omega\right)},$$

(29)

where $\eta = \left(x + y - \frac{\theta t^\beta}{\beta}\right)$

4.3 Rational type function

Finally, we use the substitution of Υ^2 from (9) that provides us the system of equations that is given below:

$$\omega^0 = -ap_0^3 + ap_0;$$

$$\omega^1 = -3ap_0^2 p_1 + ap_1;$$

$$\omega^2 = -3ap_0 p_1^2 - p_1 c + p_1\theta - \frac{3ap_0 q_1^2 - q_1\Omega}{A_1^2 - 2A_2\Omega} + \frac{2aq_1^3\Omega}{\left(A_1^2 - 2A_2\Omega\right)^2};$$

$$\omega^3 = -ap_1^3 + 2p_1 - \frac{3ap_1 q_1^2}{A_1^2 - 2A_2\Omega};$$

$$\Upsilon\omega^0 = -3ap_0^2 q_1 + aq_1 + p_1 c\Omega - p_1\Omega\theta + \frac{6ap_0 q_1^2\Omega + 2q_1\Omega^2}{A_1^2 - 2A_2\Omega} - \frac{4aq_1^3\Omega^2}{\left(A_1^2 - 2A_2\Omega\right)^2};$$

$$\Upsilon\omega^1 = -6ap_0 p_1 q_1 - 3p_1\Omega - q_1 c + q_1\theta + \frac{6ap_1 q_1^2\Omega}{A_1^2 - 2A_2\Omega};$$

$$\Upsilon\omega^2 = -3ap_1^2 q_1 + 2q_1 - \frac{aq_1^3}{A_1^2 - 2A_2\Omega};$$

After solving the above system of equations for the case of $\Lambda = 0$, we found only trivial solutions using the computing softwares.

All the graphical representations, i.e., 3D plots, density plots and line graphs, of all the solutions are given in Appendix after the references.

5 Conclusion

In this paper, we employed the $(G'/G, 1/G)$-expansion method to find the soliton solutions of the time-fractional Chaffee-Infante (TF-CI) equation. This gave us the different solutions in the form of hyperbolic and trigonometric solutions, depending upon the value of Λ. The 3D plots and density plots, along with the line graphs at a specific value of time t, have been used to show the effect of the fractional variable β in the various solutions. In the case of $\Lambda < 0$, we got the solutions in the form of hyperbolic-type functions, four of which are the dark solitons and five of that are the bright solitons. Whereas, when $\Lambda > 0$, we acquired the solutions in form of the trigonometric-type functions, whose 3D plots of the absolute values of $\mathcal{U}_{2i}$'s are supposedly the periodic breather solitons, with the exception of $\mathcal{U}_{29}$, that is to be a periodic kink soliton. This concludes that the method applied is effective in the extraction of soliton solutions of the TF-CI equation.

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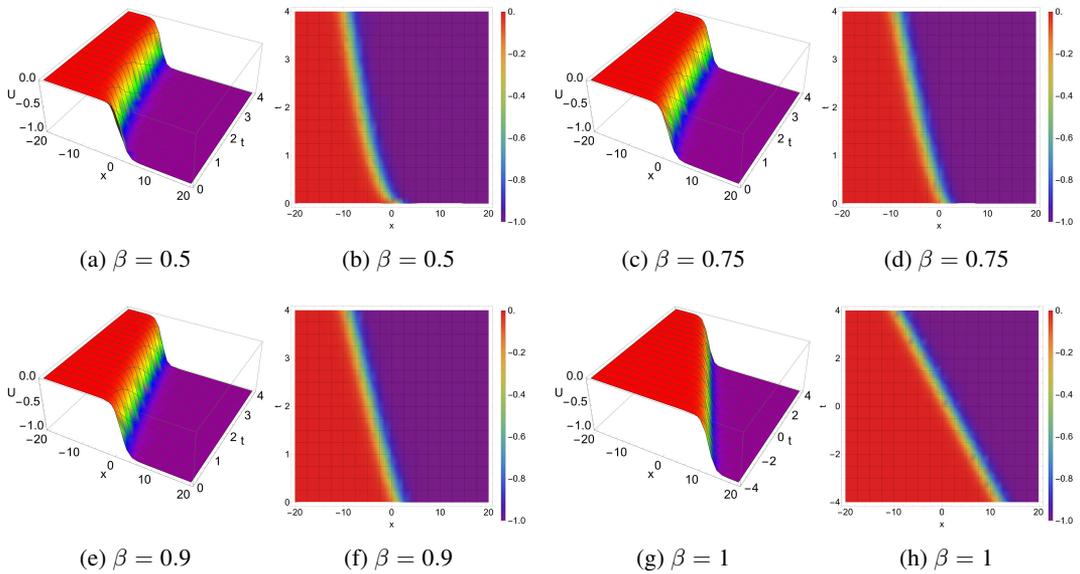
Appendix

(a) $\beta = 0.5$ (b) $\beta = 0.5$ (c) $\beta = 0.75$ (d) $\beta = 0.75$

(e) $\beta = 0.9$ (f) $\beta = 0.9$ (g) $\beta = 1$ (h) $\beta = 1$

Figure 1: 3D plots and density plots of $\mathcal{U}_{11}$ for different β when $A_1 = 8; A_2 = -7; \theta = 1; \Lambda = -1.5; \Omega = -15; y = 1$
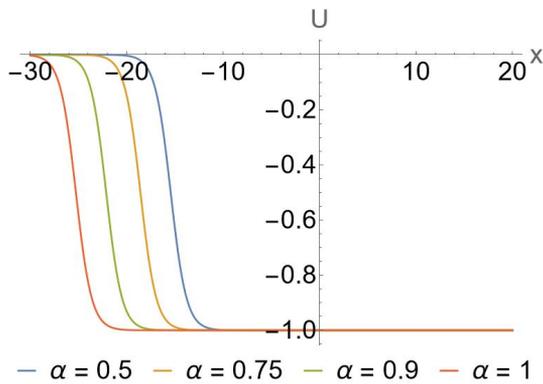

Figure 2: The line graph describing changes in $\mathcal{U}_{11}$ as β changes when $A_1 = 8; A_2 = -7; \theta = 1; \Lambda = -1.5; \Omega = -15; y = 1$ and $t = 10$

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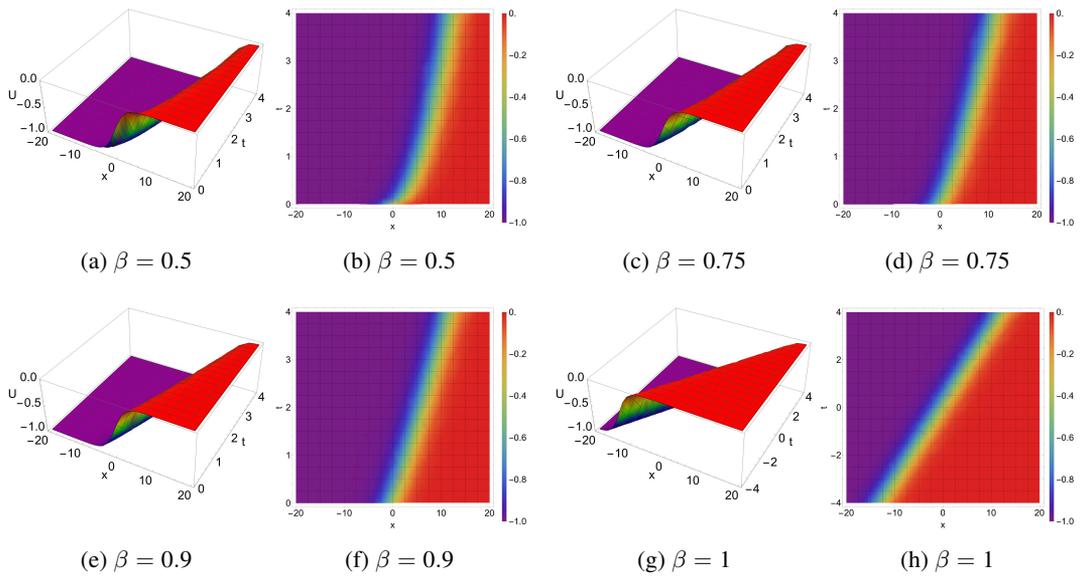
Figure 3: 3D plots and density plots of $\mathcal{U}_{15}$ for different β when $A_1 = -8; A_2 = -7; \Lambda = -0.5; \Omega = 5; \theta = 1; y = 1$
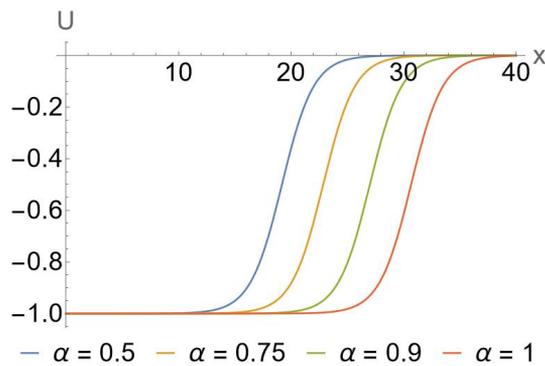

Figure 4: The line graph describing changes in $\mathcal{U}_{15}$ as β changes when $A_1 = -8; A_2 = -7; \Lambda = -0.5; \Omega = 5; \theta = 1; y = 1$ and $t = 10$
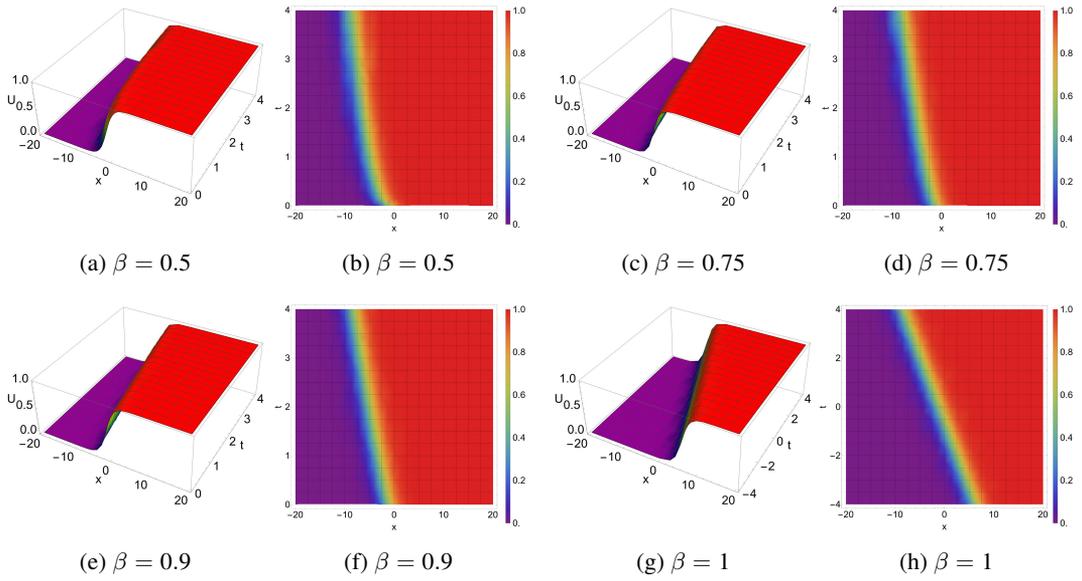
(a) $\beta = 0.5$ (b) $\beta = 0.5$ (c) $\beta = 0.75$ (d) $\beta = 0.75$

(e) $\beta = 0.9$ (f) $\beta = 0.9$ (g) $\beta = 1$ (h) $\beta = 1$

Figure 5: 3D plots and density plots of $\mathcal{U}_{19}$ for different β when $A_1 = 18; A_2 = -5; \Lambda = -0.9; \Omega = 10; \theta = 1; y = 1$
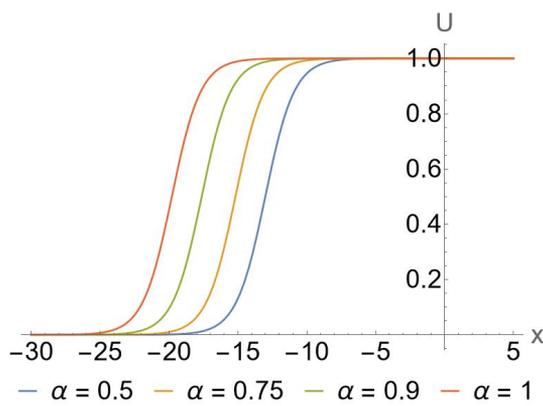
Figure 6: The line graph describing changes in $\mathcal{U}_{19}$ as β changes when $A_1 = 18; A_2 = -5; \Lambda = -0.9; \Omega = 10; \theta = 1; y = 1$ and $t = 10$

(a) $\beta = 0.5$ (b) $\beta = 0.5$ (c) $\beta = 0.75$ (d) $\beta = 0.75$

(e) $\beta = 0.9$ (f) $\beta = 0.9$ (g) $\beta = 1$ (h) $\beta = 1$
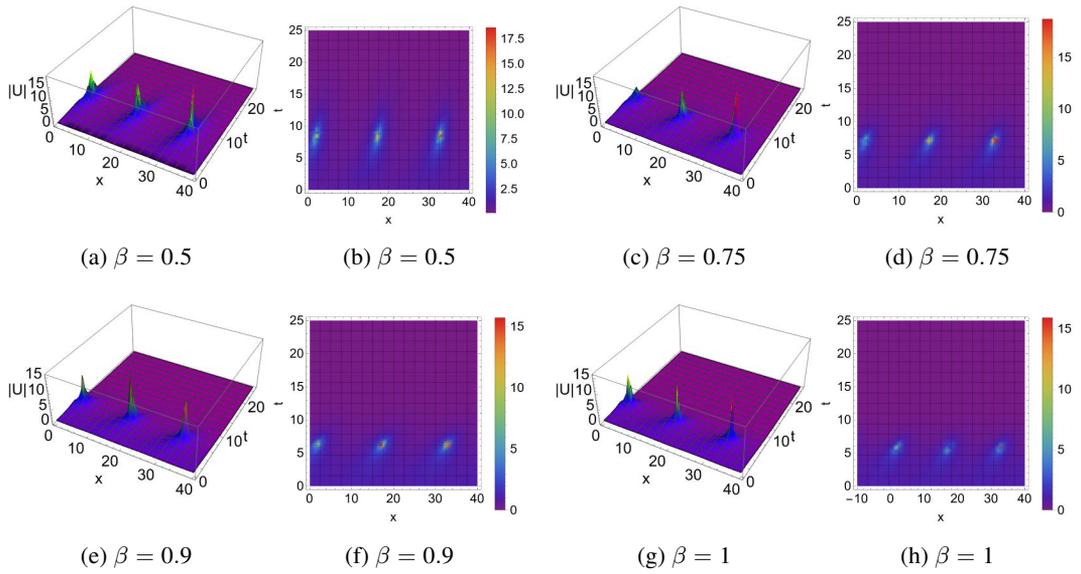
Figure 7: 3D plots and density plots of $|\mathcal{U}_{21}|$ for different β when $A_1 = 12; A_2 = -5; \Lambda = 1/6; \Omega = 20; \theta = 1; y = 1$
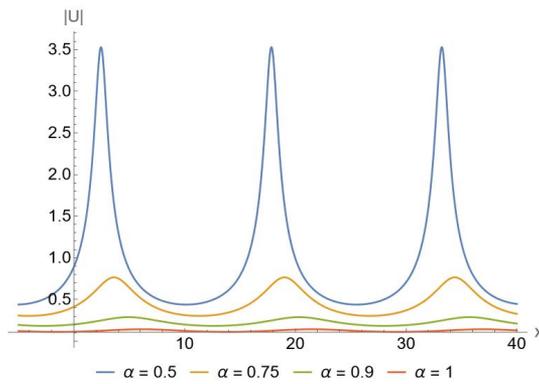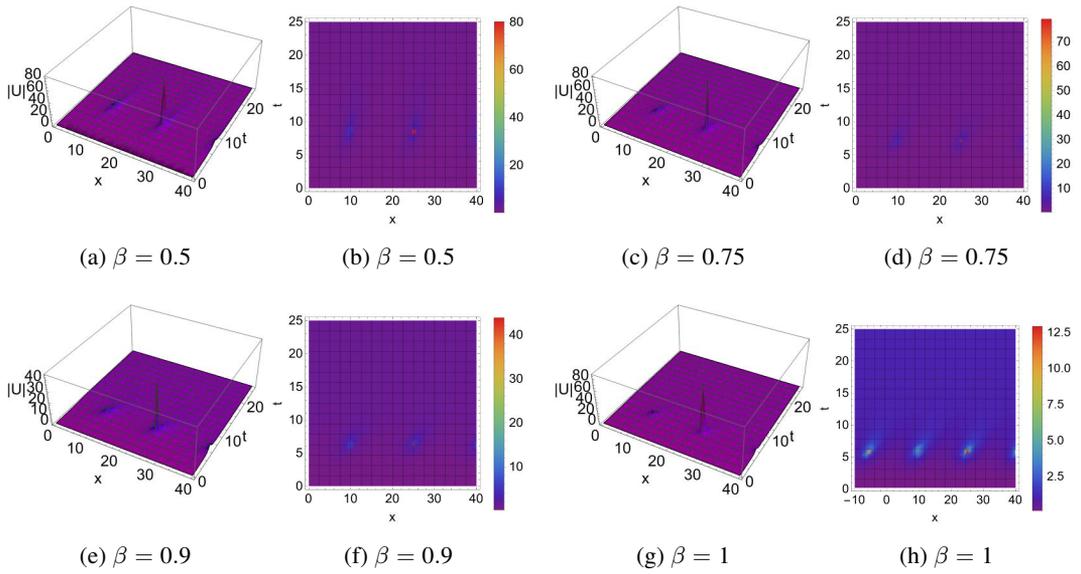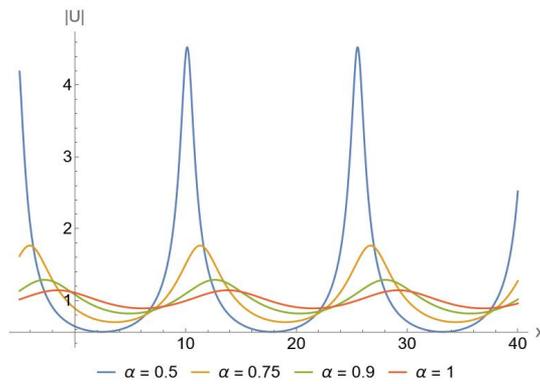

Figure 8: The line graph describing changes in $|\mathcal{U}_{21}|$ as β changes when $A_1 = 12; A_2 = -5; \Lambda = 1/6; \Omega = 20; \theta = 1; y = 1$ and $t = 10$

(a) $\beta = 0.5$ (b) $\beta = 0.5$ (c) $\beta = 0.75$ (d) $\beta = 0.75$

(e) $\beta = 0.9$ (f) $\beta = 0.9$ (g) $\beta = 1$ (h) $\beta = 1$

Figure 9: 3D plots and density plots of $|\mathcal{U}_{22}|$ for different β when $A_1 = 12; A_2 = -5; \Lambda = 1/6; \Omega = -20; \theta = 1; y = 1$

Figure 10: The line graph describing changes in $|\mathcal{U}_{22}|$ as β changes when $A_1 = 12; A_2 = -5; \Lambda = 1/6; \Omega = -20; \theta = 1; y = 1$ and $t = 10$
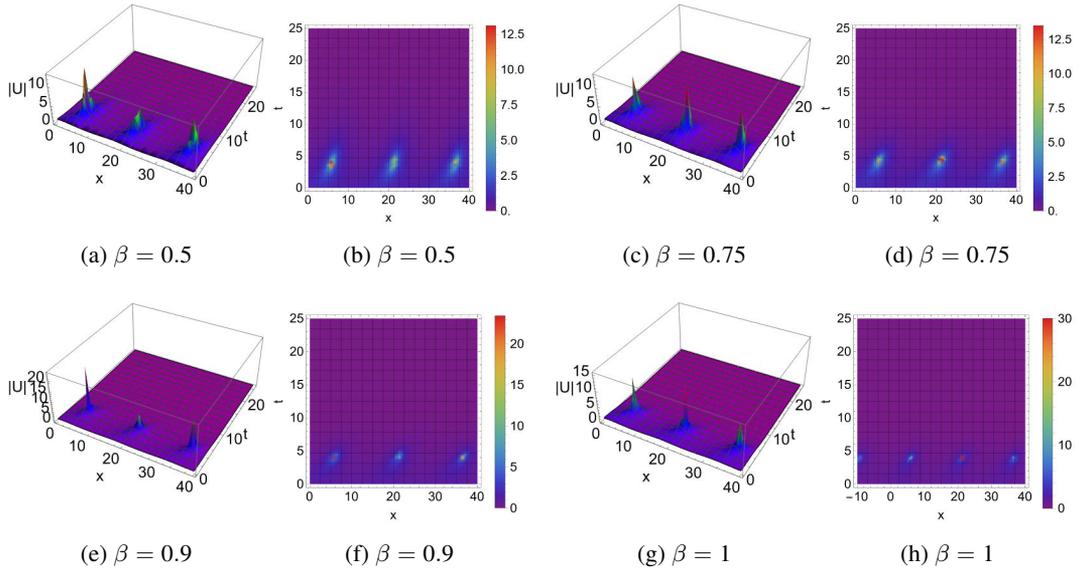
Figure 11: 3D plots and density plots of $|\mathcal{U}_{25}|$ for different β when $A_1 = -12; A_2 = -5; \Lambda = 1/6; \Omega = 8; \theta = 1; y = 1$
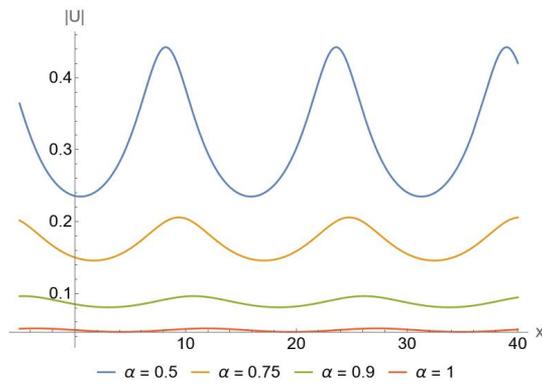

Figure 12: The line graph describing changes in $|\mathcal{U}_{25}|$ as β changes when $A_1 = -12; A_2 = -5; \Lambda = 1/6; \Omega = 8; \theta = 1; y = 1$ and $t = 10$
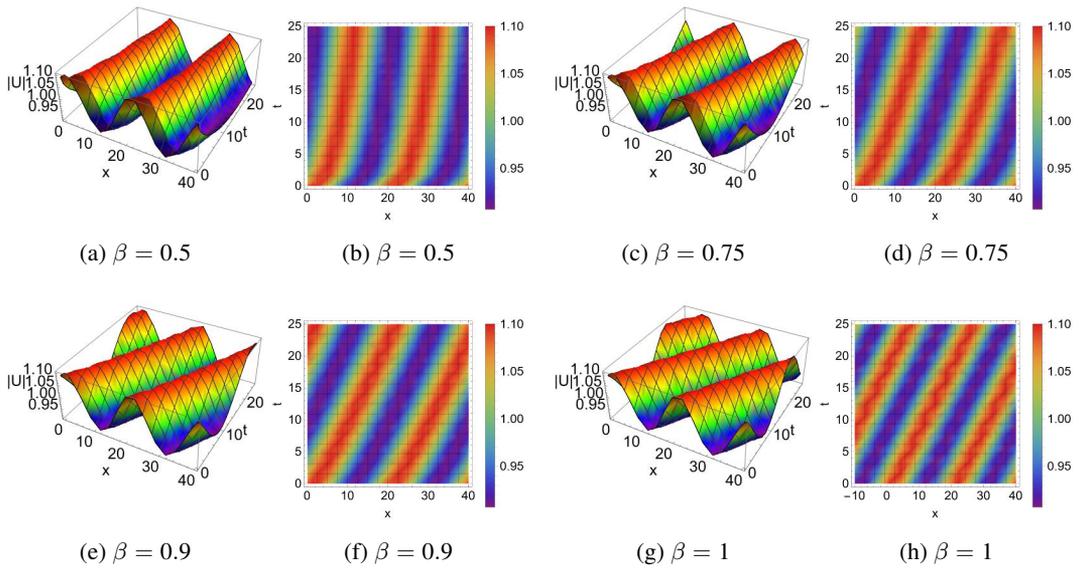
(a) $\beta = 0.5$ (b) $\beta = 0.5$ (c) $\beta = 0.75$ (d) $\beta = 0.75$

(e) $\beta = 0.9$ (f) $\beta = 0.9$ (g) $\beta = 1$ (h) $\beta = 1$

Figure 13: 3D plots and density plots of $|\mathcal{U}_{26}|$ for different β when $A_1 = 6; A_2 = 5; \Lambda = 0.1; \Omega = -8; \theta = 1; y = 1$
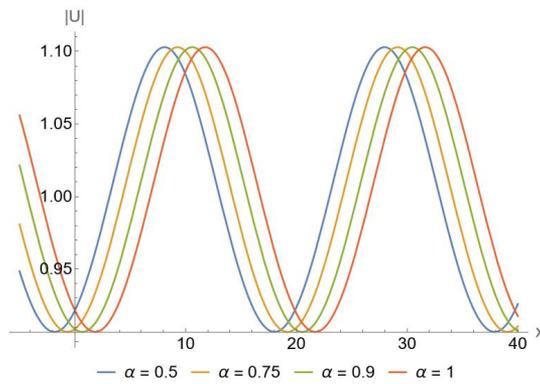

Figure 14: The line graph describing changes in $|\mathcal{U}_{26}|$ as β changes when $A_1 = 6; A_2 = 5; \Lambda = 0.1; \Omega = -8; \theta = 1; y = 1$ and $t = 10$