**Palestine Journal of Mathematics**

**Vol 14 (Special Issue IV), (2025)** , 149–176 

# A Modified MCDM Approach based on Quartic Sine Distance Measures with Application on Education System

Vanita Rani and Satish Kumar

Keywords and phrases: Quartic Fuzzy Sets, Orthopair Fuzzy Sets, Multi-Criteria Decision-Making, Linguistic Variables, Intuitionistic Fuzzy Set.

*The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.*

**Abstract** *The Quartic Fuzzy Set (QFS) represents an advanced generalization of several fuzzy set frameworks, including Fermatean Fuzzy Sets (FFSs), Pythagorean Fuzzy Sets (PFSs), Intuitionistic Fuzzy Sets (IFSs), and conventional Fuzzy Sets (FSs). Similarity measures in fuzzy set theory play an important role in diverse areas such as pattern recognition (PR), classification, and multi-criteria decision-making (MCDM). In the context of MCDM, the process of assigning weights to criteria often depends on a fuzzy framework. This study seeks to develop a similarity measure for QFS utilizing the Hausdorff distance, a powerful method for quantifying similarities between objects. Furthermore, we propose axiomatic definitions for distance measures in the QFS framework.*

*This paper presents a novel distance measure designed specifically to evaluate dissimilarities between Quartic Fuzzy Sets (QFSs). Based on this distance measure, a methodology for assessing similarity between QFSs is developed. Additionally, the study identifies and discusses key properties of the proposed similarity measures. To illustrate their practicality, we include multiple numerical examples in fields such as pattern recognition and linguistic variable analysis. The paper also introduces an original algorithm for Orthopair Fuzzy TODIM, an interactive and efficient approach for MCDM, which integrates these innovative techniques. This newly developed Orthopair Fuzzy TODIM method has been successfully applied to solve various real-world MCDM problems across different domains. The algebraic findings validate that the proposed similarity measures are both practical and effective for addressing challenges related to fuzzy linguistic variables and MCDM applications.*

## 1 Introduction

*Decision-making plays a vital role in our daily lives, influencing various situations and contexts. In an ideal scenario, all the information available to us would be accurate and reliable. However, the complexity and uncertainty of real-world situations mean that the information we rely on is often incomplete or imprecise [1, 2, 3]. This presents a considerable challenge: efficiently managing and interpreting uncertain or ambiguous data to improve the quality and effectiveness of decision-making processes [4, 5, 6, 7]. Various methodologies, such as fuzzy sets, interval-valued fuzzy sets (IVFSs), and rough sets, have been proposed to address the challenges posed by uncertainty and imprecision [8, 9, 10, 11, 12, 13, 14]. Among these, fuzzy set extensions, particularly Intuitionistic Fuzzy Sets (IFSs), stand out for their effectiveness in handling ambiguous information. IFSs distinguish themselves by representing each element with three parameters: degree of membership (DM), degree of non-membership (DNM), and hesitation degree (HD). This unique capability to capture and model uncertainty has made IFSs a popular choice in diverse fields, including pattern classification (PC) [15, 16, 17, 18], medical diagnosis [19, 20, 21], and information fusion [22, 23, 24].*

*In numerous real-world situations, individuals frequently encounter hesitation and uncertainty, complicating the process of reaching definitive decisions. To better address these challenges, Torra [25] proposed hesitant fuzzy sets (HFSs), which allow for the inclusion of multiple* *potential values within the range [0, 1] for each alternative. This approach provides a more refined framework for handling hesitation and ambiguity. Subsequently, Rodríguez et al. [26] explored the advancements and practical applications of HFSs, highlighting their relevance in various contexts.*

*The literature includes a wide range of studies focused on the distance, similarity, and entropy measures for hesitant fuzzy sets (HFSs), with significant contributions made by researchers such as [27, 28, 29, 30]. Building on the foundational concepts of fuzzy sets (FSs), Yager and Abbasov [31] and Yager [32] introduced Pythagorean fuzzy sets (PFSs), which offer greater flexibility and applicability compared to intuitionistic fuzzy sets.*

*PFSs are defined using a MD and a degree of NMD constrained by the condition that the sum of their squares does not exceed one, i.e.,* $0 \leq r^2 + l^2 \leq 1$. *This differs from IFSs, where the relationship* $0 \leq r + l \leq 1$ *must hold. For example, if r = 0.90 and l = 0.4, IFSs would be invalid because* $r + l > 1$, *but PFSs remain valid as* $0 \leq r^2 + l^2 \leq 1$. *This broader condition enables PFSs to model uncertainty more effectively in various applications. PFSs also employ unique aggregation operators, making them suitable for use in fields such as PR, image processing, and MCDM. Their enhanced ability to handle ambiguous and uncertain information allows PFSs to outperform IFSs in representing real-world scenarios more accurately and efficiently.*

*Senapati and Yager [33] introduced Fermatean fuzzy sets (FFSs) to overcome certain limitations of membership degree (MD) and non-membership degree (NMD) under the condition* $r^3 + l^3 \leq 1$. *Building on this foundation, we propose an extension called Quartic Fuzzy Sets (QFSs). This study explores the fundamental properties of QFSs to provide a deeper understanding of their structure and advantages.*

*QFSs represent a significant advancement over earlier fuzzy set models, including FSs, IFSs, PFSs, and FFSs, all of which can be considered subsets of the QFS framework. The introduction of QFSs, as proposed by Yager [34] and Yager and Alajlan [35], marks a critical step forward in generalizing fuzzy set theory. Compared to IFSs, PFSs, and FFSs, QFSs offer enhanced flexibility, robustness, and applicability, making them better suited for addressing complex and uncertain real-world scenarios.*

*To address MCDM problems involving Quartic Fuzzy (QF) data, we propose an enhanced extension of the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS). Initially introduced by Hwang [36], the TOPSIS method has consistently demonstrated its advantages over other MCDM techniques.*

*Researchers have extensively used the TOPSIS framework in a variety of decision-making settings throughout the years [37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. The applications are many and include ranking fast racing cars, choosing the best e-commerce sites, assessing the performance of heart surgeons, identifying high-performing staff, choosing plant locations, locating appropriate material suppliers, and selecting robots for particular jobs. This versatility underscores its effectiveness in tackling complex decision-making problems.*

*While the aforementioned methods prove effective in certain scenarios, they exhibit several limitations:*

- *Some distance measures for IFSs fail to fully adhere to axiomatic principles.*
- *Many existing distance measures for IFSs can produce counter-intuitive results when calculating dissimilarities between sets.*
- *Certain measures lack precision in accurately predicting the class of a query pattern in pattern classification tasks.*

*These shortcomings highlight the ongoing need for more reliable and intuitive methods to differentiate between IFSs. To address this gap, we propose a novel distance measure,* $SV_1$, *inspired by the Hellinger distance, for evaluating the dissimilarity between Quartic Fuzzy Sets (QFSs). This study explores the fundamental properties of* $SV_1$ *and validates its utility through numerical examples. Additionally, a decision-making approach based on* $SV_1$ *is introduced and applied to a real-world case study, demonstrating its practical effectiveness.*

***Contribution***

⋆ *In this study, we present a novel approach for calculating the distance between Quartic Fuzzy Sets (QFSs) based on the Hausdorff metric.*

⋆ *Our work's main goal is to provide a similarity measure for QFSs by utilizing the Hausdorff metric as a fundamental mechanism. We propose several similarity measures that help evaluate the degree of similarity between QFSs.*

⋆ *Additionally, we introduce axiomatic definitions for both the distance and similarity measures associated with QFSs. In order to ensure the measures' applicability and dependability in actual situations, these definitions specify the fundamental characteristics and requirements that they must meet.*

⋆ *We develop an orthopair fuzzy TODIM algorithm using these distance and similarity measures. To improve pattern recognition, control language variables, and address MCDM issues, this approach makes use of the suggested measures.*

***Organization***
*The fundamental concepts of $IFSs$, $PFSs$, $FFSs$, and $QFSs$, along with the Hausdorff Metric, are outlined in Section 2. In Section 3, we present a new measure for the $QFS$ framework and explore its properties and associated graphs. Section 4 introduces a similarity measure based on the Hausdorff metric. In Section 5, a PR methodology is discussed, accompanied by numerical examples. Section 6 provides a comparative analysis of linguistic variables, supported by an example. Section 7 presents a detailed comparative study. Section 8 focuses on Multi-Criteria Decision Making (MCDM) with a real-life case study. Finally, Section 9 concludes with a summary and future directions for research.*

## 2 Preliminaries

*Throughout this paper, let $P = \{p_1, p_2, ..., p_q\}$ represent a non-empty finite set. $IFS$ be the collection of Quartic fuzzy sets for $P$. Let $G_1$, $G_2$ and $G_3$ be the $QFSs$ subsets in $P$.*

**Definition 2.1.** [9] A $IFSs$ $G_1$ in $G$ is given as

$$G_1 = \{(p_n, r_{G_1}(p_n), l_{G_1}(p_n)) \mid p \in P\}, n = 1, 2, ..., q. \tag{2.1}$$

where $r_{G_1}(p_n)$ means the DM and $l_{G_1}(p_n))$ means DNM of $p_n \in P$ in $G_1$ such that $0 \leq r_{G_1}(p_n) + l_{G_1}(p_n)) \leq 1$. Here, $h^*_{G_1}(p_n)) = 1 - r_{G_1}(p_n) - l_{G_1}(p_n)$ denotes the hesitancy degree. Consider a situation where an object has a DM of 0.5 and a DNM of 0.7. If we combine the DM and DNM values (0.5 + 0.7), the resulting sum is 1.2. This does not meet the previously stated condition, which requires the sum to be less than or equal to 1.

**Definition 2.2.** [31] A $PFS$ $(G_1)$ in $P$ is specified by
$G_1 = \{(p_n, r_{G_1}(P_m), l_W(p_n)) \mid p \in P\}, n = 1, 2, ..., p$, with $0 \leq r^2_{G_1}(p_n) + l^2_{G_1}(p_n) \leq 1$, where the functions $r_{G_1} : T \longrightarrow [0, 1]$ and $l_{G_1} : P \longrightarrow [0, 1]$ indicate the DM and DNM of $p$ in $P$, for each $p \in P$, $h_{G_1} = \left(1 - \left\{r_{G_1}(p_n)^2 + l^2_{G_1}(p_n)\right\}\right)^{\frac{1}{2}}$ is the degree of hesitancy.

*Let DM be 0.7 and DNM be 0.5, we see that* $0.7^2 + 0.5^2 \leq 1$ *that satisfy the* $PFS$.

**Definition 2.3.** [33] A $FFS$ $(G_1)$ in $P$ is specified by
$G_1 = \{(p_n, r_{G_1}(p_n), l_{G_1}(p_n)) \mid p \in P\}, n = 1, 2, ..., q$, with $0 \leq r^3_{G_1}(p_n) + l^3_{G_1}(p_n) \leq 1$, where the functions $r_{G_1} : P \longrightarrow [0, 1]$ and $l_{G_1} : P \longrightarrow [0, 1]$ indicate the DM and DNM of $p$ in $P$, for each $p \in P$, $h_{G_1} = \left(1 - \left\{r^3_{G_1}(P_n) + l^3_{G_1}(P_n)\right\}\right)^{\frac{1}{3}}$ is said to be Fermatean fuzzy degree of hesitancy.
if we take DM is 0.9 and DNM is 0.6. This condition $0.9^2 + 0.6^2 \nleq 1$ for $PFSs$, is not satisfied. That means FFS is applicable.

**Definition 2.4.** A $QFS$ $(G_1)$ in $P$ is specified by
$G_1 = \{(p_n, r_{G_1}(p_n), l_{G_1}(p_n)) \mid p \in P\}, n = 1, 2, ..., q$, with $0 \leq r^4_{G_1}(p_n) + l^4_{G_1}(p_n) \leq 1$, where the functions $r_{G_1} : P \longrightarrow [0, 1]$ and $l_{G_1} : P \longrightarrow [0, 1]$ indicate the DM and DNM, for each $p \in P$, $h_{G_1} = \left(1 - \left\{r^4_{G_1}(p_n) + l^4_{G_1}(p_n)\right\}\right)^{\frac{1}{4}}$ is Quartic fuzzy degree of hesitancy.

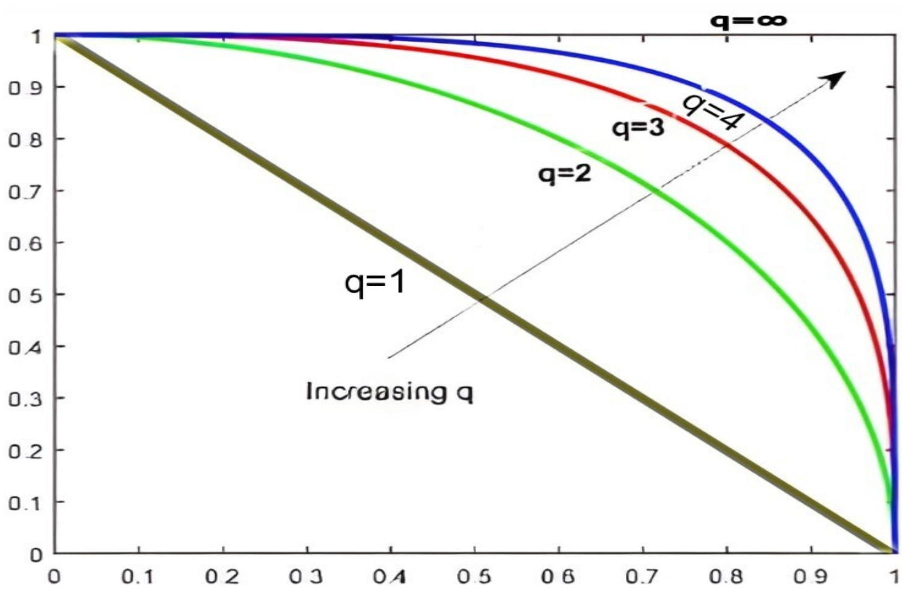


Figure 1: A Graph of Various Fuzzy Sets

In this scenario, we have an MD of 0.8 and an NMD of 0.9. When we assess the condition $0.9^3 + 0.8^3 \nleq 1$ for FFS, it is not met, indicating that FFS cannot be used here. In such cases, we turn to QFS for the solution.

*The inequality* $0 \leq (r^q_{G_1}(p_n)^p + (1-(1-l^q_{G_1})^p)^{\frac{1}{q}} \leq 1$ *is clearly true for a positive real number* $n$*. The concentration and dilatation of a* $QFSs$ $G_1$ *are explained as follows by Definition (2.4):*

$$CON(G_1) = \{(p_n, r_{CON(G_1)}(p_n), l_{CON(G_1)}(p_n)) : p \in P\} \tag{2.2}$$

*where* $r_{CON(G_1)}(p_n) = (r^4_{G_1}(p_n)^2, l_{CON(G_1)}(p_n) = (1-(1-l^4_{G_1})^2)^{\frac{1}{4}}$ *; and*

$$DIL(G_1) = \{(p_n, r_{DIL(G_1)}(p_n), l_{DIL(G_1)}(p_n)) : p \in P\} \tag{2.3}$$

*where* $r_{DIL(G_1)}(p_n) = (r^4_{G_1}(p_n)^{\frac{1}{2}}, l_{DIL(G_1)}(p_n) = (1-(1-l^4_{G_1})^{\frac{1}{2}})^{\frac{1}{4}}$.
*Let* $(r,l)$ *be a point for* $IFS$*,* $PFS$*,* $FFS$*, and* $QFS$*, where* $r$ *and* $l$ *lie in* $[0,1]$*. It can be observed that* $r^4 \leq r^3 \leq r^2 \leq r$ *and* $l^4 \leq l^3 \leq l^2 \leq l$*, which implies that* $r^4+l^4 \leq 1$*,* $r^3+l^3 \leq 1$*,* $r^2+l^2 \leq 1$*, and* $r+l \leq 1$*. The space covered by* $IFS$*,* $PFS$*,* $FFS$*, and* $QFS$ *with* $q = 1,2,3,4$ *is shown in Figure 1.*

**Definition 2.5.** [47] Some operations are given as :

(i) $G_1 \subseteq G_2$ iff $\forall\, p_n \in P$, $r_{G_1}(p_n) \leq r_{G_2}(p_n)$ and $l_{G_1}(p_n) \geq l_{G_2}(p_n)$ ;

(ii) $G_1 = G_2$ iff $\forall\, p_n \in P$, $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$;

(iii) If ${G_1}^c = \{(p_n, l_{G_1}(p_n), r_{G_1}(p_n)) \mid p \in P\}$;

**Definition 2.6.** [48] Let $SV_1$ be a mapping, $SV_1 : QFSs(P) \times QFSs(P) \to [0,1]$. $SV_1(G_1, G_2)$ is a distance measures, which holds:
(P1) $0 \leq SV_1(G_1, G_2) \leq 1$.
(P2) $SV_1(G_1, G_2) = 0$.
(P3) $SV_1(G_1, G_2) = SV_1(G_2, G_1)$.
(P4) If $G_1 \subseteq G_2 \subseteq G_3$ then $SV_1(G_1, G_3) \geq Max\,\{SV_1(G_1, G_2), SV_1(G_2, G_3))\}$.

**Definition 2.7.** [49, 50] Consider a mapping $A^*$ that operates on $QFSs$ $F_1$ and $F_2$, defined as $A^* : QFSs(P) \times QFSs(P) \to [0,1]$. The measure similarity $A^*(G_1, G_2)$ satisfies the following properties:

(i) $0 \leq A^*(G_1, G_2) \leq 1$.

(ii) $A^*(G_1, G_2) = A^*(G_2, G_1)$.

(iii) $A^*(G_1, G_1) = 1$.

(iv) If $G_1 \subseteq G_2 \subseteq G_3$, then $A^*(G_1, G_3) \leq A^*(G_1, G_2)$ and $A^*(G_1, G_3) \leq A^*(G_2, G_3)$.

***Existing Similarity Measures***
***Chen [51]***

$$S^*_{CH}(G_1, G_2) = 1 - \frac{\sum_{i=1}^{p} \left(|r_{G_1}(p_n) - r_{G_2}(p_n)| - |l_{G_1}(p_n) - l_{G_2}(p_n)|\right)}{2p} \tag{2.4}$$

***Hong and Kim [52]***

$$S^*_{HK}(G_1, G_2) = 1 - \frac{\sum_{i=1}^{p} \left(|r_{G-1}(p_n) - r_{G_2}(p_n)| + |l_{G_1}(p_n) - l_{G_2}(p_n)|\right)}{2p} \tag{2.5}$$

***Li and Xu [53]*** .

$$S^*_{LX}(G_1, G_2) = 1 - \frac{\sum_{i=1}^{p} \left(|r_{G_1}(p_m) - l_{G_1}(p_n)| + |r_{G_2}(p_n) - l_{G_2}(p_n)|\right)}{4p} - \frac{\sum_{i=1}^{p} \left(|r_{F_1}(p_n) - r_{F_2}(p_n)| + |l_{G_1}(p_n) - l_{G_2}(p_n)|\right)}{4p} \tag{2.6}$$

***Le et al. [54]*** .

$$S^*_{L}(G_1, G_2) = 1 - \left( \frac{\sum_{i=1}^{p} \left((r_{G_1}(p_n) - r_{G_2}(p_n))^2 + (l_{G_1}(p_n) - l_{G_2}(p_n))^2\right)}{2p} \right)^{\frac{1}{2}} \tag{2.7}$$

***Gupta R [55]*** .

$$S^*_{R}(G_1, G_2) = \frac{1}{p} \sum_{i=1}^{p} \frac{S^i_r(G_1, G_2) + S^i_l(G_1, G_2)}{2} \tag{2.8}$$

*Where* $S^i_r(G_1, G_2) = \frac{1}{1+|r_{G_1}(p_n) - r_{G_2}(p_n)|}$, $S^i_l(G_1, G_2) = \frac{1}{1+|l_{G_1}(p_n) - l_{G_2}(p_n)|}$.

## Hausdorff Metric

*A commonly used mathematics metric, the Hausdorff metric, evaluates the distance between two non-empty compact subsets, $G_1$ and $G_2$, in a Banach space $B$. It is calculated by taking the maximum value of the forward and backward direct Hausdorff distances.*

*The formula $d(p, q)$ also indicates the distance between a point $p$ in the set $G_1$ and a point $q$ in the set $G_2$. $k(G_1, G_2) = \max_{p \in G_1} \{\min_{q \in G_2}(||p - q||)\}$ is the forward distance, and*

$$k(G_2, G_1) = \max_{q \in G_2} \left\{ \min_{p \in G_1} (||p - q||) \right\}$$

*is the backward distance. The following is a formal definition of the Hausdorff metric:*

$$K(G_1, G_2) = \max \{k(G_1, G_2), k(G_2, G_1)\} \tag{2.9}$$

*It is important to note the asymmetry of the Hausdorff metric. $k(G_1, G_2) \neq k(G_2, G_1)$ in general,let $G_1 = [\alpha_1, \alpha_2]$ for example, if $R = S.G_2 = [\beta_1, \beta_2]$ are the two intervals, from equation* (2.9), *we have:*

$$K(G_1, G_2) = \max \{|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|\}. \tag{2.10}$$

*The first famous formula for calculating the difference between two intervals is* (2.10). *This measure of distance serves to establish a metric for similarity, playing a pivotal role in demonstrating*

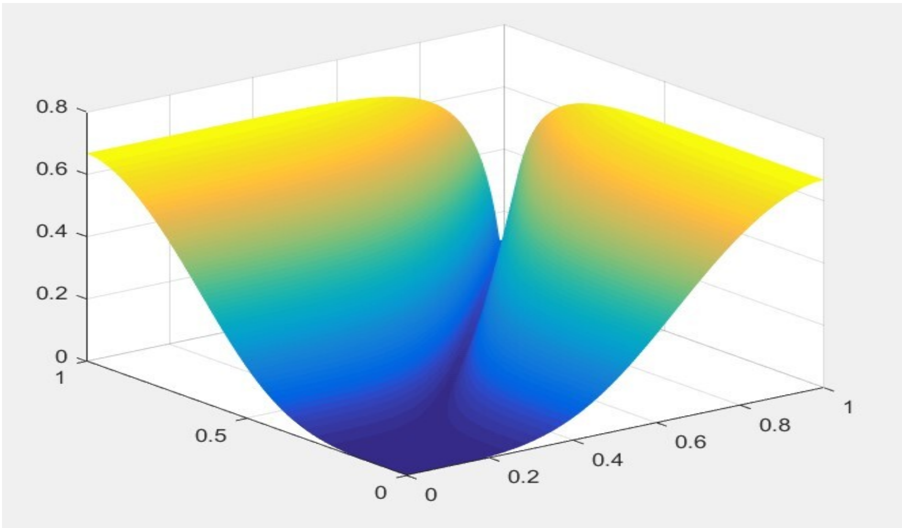


Figure 2: Quartic distance measure graph

*the likeness between two sets. Their widespread applications across diverse fields have notably boosted their prominence. Despite existing distance metrics and Hausdorff-based measures for IFS and $PFSs$ detailed in previous literature [56, 57].*

## 3 Proposed Quartic Distance Measure

*In this section, we will propose a new distance metric to address the shortcomings of current measures for comparing $QFSs$*

*To develop an innovative Hausdorff metric-based distance measure for $QFSs$, we first define a generalized interval representation of these sets. The Hausdorff metric is particularly suitable for this purpose, as it can be directly applied to intervals, making it an ideal choice for comparing $QFSs$. Let $G_1$ and $G_2$ be two Quartic, and consider two subintervals $I_{G_1}(p_n)$ and $I_{G_2}(p_n)$ on the interval $[0,1]$, denoted as $I_{G_1}(p_n) = \left[r^4_{G_1}(p_n), 1 - l^4_{G_1}(p_n)\right]$ and $I_{G_2}(p_n) = \left[r^4_{G_2}(p_n), 1 - l^4_{G_2}(p_n)\right]$, respectively. Here, $K^*(I_{G_1}(p_n), I_{G_2}(p_n))$ is given by $K^*(I_{G_1}(p_n), I_{G_2}(p_n)) = \max\left\{|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |(1 - l^4_{G_1}(p_n)) - (1 - l^4_{G_2}(p_n))|\right\}$. Thus we introduce new Hausdorff metric $SV_1(G_1, G_2)$ between the $QFSs$ $G_1$ and $G_2$ as follows:*

$$SV_1(G_1, G_2) = \frac{1}{q}\sum_{n=1}^{q}\left(\frac{2\sin\left(\frac{\pi}{2}\max\left(|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|\right)\right)}{1 + \sin\left(\frac{\pi}{2}\max(|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|)\right)}\right) \tag{3.1}$$

**Note.** The concepts of distance measures and metrics differ fundamentally. There exists an axiomatic contrast between these measurement tools. The transfer of a metric using the function $k(d^*) = \frac{ad^*}{1+d^*}$ consistently results in a metric. However, it's not always the case that transforming a distance measure yields another distance measure.

For example, if $d^*$ is a measure of distance, then it falls inside the range $d^* \in [0,1]$. Here, $0 \leqslant k(d^*) \leqslant 1.5$ for a given value of $a = 3$. As a result, the function $k(d^*)$ violates property $P1$ of a distance measure. Figure 2 shows the graph of proposed measure.

**Graphs of Quartic distance measure**

**Property 1.** $0 \leq SV_1(G_1, G_2) \leq 1$.

**Proof.** For $G_1, G_2 \in QFS(P)$, it is evident that

$0 \leqslant |r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)| \leqslant 1$.

$0 \leqslant \max\left(|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|\right) \leqslant 1$.

$0 \leqslant \left(\frac{\pi}{2}\max\left(|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|\right)\right) \leqslant \frac{\pi}{2}$

This implies that $0 \leqslant \sin\left(\frac{\pi}{2}\max\left(|r^4_{G_1}(t_m) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l_{G_2}(p_n)|\right)\right) \leqslant \sin\frac{\pi}{2}$

$$0 \leqslant \sin\left(\frac{\pi}{2}\max\left(|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|\right)\right) \leqslant 1 \tag{3.2}$$

This implies that $0 \leqslant 2\sin\left(\frac{\pi}{2}\max\left(|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|\right)\right) \leqslant 2$
From equation (4.3) $1 \leqslant 1 + \sin\left(\frac{\pi}{2}\max\left(|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|\right)\right) \leqslant 2$
Suppose $z_i = \sin\left(\frac{\pi}{2}\max\left(|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|\right)\right)$
then we have $0 \leq \frac{2z_i}{1+z_i} \leq 1$ for all $z_i \in [0,1]$.
Therefore by equation (4.1), we get $0 \leq SV_1(G_1, G_2) \leq 1$.

**Property 2.** $SV_1(G_1, G_2) = 0$.

**Proof.** Suppose $G_1 = G_2$ then $r^4_{G_1}(p_n) = r^4_{G_2}(p_n), l^4_{G_1}(p_n) = l^4_{G_2}(p_n)$ for all $p_n$.
$r^4_{G_1}(p_n) - r^4_{G_2}(p_n) = 0, l^4_{G_1}(p_n) - l^4_{G_2}(p_n) = 0$ for all $p_n$.
therefore, $|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)| = |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)| = 0$,
This gives $SV_1(G_1, G_2) = 0$.
**Property 3.** $SV_1(G_1, G_2) = SV_1(G_2, G_1)$.

**Proof.** It is obvious that $SV_1(G_1, G_2) = SV_1(G_2, G_1)$ holds because, for each $p_n \in P$, $|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)| = |r^4_{G_2}(p_n) - r^4_{G_1}(p_n)|$, $|l^4_{G_1}(p_n) - l^4_{G_2}(p_n)| = |l^4_{G_2}(p_n) - l^4_{G_1}(p_n)|$ and hence, property 3. is proved.
**Property 4.** If $G_1 \subseteq G_2 \subseteq G_3$ then $SV_1(G_1, G_3) \geq Max\{SV_1(G_1, G_2), SV_1(G_2, G_3)\}$.

**Proof.** If $G_1 \subseteq G_2 \subseteq G_3$ then $r^4_{G_1}(p_n) \leq r^4_{G_2}(p_n) \leq r^4_{G_3}(p_n)$, $l^4_{G_1}(p_n) \geq l^4_{G_2}(p_n) \geq l^4_{G_3}(p_n)$, for all $p_n \in P$. Thus, we can get $SV_1(G_1, G_2) = \max\left\{|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|\right\}$, $SV_1(G_1, G_3) = \max\left\{|r^4_{G_1}(p_n) - r^4_{G_3}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_3}(p_n)|\right\}$, and
$SV_1(G_2, G_3) = \max\left\{|r^4_{G_2}(p_n) - r^4_{G_3}(p_n)|, |l^4_{G_2}(p_n) - l^4_{G_3}(p_n)|\right\}$. We arises two cases: If (i) $| r^4_{G_1}(p_n) - r^4_{G_3}(p_n) | \geq | l^4_{G_1}(p_n) - l^4_{G_3}(p_n) |$, then $K^*(G_1, G_3) = |r^4_{G_1}(p_n) - r^4_{G_3}(p_n)|$. However, we have $| l^4_{G_1}(p_n) - l^4_{G_2}(p_n) | \leq | l^4_{G_1}(p_n) - l^4_{G_3}(p_n) | \leq | r^4_{G_1}(p_n) - r^4_{G_3}(p_n) |$ and $| l^4_{G_2}(p_n) - l^4_{G_3}(p_n) | \leq | l^4_{G_1}(p_n) - l^4_{G_3}(p_n) | \leq | r^4_{G_1}(p_n) - r^4_{G_3}(p_n) |$. On the other hand, we have $| r^4_{G_1}(p_n) - r^4_{G_2}(p_n) | \leq | r^4_{G_1}(p_n) - r^4_{G_3}(p_n) |$ and $| r^4_{G_2}(p_n) - l^4_{G_3}(p_n) | \leq | r^4_{G_1}(p_n) - r^4_{G_3}(p_n) |$. By the findings from earlier analyses, we can deduce a novel outcome, $K(I_{G_1}, I_{G_2}) \leq K(I_{G_1}, I_{G_3})$ and $K(I_{G_2}, I_{G_3}) \leq K(I_{G_1}, I_{G_3})$. Hence, we have $SV_1(G_1, G_2) \leq SV_1(G_1, G_3)$.
(ii) If $| r^4_{G_1}(p_n) - r^4_{G_3}(p_n) | \leq | l^4_{G_1}(p_n) - l^4_{G_3}(p_n) |$, then $K(G_1, G_3) = |l^4_{G_1}(p_n) - l^4_{G_3}(p_n)|$. However, we have $| r^4_{G_1}(p_n) - r^4_{G_2}(p_n) | \leq | r^4_{G_1}(p_n) - r^4_{G_3}(p_n) | \leq | l^4_{G_1}(p_n) - l^4_{G_3}(p_n) |$ and $| r^4_{G_2}(p_n) - r^4_{G_3}(p_n) | \leq | r^4_{G_1}(p_n) - r^4_{G_3}(p_n) | \leq | l^4_{G_1}(p_n) - l^4_{G_3}(p_n) |$. On the other hand, we have $| l^4_{G_1}(p_n) - l^4_{G_2}(p_n) | \leq | l^4_{G_1}(p_n) - l^4_{G_3}(p_n) |$ and $| l^4_{G_2}(p_n) - l^4_{G_3}(p_n) | \leq | l^4_{G_1}(p_n) - l^4_{G_3}(p_n) |$. By amalgamating the findings from earlier analyses, we can deduce a novel outcome, $K(I_{G_1}, I_{G_2}) \leq K(I_{G_1}, I_{G_3})$ and $K(I_{G_2}, I_{G_3}) \leq K(I_{G_1}, I_{G_3})$. Hence, we have $SV_1(G_1, G_2) \leq SV_1(G_1, G_3)$. Hence, cases (i) and (ii) satisfied the (P4).
If $G_1 \subseteq G_2 \subseteq G_3$ then $r^4_{G_1}(p_n) \leq r^4_{G_2}(p_n) \leq r^4_{G_3}(p_n)$, $l^4_{G_1}(p_n) \leq l^4_{G_2}(p_n) \leq l^4_{G_3}(p_n)$, for all $p_n \in P$.
$|r^4_{G_1}(p_n) - r^4_{G_3}(p_n)| \geq |r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|$, $|l^4_{G_1}(p_n) - l^4_{G_3}(p_n)| \geq |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|$.

$$\left(|r^4_{G_1}(p_n) - r^4_{G_3}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_3}(p_n)|\right) \geqslant \left(|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|\right)$$
$\max\left(|r^4_{G_1}(p_n) - r^4_{G_3}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_3}(p_n)|\right) \geqslant \max\left(|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|\right)$
$\frac{\pi}{2}\max\left(|r^4_{G_1}(p_n) - r^4_{G_3}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_3}(p_n)|\right) \geqslant \frac{\pi}{2}\max\left(|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|\right)$
$\sin\left(\frac{\pi}{2}\max\left(|r^4_{G_1}(p_n) - r^4_{G_3}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_3}(p_n)|\right)\right) \geqslant$
$\sin\left(\frac{\pi}{2}\max\left(|r^4_{G_1}(p_n) - r^4_{G_2}(p_n)|, |l^4_{G_1}(p_n) - l^4_{G_2}(p_n)|\right)\right)$.
Since for all $u_i, z_i \in [0,1]$ and $u_i \leq z_i$,
Now $\frac{2u_i}{1+u_i} \leq \frac{2z_i}{1+z_i}$.
Hence $SV_1(G_1, G_3) \leq SV_1(G_1, G_2)$,
Similarly, we can prove $SV_1(G_1, G_3) \leq SV_1(G_2, G_3)$.
**Corollary 1.** If $G_1 = (1,0)$ and $G_2 = (0,1)$, then $SV_1(G_1, G_2) = 1$.

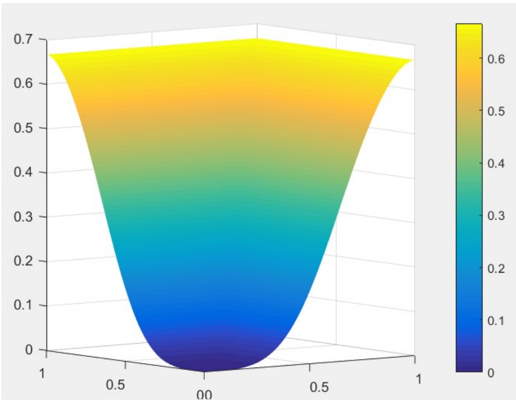

Figure 3: Graph of $SV1$ verifying the value $(0,0)$

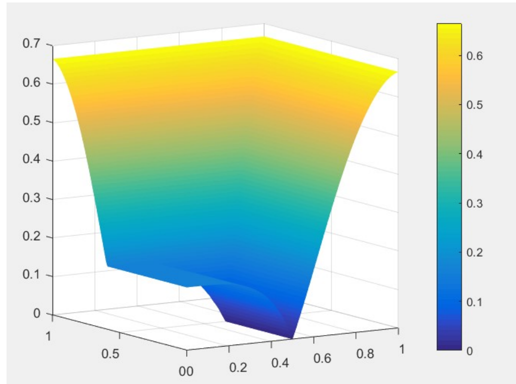

Figure 4: Graph of $SV1$ verifying the value $(0.5,0.5)$

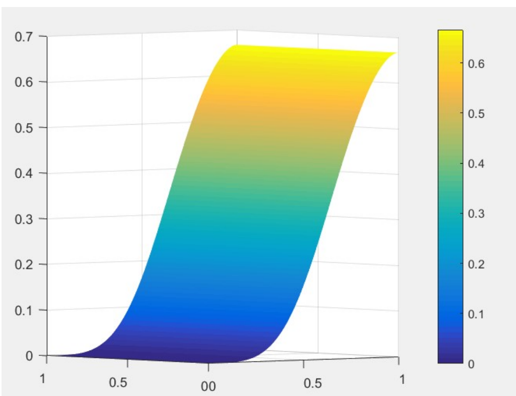

Figure 5: $SV1$'s graph showing the value $(0,1)$

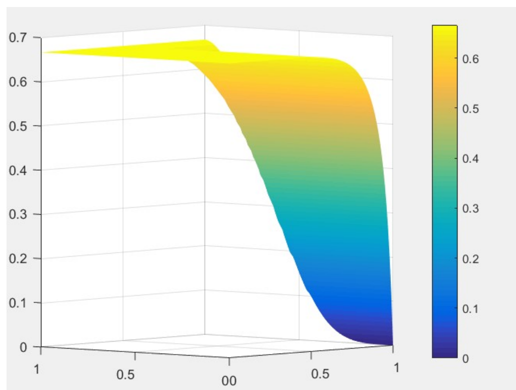

Figure 6: $SV1$'s graph showing the value $(1,0)$

**Proof:** Proof is clear.
**Corollary 2.** If $G_1 = (0,0)$ and $G_2 = (0,0)$, then $SV_1(G_1, G_2) = 0$.
**Proof.** Proof is clear.

**Example 3.1.** Given two $QFSs$ $G_1$, $G_2$ in $P = \{p_i\}$ these $QFSs$ are expressed as follows:
$G_1 = \{(p_i, r, l)\}$, $G_2 = \{(p_i, l, r)\}$
The variables $r$ and $l$ can only have values within [0,1], and their sum, $r + l$, must not be more than 1. Figure 4 illustrates this link. Figure 6 illustrates the measure of the $SV_1$ between $QFSs$ $G_1$ and $G_2$. It is evident from looking at Figure 3 that the $SV_1$ value constantly falls between 0 and 1 when r and l change. Thus, $SV_1$'s Property 1 is confirmed. Graphs 3-7 illustrate the influence of $r$ and $l$, along with the variables $\alpha$ and $\beta$, on $SV_1$ across different cases. Upon closer inspection of Graphs 3-7, it is clear that the correlation between $SV_1$ and the $r$ and $l$ variables of $QFSs$ is non-linear, effectively capturing the non-linear attributes of $SV_1$. Moreover, the maximum value of $SV_1$ derived from $r$ and $l$ in diverse cases is as follows:
$* \; G_2 = \{(p_i, 1, 0)\} \Rightarrow SV_1(G_1, G_2) = 1, forG_1 = \{p_i, 0, \beta)\}, \beta \in [0,1]$.
$* \; G_2 = \{(p_i, 0, 1)\} \Rightarrow SV_1(G_1, G_2) = 1, forG_1 = \{p_i, \alpha, 0)\}, \alpha \in [0,1]$.
$* \; G_2 = \{(p_i, 0.5, 0.5)\} \Rightarrow SV_1(G_1, G_2) = 1, forG_1 = \{p_i, 0, 0)\}$.

**Graphs of Quartic distance measure for different values**

For each $p_m \in P$, a weight $w_m$ is assigned, where $0 \leqslant w_m \leqslant 1$ and $\sum_{m=1}^{q} w_m = 1$ holds. Let's delve into the method of establishing a weighted distance measure for IFSs: Distance Measures $SV_w : QFSs(P) \times QFSs(P) \to [0,1]$ is given as

$$SV_w(G_1, G_2) = \sum_{m=1}^{q} w_m \left( \frac{2\sin\left(\frac{\pi}{2}\max\left(|r^4_{G_1}(p_m) - r^4_{G_2}(p_m)|, |l^4_{G_1}(p_m) - l^4_{G_2}(p_m)|\right)\right)}{1 + \sin\left(\frac{\pi}{2}\max(|r^4_{G_1}(p_m) - r^4_{G_2}(p_m)|, |l^4_{G_1}(t_m) - l^4_{G_2}(p_m)|)\right)} \right) \tag{3.3}$$

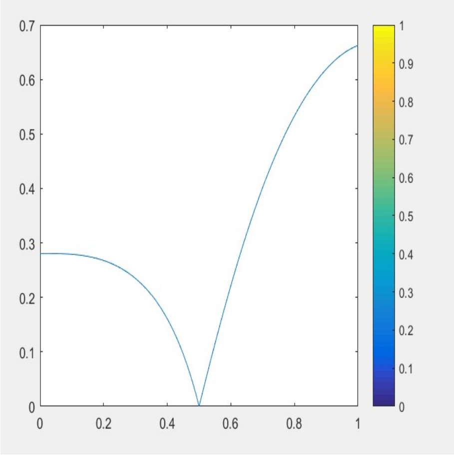


Figure 7: $SV1$'s line graph showing the value $(0, 1)$

**Note:** Equation 3.3 becomes Equation 3.1 if we change $w_m = 1/q$.

## 4 Quartic Similarity measure

The well-known notion of the dual distance and similarity measure allows us to determine the similarity among two $QFSs$ by considering their distance apart and the Hausdorff metric. Let's suppose that $h$ represents a function that exhibits a monotonic decrease. Since $0 \leq SV_1(G_1, G_2) \leq 1$, $h(1) \leq h(SV_1(G_1, G_2)) \leq h(0)$. This implies $0 \leq (h(SV_1(G_1, G_2)) - h(1))/(h(0) - h(1)) \leq 1$.

**Definition 4.1.** Let $P = \{p_1, p_2, ..., p_q\}$ be the universal set and $G_1 = \{(p_n, r_{G_1}(p_n), l_{G_1}(p_n)) \mid p_n \in P\}$, $G_2 = \{(p_n, r_{G_2}(p_n), l_{G_2}(p_n)) \mid p_n \in P\}$ be two $QFSs$ on $P$. Let us suppose that $k$ be a monotone decreasing function. Then, a new similarity measure $V_S(G_1, G_2)$ between two $QFSs$ $G_1$ and $G_2$ is defined as:

$$V_S(G_1, G_2) = \frac{k(SV_1(G_1, G_2)) - k(1)}{k(0) - k(1)} \tag{4.1}$$

Equation $(4.1)$ allows for the computation of different similarity metrics by choosing a suitable value for $k$. A simple approach is to select the linear function $k(z) = 1 - z$. Using this, the similarity between $QFSs$ can then be determined.

$$V_{S1}(G_1, G_2) = 1 - SV_1 \tag{4.2}$$

Moreover, an alternative option is to opt for a straightforward rational function such as $k(z) = 1/(1 + z)$. In this case, the similarity metric between $QFSs$ $G_1$ and $G_2$ is defined as follows:

$$V_{S2}(G_1, G_2) = \frac{1 - SV_1}{1 + SV_1} \tag{4.3}$$

$$V_{S3}(G_1, G_2) = \frac{e^{-SV_1} - e^{-1}}{1 - e^{-1}} \tag{4.4}$$

## 5 An Analytical Comparison

In the comparative analysis, we illustrate the practical utility of the proposed methods. To achieve this, we present examples related to pattern recognition and subsequently apply them to queries involving fuzzy linguistic variables.

Table 1: Values of similarity measures

| $ProposedMeasures$ | | | | |
|---|---|---|---|---|
| | $S_1$ | $S_2$ | $S_3$ | $Result$ |
| $V_{S1}$ | $V_{S1}(G_1,R)=0.6730$ | $V_{S1}(G_2,R)=0.6909$ | $V_{S1}(G_3,R)=0.7440$ | $G_3$ |
| $V_{S2}$ | $V_{S2}(G_1,R)=0.3206$ | $V_{S2}(G_2,R)=0.5531$ | $V_{S2}(G_3,R)=0.5748$ | $G_3$ |
| $V_{S3}$ | $V_{S3}(G_1,R)=0.3540$ | $V_{S3}(G_2,R)=0.5934$ | $V_{S3}(G_3,R)=0.6134$ | $G_3$ |
| $ExitingMeasures$ | | | | |
| $S^*_{CH}$[51] | $S^*_{CH}(G_1,R)=0.8092$ | $S^*_{CH}(G_2,R)=0.8296$ | $S^*_{CH}(G_3,R)=0.8954$ | $G_3$ |
| $S^*_{HK}$[52] | $S^*_{HK}(G_1,R)=0.8759$ | $S^*_{HK}(G_2,R)=0.8407$ | $S^*_{HK}(G_3,R)=0.9092$ | $G_3$ |
| $S^*_{LX}$[53] | $S^*_{LX}(G_1,R)=0.7426$ | $S^*_{CH}(G_2,R)=0.7630$ | $S^*_{CH}(G_3,R)=0.7815$ | $G_3$ |
| $S^*_{L}$[54] | $S^*_{L}(G_1,R)=0.8092$ | $S^*_{L}(G_2,R)=0.8296$ | $S_{L}(G_3,R)=0.8954$ | $G_3$ |
| $S^*_{R}$[55] | $S^*_{R}(G_1,R)=0.8838$ | $S^*_{R}(G_2,R)=0.8203$ | $S^*_{R}(G_3,R)=0.8960$ | $G_3$ |

### 5.1 Pattern Recognition

Using Equations (4.2-4.4), a number of pattern recognition cases are shown.

**Example 5.1.** Let three $QFSs$ in $P=\{p_1,p_2,p_3\}$. The three $QFSs$ are:
$G_1=\{(p_1,0.3,0.7),(p_2,0.9,0.9),(p_3,0.7,0.9)\}$, $G_2=\{(p_1,0.7,0.9),(p_2,0.2,0.9),(p_3,0.5,0.7)\}$,
$G_3=\{(p_1,0.9,0.5),(p_2,0.7,0.8),(p_3,0.5,0.4)\}$.
Let a given sample be
$A=\{(p_1,0.7,0.7),(p_2,0.9,0.9),(p_3,0.7,0.2)\}$. Using Equations (4.2-4.3), the computed values are summarized in Table 1. From the table, it is clear that $A$ shows the highest similarity to $G_3$. This indicates that the proposed similarity metrics effectively classify the $FSs$ based on the maximum similarity value.

**Example 5.2.** Let three $QFSs$ in $T=\{p_1,p_2,p_3\}$. The three $QFSs$ are:
$G_1=\{(p_1,0.4,0.8),(p_2,0.6,0.5),(p_3,0.8,0.9)\}$, $G_2=\{(p_1,0.8,0.9),(p_2,0.3,0.9),(p_3,0.6,0.8)\}$,
$G_3=\{(p_1,0.7,0.6),(p_2,0.8,0.9),(p_3,0.6,0.5)\}$.
Let a given sample be
$A=\{(p_1,0.8,0.8),(p_2,0.6,0.7),(p_3,0.8,0.3)\}$.

The provided examples of different types highlight the practicality and effectiveness of the proposed similarity measures. Additionally, an illustrative case is presented to evaluate the similarity between linguistic variables.

## 6 Linguistic Variables

An accompanying example is provided to illustrate how linguistic variables can be evaluated for similarity using particular measures. The example uses these measures, which are described in Equations (4.2) to (4.4), to evaluate similarities between linguistic variables. Tahani initially introduced the use of fuzzy sets to establish a framework for fuzzy query processing [58]. Expanding on Tahani's work, Kacprzyk and Ziolkowski further developed the concept by integrating fuzzy linguistic quantifiers into database queries [59]. Furthermore, Petry carried out a thorough investigation into fuzzy databases, including both their applications and underlying principles [60]. Candan et al. highlighted the crucial role of employing similarity measures to effectively

Table 2: Values of similarity measures

| $ProposedMeasures$ | | | | |
|---|---|---|---|---|
| | $S_1$ | $S_2$ | $S_3$ | $Result$ |
| $V_{S1}$ | $V_{S1}(G_1, R) = 0.6187$ | $V_{S1}(G_2, R) = 0.6484$ | $V_{S1}(G_3, R) = 0.6948$ | $G_3$ |
| $V_{S2}$ | $V_{S2}(G_1, R) = 0.3625$ | $V_{S2}(G_2, R) = 0.4666$ | $V_{S2}(G_3, R) = 0.5176$ | $G_3$ |
| $V_{S3}$ | $V_{S3}(G_1, R) = 0.3980$ | $V_{S3}(G_2, R) = 0.5064$ | $V_{S3}(G_3, R) = 0.5575$ | $G_3$ |
| $ExitingMeasures$ | | | | |
| $S^*_{CH}$[51] | $S^*_{CH}(G_1, R) = 0.7926$ | $S^*_{CH}(G_2, R) = 0.8245$ | $S^*_{CH}(G_3, R) = 0.8939$ | $G_3$ |
| $S^*_{HK}$[52] | $S^*_{HK}(G_1, R) = 0.8815$ | $S^*_{HK}(G_2, R) = 0.8537$ | $S^*_{HK}(G_3, R) = 0.9056$ | $G_3$ |
| $S_{LX}$[53] | $S^*_{LX}(G_1, R) = 0.7648$ | $S^*_{CH}(G_2, R) = 0.7606$ | $S^*_{CH}(G_3, R) = 0.7848$ | $G_3$ |
| $S_L$[54] | $S^*_L(G_1, R) = 0.5963$ | $S^*_L(G_2, R) = 0.6205$ | $S^*_L(G_3, R) = 0.6648$ | $G_3$ |
| $S_R$[55] | $S^*_R(G_1, R) = 0.8621$ | $S^*_R(G_2, R) = 0.8353$ | $S^*_R(G_3, R) = 0.8960$ | $G_3$ |

query a database [61]. Improving the usefulness of fuzzy queries requires an understanding of the degree of similarity between fuzzy sets. Additionally, Hussain and Yang examined similarity metrics for language variables [62]. The recommended similarity metrics between $QFSs$ are used in the example (6.1) to describe the similarities between linguistic variables.

**Example 6.1.** Let $G_1 = \left\{(p_n, r^4_{G_1}(p_n), l^4_{G_1}(p_n)) \mid p_n \in P\right\}$ be a $QFS$ on $K$. $\forall$ positive real number $p$, we have $QFSs$ $G^d_1$ from the definition 2.4, with
$G^d_1 = \left\{\left(p_n, (r^4_{G_1}(p_n))^d, (1-(1-l^4_{G_1})^d)^{\frac{1}{4}}\right) : p_n \in P\right\}, d > 0, 2 < q < \infty.$
The $QFS$ $G_1$ has two linguistic operators, dilation and concentration. The dilation of $G_1$ is represented as $DIL(G_1) = G_1^{\frac{1}{2}}$ and the concentration of $G_1$ is $CON(G_1) = G^2_1$. The parameters "very $(G_1)$" and "more or less $(G_1)$" can be used to describe these procedures, respectively. In this case, the $QFS$ is contained in the set $P = p_1, p_2, p_3, p_4, p_5$.
$G_1 = (p_1, 0.7, 0.5), (p_2, 0.6, 0.9), (p_3, 0.8, 0.6), (p_4, 0.9, 0.5), (p_5, 1.0, 0.0)$.
$G_1^{\frac{1}{2}} = (p_1, 0.84, 0.71), (p_2, 0.77, 0.95), (p_3, 0.89, 0.77), (p_4, 0.95, 0.71), (p_5, 1.00, 0.00)$.
$G^2_1 = (p_1, 0.49, 0.25), (p_2, 0.36, 0.81), (p_3, 0.64, 0.36), (p_4, 0.81, 0.25), (p_5, 1.00, 0.00)$.
$G^4_1 = (p_1, 0.24, 0.06), (p_2, 0.13, 0.66), (p_3, 0.41, 0.13), (p_4, 0.66, 0.06), (p_5, 1.00, 0.00)$.
Within the framework of set $K$, the $QFS$ $G_1$ embodies the concept of "Wide." It has been established that operations like $CON(G_1)$ and $DIL(G_1)$ serve as linguistic hedge expressions such as "Somewhat Wide," "Wider," and "Widest."

Hence, we establish the following associations:

$G_1^{\frac{1}{2}}$ signifies "Somewhat Wide,"
$G^2_1$ signifies "Wider,"
$G^4_1$ signifies "Widest".

The abbreviations for the terms "Wide," "Somewhat Wide," "Wider," and "Widest" are denoted as W, S.W, WI., and WID. respectively. The similarity measures described in Equations (4.2) to (4.4) are utilized to evaluate the similarity between $QFS$. The outcomes of this comparison are presented in Table 3, enabling assessments to be drawn regarding the similarities among different $QFSs$.

Table 3: Similarity Measures

| | $L$ | $S.W.$ | $WI.$ | $WID.$ |
|---|---|---|---|---|
| | 1.0000 | 0.9763 | 0.9775 | 0.9809 |
| $W$ | 1.0000 | 0.7775 | 0.7817 | 0.7943 |
| | 1.0000 | 0.8028 | 0.8055 | 0.8171 |
| | 0.9763 | 1.0000 | 0.9807 | 0.9841 |
| $S.W.$ | 0.7775 | 1.0000 | 0.7966 | 0.8037 |
| | 0.8028 | 1.0000 | 0.8190 | 0.8251 |
| | 0.9775 | 0.9807 | 1.0000 | 0.9766 |
| $WI.$ | 0.7817 | 0.7966 | 1.0000 | 0.7726 |
| | 0.8055 | 0.8190 | 1.0000 | 0.7989 |
| | 0.9809 | 0.9841 | 0.9766 | 1.0000 |
| $WID.$ | 0.7943 | 0.8037 | 0.7726 | 1.0000 |
| | 0.8171 | 0.8251 | 0.7989 | 1.0000 |

$G_1(W, WID.) > G_1(W, WI.) > G_1(W, S.W.), G_1(S.W., WID.) > G_1(S.W., WI.) > G_1(S.W., W.),$
$G_1(WI., S.W.) > G_1(WI., W.) > G_1(WI., WID.),$
$G_1(WID., S.W.) > G_1(WID., W.) > G_1(WID., WI.).$
The proposed similarity measures (4.2) to (4.4) have been demonstrated to fulfill essential criteria (Table 3), enabling precise comparisons among W, S.W., WI., and WID. This substantiates their utility and practical applicability. Figure 8 shows the graph of linguistic variables.

## 7 Comparative Analysis

A similarity measure [63] in a recent study.

$$S^*(G_1, G_2) = \frac{S_P^*(G_1, G_2) + 1 - d_w^*}{2} \tag{7.1}$$

where $S_P^*(G_1, G_2) = 1 - \frac{1}{2p}\sum_{i=1}^{p}\left\{|r_{G_1}^q - r_{G_2}^q| + |l_{G_1}^q - l_{G_2}^q| + |m_{G_1}^q - m_{G_2}^q|\right\}$ and
$d_w^* = 1 - \frac{1}{p}\sum_{i=1}^{p}\cos\left\{\frac{\pi}{2}\left\{\max\left(|r_{G_1}^q - r_{G_2}^q|, |l_{G_1}^q - l_{G_2}^q|\right)\right\}\right\}$
Now, we conduct a quantitative analysis of the similarity measures we introduced and compare them with those proposed by [63]. The goal is to highlight the advantages and superior performance of our suggested approaches.

**Example 7.1.** Let $P = \{p_1\}$ be the universe of discourse. Suppose $G_1 = \{(p_1, 0.3, 0.4)\}$, $G_2 = \{(p_1, 0.4, 0.3)\}$, and $G_3 = \{(p_1, 0.0, 0.45)\}$. It becomes evident that $G_1$ is more similar to $G_2$ than it is to $G_3$. Therefore, similarity measure between $G_1$ and $G_2$ would be greater than similarity measure between $G_1$ and $G_3$. According to the $S^*(G_1, G_2)$ [36], we obtain $S^*(G_1, G_2) = 0.8718 > S^*(G_1, G_3) = 0.6230$. And $S_{V1}(G_1, G_2) = 0.7294 > S_{V1}(G_1, G_3) = 0.3755$; $S_{V2}(G_1, G_2) = 0.5741 > S_{V1}(G_1, G_3) = 0.2312$; $S_{V3}(G_1, G_2) = 0.6250 > S_{V1}(G_1, G_3) = 0.2652$. Equation 7.1's conclusion is supported by the results of equations 4.2 – 4.4, which show that $G_1$ is more like $G_2$ than $G_3$. Therefore, the three QFSs ($G_1$, $G_2$, and $G_3$) are correctly classified by the proposed similarity measures.

## 8 Application to MCDM

Decision-making holds immense significance in our day-to-day existence, given the frequent cases demanding our selection of the most favourable choice among a finite set of options. In a wide range of fields, including the biological sciences, engineering, computer science, business intelligence, medical sciences, financial management, and the social and political sciences, this

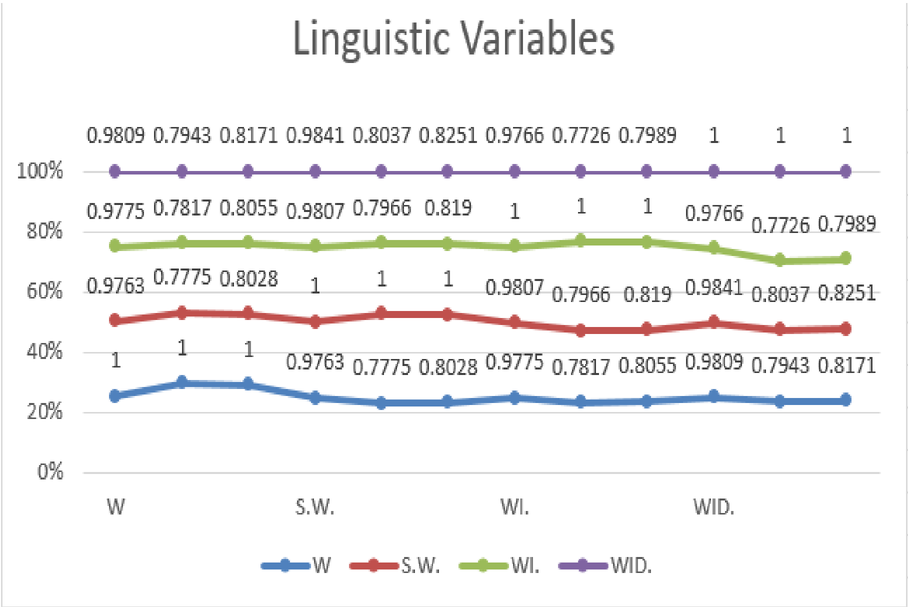


Figure 8: Graph of Linguistic Variables

process is extremely important. A well-known technique called MCDM is intended to make decision-making easier by analyzing a variety of factors and selecting the best choice. Integrating decision-makers' (DMs') preference data into the assessment of the available options is a crucial component of MCDM. Uncertainties, ambiguities, and lack of expertise, however, are common in real-world scenarios and can result in mistakes and inaccuracies while making decisions. Because these uncertainties can significantly affect the efficacy and outcomes of steps, it is necessary to identify and manage them.

QFSs have proven to be a useful tool for decision-making when faced with unclear or inadequate information. They provide a high degree of accuracy and precision, which aids decision-makers in navigating the complexities of unclear or inadequate data. Using QFSs makes the decision-making process more solid and dependable. First introduced in [64], the TODIM approach is an interactive MCDM methodology that has shown remarkable efficacy in real-life problems. Further research in this area has been expanded by other scholars [65, 66].

This research introduces an advanced version of the TODIM method, referred to as Orthopairian Fuzzy TODIM. The approach integrates the similarity measures outlined in Equations (4.2)-(4.4) to manage situations involving QFSs effectively. By applying the Orthopairian Fuzzy TODIM framework, we can efficiently tackle MCDM challenges across a variety of real-world scenarios. The set of alternatives is represented as $G = \{G_1, G_2, ..., G_i\}$, while the set of criteria is denoted as $V = \{V_1, V_2, ..., V_i\}$.

Following the TODIM methodology, the evaluation begins by analyzing the gains and losses of each alternative $V_i$ with respect to every criterion $V_j$. Subsequently, the dominance level of each alternative $G_i$ over another alternative $G_t$ is determined for each criterion $V_j$. During the completion of these computations, the options $G_i$ are arranged according to their total performance scores in descending order. To solve real-world problems using the proposed similarity measures from Equations (4.2)-(4.4), the following steps outline the process for developing the algorithm:

**1. Construction of Fuzzy Decision Matrix (FDM)**

The decision-making process involves a set of alternatives, represented as $G = \{G_1, G_2, ..., G_i\}$, and criteria denoted by $V = \{V_1, V_2, ..., V_i\}$. To facilitate this process, we define an orthopair fuzzy decision matrix (OFDM), $G = [g_{ij}]p \times q$, gave by the decision-makers involved in a MCDM issue. Each element $g_{ij}$ within the matrix is a Quartic Fuzzy Number (QFN), expressed as $h_{ij} = (r_{ij}, l_{ij})$. Here, $r_{ij}$ and $l_{ij}$ represent the degree to which the alternative $G_i$ satisfies the criterion $V_j$ and the degree to which it does not, respectively. The matrix must adhere to the condition $0 \leqslant r^4G_1(t_m)+l^4G_1(t_m) \leq 1$ is satisfied, where $r^4G_1(t_m) \in [0,1]$ and $l^4G_1(t_m) \in [0,1]$, ensuring consistency and validity. In general, the QFNs supplied by the decision-makers are integrated into the decision matrix $A$. These ambiguous numbers successfully handle the uncertainties and ambiguities included in the decision-making process by encapsulating the degrees of pleasure and discontent for each alternative with regard to the criteria.

$$D = [d_{ij}]_{m\times n} = \begin{array}{c} \\ B_1 \\ B_2 \\ \vdots \\ B_i \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \dots & E_n \end{array} \\ \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1j} \\ d_{21} & d_{22} & \cdots & d_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ d_{i1} & d_{i2} & \cdots & d_{ij} \end{bmatrix} \end{array} \tag{8.1}$$

**2. construction of Normalised FDM (NOFDM)**

$$\overline{W} = (w_{i,j})_{m\times n} = \begin{cases} u_{ij}, & \text{for benefit attribute } t_m, \\ u_{ij}^c, & \text{for cost attribute } t_m, \end{cases} \tag{8.2}$$

During this phase, we transform the original $OFDM\ D = [d_{ij}]$ into a $NOFDM$. However, if cost serves as the criterion, then we consider $d_{ij}^c$.

**3. Relative weight**

Step 3 involves computing the relative weight $v_{jr}$ for each criterion $V_j$. This calculation, done using the formula $v_{jr} = v_j/v_r$, derives the individual weights $v_j$ for each criterion $V_j$, where $j$ ranges from 1 to $n$. These weights $v_j$ follow the condition $\sum_{j=1}^{n} v_j = 1$.

In the Orthopairian Fuzzy TODIM methodology, the highest weight $v_r$ is chosen as the reference weight. Consequently, dividing this reference weight by the other weights $v_j$ yields the relative weights $v_{jr}$.

This computation allows us to gauge the relative significance of each criterion concerning the reference weight. It's a vital step that incorporates criterion weightings into the decision-making process.

$v_r = \max(v_j : j = \{1,2,...,n\})$

**4. Degree of dominance $G_i$**

We establish the level of superiority of each option $G_i$ over another option $G_t$ concerning the criterion $V_j$ in the following manner:

$$\psi_j(G_i,G_j) = \begin{cases} \sqrt{\frac{v_{jr}S^*(I_{ij},I_{tj})}{\sum_{j=1}^{n} v_j}}, & I_{ij} > I_{tj} \\ 0, & I_{ij} = I_{tj} \\ \frac{-1}{\theta}\sqrt{\frac{\sum_{j=1}^{n} v_j S^*(I_{ij},I_{tj})}{v_{jr}}}, & I_{ij} < I_{tj} \end{cases} \tag{8.3}$$

Determining the extent of one alternative $G_i$ dominating another alternative $G_t$ concerning criterion $V_j$ involves intricate calculations using specific formulas. Based on whether it shows a gain or a loss, dominance is determined. When $I_{ij} > I_{tj}$ or $I_{ij} - I_{tj} > 0$, it signifies a gain. In such cases, we apply the formula $\sqrt{\frac{v_{jr}S(I_{ij},I_{tj})}{\sum_{j=1}^{n} v_j}}$, where $S(I_{ij},I_{tj})$ denotes similarity metrics, and $v_{jr}$ is the relative weight of criterion $V_j$.

Conversely, if $I_{ij} < I_{tj}$ or $I_{ij} - I_{tj} < 0$, it indicates a loss. Here, the formula used is $\frac{-1}{\theta}\sqrt{\frac{\sum_{j=1}^{n} v_j S^*(I_{ij},I_{tj})}{v_{jr}}}$, where $v_{jr}$ represents the relative weight of criterion $V_j$, and $\theta$ denotes the attenuation factor of the loss.

It's crucial to note that when $I_{ij} = I_{tj}$, it's considered inconclusive.

These detailed computations enable the evaluation of dominance levels for each pair of alternatives and their corresponding criteria, taking into account both gains and losses while incorporating the relative importance of the criteria. This process plays a crucial role in analyzing and ranking the alternatives in the decision-making framework.

**5. Overall dominance**
The measure $\chi(G_i, G_t)$ is employed to assess the overall dominance level of $N_i$ with respect to each alternative $G_t$. The value of $\chi(G_i, G_t)$ reflects the degree to which alternative $G_i$ outperforms alternative $G_t$, signifying its relative superiority.
**6. Overall value of** $G_i$ We use this formula:

$$\lambda(G_i) = \frac{\sum_{j=1}^{n} \chi(G_i, G_t) - \min(\sum_{i=1}^{m} \chi(G_i, G_t))}{\max\left\{\sum_{j=1}^{n} \chi(G_i, G_t)\right\} - \min(\sum_{i=1}^{m} \chi(G_i, G_t))}$$

Since $0 \leqslant \lambda \leqslant 1$, it is obvious that we must choose the larger value of $\lambda_i$, which will be regarded as the optimal option $G_i$.
**7. Ranking**
At this step, alternatives are arranged in descending order based on their overall scores. The alternative with the highest $\lambda_i$ value is deemed the most preferable. The $\lambda_i$ values reflect the comprehensive assessment or preference associated with each alternative. This ranking approach enables the identification and prioritization of the most suitable options, aligning with the decision criteria and the preferences of the decision-makers. Figure 9 shows that the experts make framework how we work on MCDM problems.

## A Case Study

In the modern era, we are bound by the internet for our daily basic needs, which eventually made ample tech companies boom around the world. We turn on the Internet for fast and informative resources through uninterrupted Internet services. Some companies around the globe are providing the basic internet services NPC (Network Parameter Control). Some of the best NPC, which are providing an exceptional boost among all the different working companies, are mentioned below which we are considering in our case study (Google, Amazon, TCS, Open A.I and Apple) such as $\{G_1, G_2, G_3, G_4, G_5\}$

The internet has become a crucial element of daily life, enabling global connectivity and offering access to vast information and resources. The necessity for reliable and fast internet services increased during the COVID-19 epidemic, especially in underdeveloped countries. The efforts of a planning commission in a developing nation to select the best Network Provider Company (NPC) for enhancing online education are examined in this case study. Five NPCs are evaluated as part of the evaluation process using a set of predetermined criteria.

In a developing nation, a planning commission recognizes the value of reliable and effective internet services in enabling the sharing of information and data online. To address this, the commission has shortlisted five potential Network Provider Companies (NPCs): $\{G_1, G_2, G_3, G_4, G_5\}$.

Evaluation Criteria: The commission has outlined five specific criteria to evaluate and determine the most suitable NPC to improve online education, as detailed in Table 4.
Assessment Procedure and Resolution: The planning commission intends to conduct a thorough assessment of each NPC using the established criteria. This evaluation will include reviewing performance history, gathering insights through surveys, and dependability of each provider.

The findings from this process will guide the commission in selecting the NPC that offers the most suitable internet services for advancing e-learning within the country. The selected NPC will be responsible for working closely with educational institutions, delivering cost-effective and reliable internet solutions, and ensuring robust technical support and data protection measures.

Table 5 presents the Orthopair FDM for the alternatives across the evaluation criteria. Table 6 shows the Criterion's weight using $w_j = \frac{\max(r^4_{G_1}, r^4_{G_2}) + \min(r^4_{G_1}, r^4_{G_2})}{2}$, and Table 7 shows the relative weights.
To assess the dominance degree of $G_i$ over $G_t$ for each criterion, Equation (4.2) is applied individually. In Table 8, and the dominance degree that corresponds to criterion $V_1$ is written as $\psi_1(G_i, G_t)$. Similarly, the dominance degrees for criteria $V_2$, $V_3$, $V_4$, and $V_5$,are calculated as

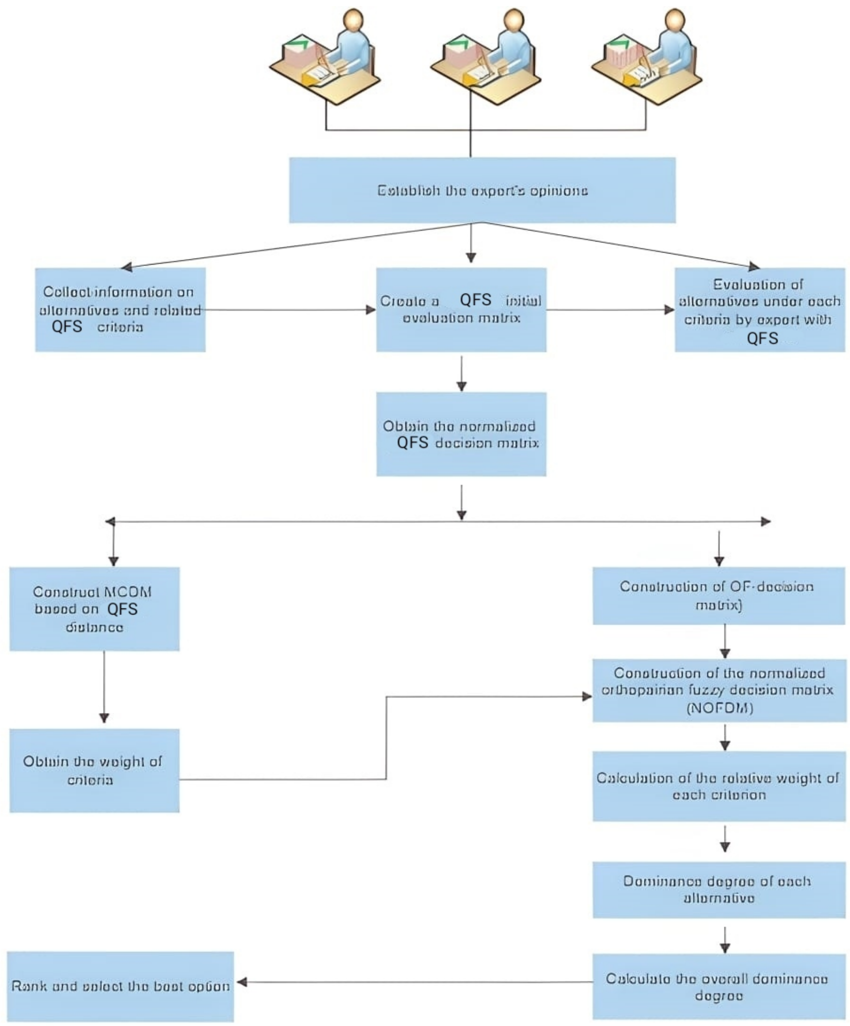


Figure 9: Graph of Framework

Table 4: Overview of the Criteria

| Criteria | Representation of Criteria |
|---|---|
| 1. Government-recognized (V1) | The commission evaluates the companies should be a Government-approved company for better providence and unobstructed provi--dence of sources. It also makes a point clear about scams on the internet which eventually makes a lot of difference in terms of acceptance of resources. Ice cream on the cake being it has a certification from the respective government. |
| 2. Internationally approved (V2) | The following companies should be internationally approved for internet acceptance world-wide. For better providence of international education among the world. |
| 3. Approved certification (V3) | This criterion determines the companies should be certified and have a certification in every aspect for better. Eventually, a cert--ified company has better feedback than other companies. |
| 4. Network providence (V4) | The commission evaluates the companies have the best possible network providence for a smooth and better functional providence in data and inormation. An unobstructed network has a better result among NPC, education will be easier with more network providence. |
| 5. Variety of content (V5) | This criteria defines that the companies have a variety of content for better providence and students of online data and information will use that content for their education good content, and quality of content are importance among students, a platform with good content attracts a lot of students and other needy peoples around the globe. |

Table 5: QFSs decision matrix

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | (0.8,0.4) | (0.6,0.5) | (0.5,0.6) | (0.6,0.7) | (0.7,0.6) |
| $G_2$ | (0.8,0.6) | (0.7,0.5) | (0.5,0.8) | (0.2,0.8) | (0.5,0.7) |
| $G_3$ | (0.7,0.7) | (0.4,0.7) | (0.6,0.8) | (0.3,0.9) | (0.6,0.5) |
| $G_4$ | (0.5,0.7) | (0.7,0.2) | (0.5,0.2) | (0.5,0.2) | (0.4,0.8) |
| $G_5$ | (0.7,0.5) | (0.6,0.6) | (0.6,0.5) | (0.6,0.8) | (0.2,0.9) |

Table 6: Criterion's weight

| $V$ | $w1$ | $w2$ | $w3$ | $w4$ | $w5$ |
|---|---|---|---|---|---|
| $max/min$ | 0.280 | 0.260 | 0.110 | 0.110 | 0.230 |

Table 7: Relative weight

| $V$ | $Rw1$ | $Rw2$ | $Rw3$ | $Rw4$ | $Rw5$ |
|---|---|---|---|---|---|
| $max/min$ | 0.280 | 0.260 | 0.110 | 0.110 | 0.230 |

Table 8: The matrix for $V_1$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.3722 | 0.2043 | 0.2043 | 0.3501 |
| $G_2$ | 0.3864 | 0.0000 | -0.7351 | -0.3898 | 0.0000 |
| $G_3$ | 0.3259 | -0.6112 | 0.0000 | -0.6112 | -0.6030 |
| $G_4$ | 0.3259 | -0.6112 | -0.4970 | 0.0000 | -0.7089 |
| $G_5$ | 0.4542 | 0.2422 | -0.8716 | 0.0000 | 0.0000 |

Table 9: The matrix for $V_2$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | -0.5672 | 0.0000 | 0.0000 | 0.0000 |
| $G_2$ | -0.5464 | 0.0000 | -0.7351 | -0.4970 | 0.0000 |
| $G_3$ | 0.3864 | -0.4784 | 0.0000 | -0.8749 | 0.3501 |
| $G_4$ | -0.5464 | -0.4784 | 0.0000 | 0.0000 | 0.1565 |
| $G_5$ | -0.6423 | 0.4376 | 0.2423 | 0.1082 | 0.0000 |

$\psi_2(G_i,G_t)$, $\psi_3(G_i,G_t)$, $\psi_4(G_i,G_t)$, and $\psi_5(G_i,G_t)$, with the evaluations detailed in Tables 9, 10, 11, and 12, respectively. These computations show how much $G_i$ outperforms each alternative $G_t$ in terms of the corresponding criterion. This method makes understanding the relative performance and dominance of the alternatives according to each criterion easier.
To assess the dominance degree of $G_i$ over $G_t$ for each criterion, Equation (4.3) is applied individually. In Table 13, and the dominance degree that corresponds to criterion $V_1$ is written as $\psi_1(G_i,G_t)$. Similarly, the dominance degrees for criteria $V_2$, $V_3$, $V_4$, and $V_5$,are calculated as $\psi_2(G_i,G_t)$, $\psi_3(G_i,G_t)$, $\psi_4(G_i,G_t)$, and $\psi_5(G_i,G_t)$, with the evaluations detailed in Tables 14, 15, 16, and 17, respectively. These computations reveal how much $G_i$ outperforms each alternative $G_t$ in terms of the corresponding criterion. This approach facilitates comprehension of the alternatives' relative dominance and performance concerning each criterion.

To assess the dominance degree of $G_i$ over $G_t$ for each criterion, Equation (4.4) is applied individually. The results are shown in Table 18, and the dominance degree that corresponds to criterion $V_1$ is written as $\psi_1(G_i,G_t)$. Similarly, the dominance degrees for criteria $V_2$, $V_3$, $V_4$, and $V_5$,are calculated as $\psi_2(G_i,G_t)$, $\psi_3(G_i,G_t)$, $\psi_4(G_i,G_t)$, and $\psi_5(G_i,G_t)$, with the evaluations detailed in Tables 19, 20, 21, and 22, respectively.

**Sensitive analysis under similarity measures**

Case Study Conclusion: The advancement of online education in developing nations depends on the choice of internet service provider. The planning commission can determine which NPC best suits the demands of the populace by assessing important elements including coverage, price, connectivity, technical assistance, and data security. Better access to online education will

Table 10: The matrix for $V_3$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.0000 | -0.7351 | 0.0000 | 0.0000 |
| $G_2$ | -0.4608 | 0.0000 | 0.2042 | -0.7351 | 0.4116 |
| $G_3$ | 0.3864 | 0.0000 | 0.0000 | -0.7351 | 0.2953 |
| $G_4$ | -0.6423 | -0.2537 | -0.3898 | 0.0000 | -0.2697 |
| $G_5$ | -0.5464 | 0.4376 | 0.2042 | 0.1082 | 0.0000 |

Table 11: The matrix for $V_4$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | -0.5672 | 0.0000 | -0.7351 | 0.0000 |
| $G_2$ | -0.5464 | 0.0000 | 0.2042 | 0.1382 | 0.2953 |
| $G_3$ | 0.4542 | 0.3723 | 0.0000 | 0.1382 | 0.2456 |
| $G_4$ | -0.6423 | 0.1664 | 0.0800 | 0.0000 | -0.2697 |
| $G_5$ | -0.5464 | 0.2610 | 0.1700 | 0.1082 | 0.0000 |

Table 12: The matrix for $V_5$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | -0.6030 |
| $G_2$ | -0.6423 | 0.0000 | -1.0246 | -1.0246 | 0.0000 |
| $G_3$ | 0.4542 | -0.6667 | 0.0000 | -0.6112 | -0.5086 |
| $G_4$ | -0.5464 | -0.3234 | -0.7351 | 0.0000 | 0.1565 |
| $G_5$ | -0.4542 | 0.4376 | 0.0000 | 0.2423 | 0.0000 |

Table 13: The matrix for $V_1$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.3068 | 0.1603 | 0.1603 | 0.2886 |
| $G_2$ | 0.3185 | 0.0000 | -0.5768 | -0.2831 | 0.0000 |
| $G_3$ | 0.2557 | -0.3016 | 0.0000 | -0.4634 | -0.4970 |
| $G_4$ | 0.2557 | -0.3016 | -0.3674 | 0.0000 | -0.6289 |
| $G_5$ | 0.4030 | 0.3068 | -0.7184 | 0.0000 | 0.0000 |

Table 14: The matrix for $V_2$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | -0.4675 | 0.0000 | 0.0000 | 0.0000 |
| $G_2$ | -0.5036 | 0.0000 | -0.5768 | -0.3674 | 0.0000 |
| $G_3$ | 0.3185 | -0.3754 | 0.0000 | -0.7184 | 0.2886 |
| $G_4$ | -0.5036 | -0.3754 | 0.0000 | 0.0000 | 0.1137 |
| $G_5$ | -0.5698 | 0.3882 | 0.1997 | 0.0784 | 0.0000 |

Table 15: The matrix for $V_3$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.0000 | -0.5768 | 0.0000 | 0.0000 |
| $G_2$ | -0.3616 | 0.0000 | 0.1603 | -0.5768 | 0.3651 |
| $G_3$ | 0.3185 | 0.0000 | 0.0000 | -0.5768 | 0.2317 |
| $G_4$ | -0.5698 | -0.1842 | -0.2831 | 0.0000 | -0.1959 |
| $G_5$ | -0.4504 | 0.1997 | 0.1603 | 0.0787 | 0.0000 |

Table 16: The matrix for $V_4$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.0000 | 0.0000 | -0.5768 | 0.0000 |
| $G_2$ | -0.4504 | 0.0000 | 0.1603 | 0.1021 | 0.2317 |
| $G_3$ | 0.4030 | 0.3068 | 0.0000 | 0.1603 | 0.1861 |
| $G_4$ | -0.5741 | 0.1209 | 0.0575 | 0.0000 | -0.1959 |
| $G_5$ | -0.4504 | 0.1978 | 0.1287 | 0.0787 | 0.0000 |

Table 17: The matrix for $V_5$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | -0.4970 |
| $G_2$ | -.5698 | 0.0000 | -0.9090 | -0.5768 | 0.0000 |
| $G_3$ | 0.4030 | -0.5915 | 0.0000 | -0.4634 | -0.3991 |
| $G_4$ | -0.4504 | -0.2391 | -0.5768 | 0.0000 | 0.1137 |
| $G_5$ | -0.5698 | 0.3882 | 0.0000 | 0.1997 | 0.0000 |

Table 18: The matrix for $V_1$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.3259 | 0.1717 | 0.1717 | 0.3065 |
| $G_2$ | 0.3383 | 0.0000 | -0.6178 | -0.3053 | 0.0000 |
| $G_3$ | 0.2739 | -0.3242 | 0.0000 | -0.4982 | -0.5280 |
| $G_4$ | 0.2739 | -0.3242 | -0.3959 | 0.0000 | -0.6562 |
| $G_5$ | 0.4205 | 0.3259 | -0.7631 | 0.0000 | 0.0000 |

Table 19: The matrix for $V_2$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | -0.4966 | 0.0000 | 0.0000 | 0.0000 |
| $G_2$ | -0.4784 | 0.0000 | -0.6178 | -0.3959 | 0.0000 |
| $G_3$ | 0.3383 | -0.4020 | 0.0000 | -0.7631 | 0.3065 |
| $G_4$ | -0.4784 | -0.4020 | 0.0000 | 0.0000 | 0.1227 |
| $G_5$ | -0.5946 | 0.4050 | 0.2121 | 0.0849 | 0.0000 |

Table 20: The matrix for $V_3$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.0000 | -0.4020 | 0.0000 | 0.0000 |
| $G_2$ | -0.3873 | 0.0000 | 0.1717 | -0.6178 | 0.3810 |
| $G_3$ | 0.3383 | 0.0000 | 0.0000 | -0.6178 | 0.2482 |
| $G_4$ | -0.5946 | -0.1987 | -0.3053 | 0.0000 | -0.2113 |
| $G_5$ | -0.5946 | 0.3259 | 0.1717 | 0.0849 | 0.0000 |

Table 21: The matrix for $V_4$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.0000 | 0.0000 | -0.3873 | 0.0000 |
| $G_2$ | -0.4784 | 0.0000 | 0.1717 | 0.1100 | 0.2482 |
| $G_3$ | 0.4205 | 0.3259 | 0.0000 | 0.1717 | 0.2002 |
| $G_4$ | -0.5945 | 0.1304 | 0.0619 | 0.0000 | -0.2113 |
| $G_5$ | -0.4784 | 0.2128 | 0.1385 | 0.0849 | 0.0000 |

Table 22: The matrix for $V_5$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | -0.5280 |
| $G_2$ | -0.5945 | 0.0000 | -0.9484 | -0.6178 | 0.0000 |
| $G_3$ | 0.4205 | -0.6172 | 0.0000 | -0.4982 | -0.4274 |
| $G_4$ | -0.4784 | -0.2576 | -0.6178 | 0.0000 | 0.1227 |
| $G_5$ | -0.5945 | 0.4050 | 0.0000 | 0.2121 | 0.0000 |

Table 23: Overall dominance degree of $V_{S1}$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ | $\sum_{j=1}^{n} \chi(V_i, V_t)$ |
|---|---|---|---|---|---|---|
| $G_1$ | 0.0000 | -0.1950 | -1.2659 | -0.5308 | -0.2529 | -2.2446 |
| $G_2$ | -1.8095 | 0.0000 | -2.0864 | -2.5083 | 0.6709 | -5.7333 |
| $G_3$ | 2.0071 | -1.3840 | 0.0000 | -2.6942 | -0.2206 | -2.2917 |
| $G_4$ | -2.0515 | -0.8891 | -1.5419 | 0.0000 | -0.9353 | -5.4178 |
| $G_5$ | -1.7351 | 1.8160 | -0.2551 | 0.5669 | 0.0000 | 0.3927 |

Table 24: Overall dominance degree of $V_{S2}$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ | $\sum_{j=1}^{n} \chi(V_i, V_t)$ |
|---|---|---|---|---|---|---|
| $G_1$ | 0.0000 | -0.1607 | -0.4165 | -0.4165 | -0.2084 | -1.2021 |
| $G_2$ | -1.5669 | 0.0000 | -1.7420 | -1.7020 | 0.5968 | -4.4141 |
| $G_3$ | 1.6987 | -0.9617 | 0.0000 | -2.0617 | -0.1897 | -1.5144 |
| $G_4$ | -1.8422 | -0.9794 | -1.1698 | 0.0000 | -0.7933 | -4.7847 |
| $G_5$ | -1.6374 | 1.4807 | -0.2297 | 0.4358 | 0.0000 | 0.0494 |

Table 25: Overall dominance degree of $V_{S3}$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ | $\sum_{j=1}^{n} \chi(V_i, V_t)$ |
|---|---|---|---|---|---|---|
| $G_1$ | 0.0000 | -0.1707 | -0.2303 | -0.2156 | -0.2215 | -0.8381 |
| $G_2$ | -1.6003 | 0.0000 | -1.8406 | -1.8268 | 0.6292 | -4.6385 |
| $G_3$ | 1.7915 | -1.0175 | 0.0000 | -2.2056 | -0.2005 | -1.6321 |
| $G_4$ | -1.8720 | -1.0521 | -1.2571 | 0.0000 | -0.8334 | -5.0146 |
| $G_5$ | -1.8416 | 1.6746 | -0.2408 | 0.4668 | 0.0000 | 0.0590 |

Table 26: $\lambda_i$ values of $V_i$ over each $V_t$

| $S_1$ | $\lambda_1$ | $S_2$ | $\lambda_1$ | $S_3$ | $\lambda_1$ |
|---|---|---|---|---|---|
| $G_1$ | 0.8231 | $G_1$ | 0.8135 | $G_1$ | 0.7561 |
| $G_2$ | 0.4782 | $G_2$ | 0.3691 | $G_2$ | 0.3342 |
| $G_3$ | 0.7567 | $G_3$ | 0.6806 | $G_3$ | 0.6678 |
| $G_4$ | 0.0000 | $G_4$ | 0.0000 | $G_4$ | 0.0000 |
| $G_5$ | 1.0000 | $G_5$ | 1.0000 | $G_5$ | 1.0000 |

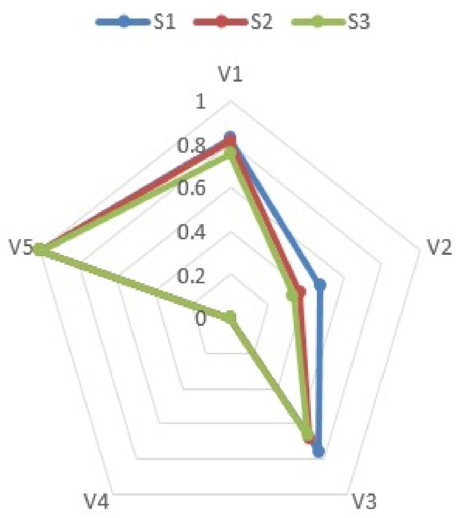


Figure 10: Graph of Quartic Fuzzy Sets with Sensitive Analysis

Table 27: Ranking of Similarity Measures

| $Similarity Measures$ | Ranking |
|---|---|
| $V_{S1}$ | $G_5 > G_1 > G_3 > G_2 > G_4$ |
| $V_{S2}$ | $G_5 > G_1 > G_3 > G_2 > G_4$ |
| $V_{S3}$ | $G_5 > G_1 > G_3 > G_2 > G_4$ |

Table 28: The matrix for $V_1$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.8755 | 0.7045 | 0.8205 | 0.9005 |
| $G_2$ | 0.8255 | 0.0000 | -0.6795 | -0.4285 | 0.6795 |
| $G_3$ | 0.7705 | -0.6295 | 0.0000 | -0.7223 | -0.7205 |
| $G_4$ | 0.7045 | -0.6322 | -0.5101 | 0.0000 | -0.9438 |
| $G_5$ | 0.9438 | 0.8005 | -0.7205 | 0.0000 | 0.0000 |

Table 29: The matrix for $V_2$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | -0.8755 | 0.0000 | 0.0000 | 0.0000 |
| $G_2$ | -0.7255 | 0.0000 | -0.6795 | -0.6351 | 0.0000 |
| $G_3$ | 0.7205 | -0.7705 | 0.0000 | -0.7180 | 0.8205 |
| $G_4$ | -0.7955 | -0.8943 | 0.0000 | 0.0000 | 0.5287 |
| $G_5$ | -0.9438 | 0.9005 | 0.8205 | 0.4787 | 0.0000 |

be made possible by this decision, which will aid in closing the digital gap and advancing the nation's socioeconomic development. The graph of quartic fuzzy sets with sensitivity analysis is shown in Figure 10, and the ranking of similarity measures is shown in Table 27.

Case Study Analysis: In this study, the proposed similarity measures were compared with the similarity metric $S^*(G_1, G_2)$introduced by [63]. To demonstrate this comparison, a case study was used as a practical example. The initial steps (Steps 1, 2, and 3) of both approaches are consistent, as they rely on the similarity $S^*(G_1, G_2)$ as defined by [63]. However, in Steps 4 through 7, the similarity measure $S^*(G_1, G_2)$ was specifically applied as follows:
In Step 4, the dominance degree of $N_i$ over $N_t$ was determined for the criteria $V_1$, $V_2$, $V_3$, $V_4$, and $V_5$. The similarity measure $S^*(G_1, G_2)$. was employed to calculate these dominance degrees. The outcomes for each criterion are provided in Tables 28-33.

Step 5: The overall dominance degrees of $V_i$ over $V_t$ calculated using the similarity measure $S^*(G_1, G_2)$, are summarized in Table 33. This table highlights the extent to which each $V_i$ over $V_t$, offering a clear perspective on their relative strengths and comparative advantages.

Table 34 shows the overall values of $V_i$ over $V_t$ using $\lambda_i$ for the similarity measure $S^*(G_1, G_2)$ in Step 6.

Final Ranking: The possibilities are ranked in descending order based on their $\lambda_i$. values, as shown in Table 35. The table provides the final ranking, which is as follows: $G_5 > G_4 > G_3 > G_1 > G_2$.

The correlation factors for each alternative are displayed in Figure 11. Comparison of Meth-

Table 30: The matrix for $V_3$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.0000 | -0.7045 | 0.0000 | 0.0000 |
| $G_2$ | -0.6545 | 0.0000 | 0.6795 | -0.8693 | 0.8755 |
| $G_3$ | 0.7205 | 0.0000 | 0.0000 | -0.7894 | 0.6545 |
| $G_4$ | -0.9438 | -0.5037 | -0.4537 | 0.0000 | -0.5787 |
| $G_5$ | -0.7955 | 0.8255 | 0.6545 | 0.4287 | 0.0000 |

Table 31: The matrix for $V_4$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.0000 | 0.0000 | -0.8205 | 0.0000 |
| $G_2$ | -0.8005 | 0.0000 | 0.8693 | 0.6351 | 0.8605 |
| $G_3$ | 0.8938 | 0.9193 | 0.0000 | 0.7894 | 0.4414 |
| $G_4$ | -0.9005 | 0.5387 | 0.2916 | 0.0000 | -0.5787 |
| $G_5$ | -0.8205 | 0.7572 | 0.4414 | 0.4287 | 0.0000 |

Table 32: The matrix for $V_5$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | -0.9005 |
| $G_2$ | -0.8688 | 0.0000 | -0.8755 | -0.8605 | 0.0000 |
| $G_3$ | 0.8938 | -0.9438 | 0.0000 | -0.4414 | -0.6545 |
| $G_4$ | -0.8205 | -0.7072 | -0.8355 | 0.0000 | 0.5787 |
| $G_5$ | -0.9005 | 0.9255 | 0.0000 | 0.8255 | 0.0000 |

Table 33: Overall dominance degree of $S^*(F_1, F_2)$

| $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ | $\sum_{j=1}^{n} \chi(V_i, V_t)$ |
|---|---|---|---|---|---|---|
| $G_1$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $G_2$ | -2.2238 | 0.0000 | -.6857 | -2.1585 | 2.4155 | -2.9130 |
| $G_3$ | 3.9991 | -1.4245 | 0.0000 | -1.8817 | 0.5414 | 1.2343 |
| $G_4$ | -2.7558 | -2.1987 | -1.5077 | 0.0000 | -0.9938 | -7.4560 |
| $G_5$ | -2.5165 | 4.2092 | 1.1959 | 2.1616 | 0.0000 | 5.0502 |

Table 34: Overall outcomes of $V_i$ over each alternative $V_t$

| $S^*(G_1, G_2)$ | $G$ | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ |
|---|---|---|---|---|---|---|
| $\lambda_i$ | 0.5962 | 0.3632 | 0.6949 | 0.0000 | 1.0000 | |

Table 35: Ranking of Alternatives

| $Similarity Measures$ | Ranking |
|---|---|
| $S^*$[63] | $G_5 > G_4 > G_3 > G_1 > G_2$ |
| $S^*_{CH}$[51] | $G_5 > G_3 > G_4 > G_2 > G_1$ |
| $S^*_{HK}$[52] | $G_5 > G_2 > G_3 > G_1 > G_4$ |
| $S^*_{LX}$[53] | $G_5 > G_2 > G_3 > G_1 > G_4$ |
| $S^*_{L}$[54] | $G_5 > G_2 > G_3 > G_1 > G_4$ |
| $S^*_{R}$[55] | $G_5 > G_2 > G_3 > G_1 > G_4$ |

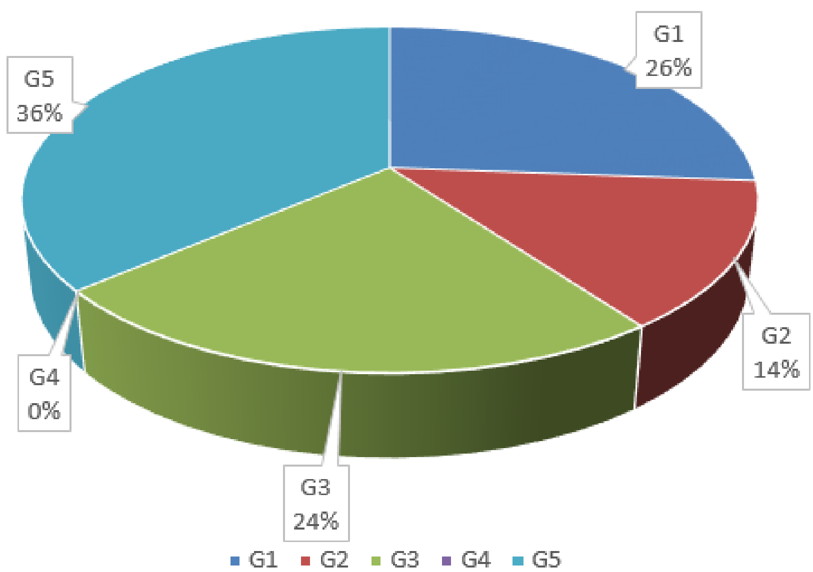


Figure 11: correlation coefficients for each alternative's proximity

ods: When comparing the proposed measures (4.2)-(4.4) with the approach introduced by [63], it is evident that both methodologies produce identical outcomes for the given application. $G_5$ is as the best option, with the final ranking order being $G_5 > G_4 > G_3 > G_1 > G_2$. Notably, an analysis of Equations (4.2)-(4.4) compared to those in [63] demonstrates that the proposed methods are more straightforward and intuitive. Additionally, a comparative evaluation using Example 7 reveals that the proposed methods outperform the approach by [63].

## 9 Conclusion

The innovative idea of QFS is presented in this study as a remedy for the drawbacks and restrictions of the IFS, PFS, and FFS models. Through a detailed numerical analysis, the paper critically examines the weaknesses of IFS, PFS, and FFS, highlighting their limitations.

The primary benefit of the suggested strategy is its successful application of QFSs, which work incredibly well in unpredictable situations. Because of QFSs' adaptability, DMs can more accurately depict information, reducing the uncertainty that comes with decision-making. Additionally, the method exhibits extensive applicability to a range of MCDM issues.
The following are some major areas where the study contributes:

- A new definition of QFS is introduced, incorporating the concept of hesitation grade.
- A geometric comparison is made between QFSs and IFSs, PFSs, and FFSs, highlighting the superior features of QFS.
- The basic operations and accuracy functions for QFSs are explored.
- A QF TOPSIS method is developed to solve MCDM problems, particularly for selecting the optimal NPC.
- Distance measures for QFSs are analyzed and compared, demonstrating the advantages of QFSs and evaluating their performance relative to existing distance and similarity measures.

The proposed approach has some notable limitations that must be acknowledged. A significant drawback is its inability to address varying degrees of uncertainty in the evaluation results, which is a critical aspect of decision-making. This omission limits the method's capacity to fully capture the complexities of real-world scenarios. Additionally, the computational demands of working with Quartic fuzzy sets, especially when dealing with extensive or high-dimensional datasets, can pose challenges, potentially restricting its practicality in certain applications. Future studies should concentrate on improving the method by combining it with multi-decision-maker group decision-making processes, taking into account a range of perspectives and knowledge, in order to overcome these difficulties. Moreover, the distance measures developed for the Quartic TOPSIS method could be adapted to broader decision-making systems across various fields. Applying more MCDM methods, like COPRAS and ELECTRE-I, could increase the process's efficacy and dependability even further.

## Declarations

**Availability of data and material**
This article contains all of the data that was analyzed for the study.
**Code availability**
Not applicable.
**Conflicts of interest**
No conflicting interests are stated by the authors.
**Funding**
No funding is available.
**Author Contribution**

**Vanita Rani**: Prepared the initial document, wrote the technique, used software, performed formal analysis, and created the visualization.
**Satish Kumar**: Investigation, original draft review, editing, supervision, visualization, and conceptual work. The author(s) evaluated and approved the completed paper.

## References

[1] Keith, A.J. and Ahner, D.K.: A survey of decision making and optimization under uncertainty. Ann. Oper. Res. **300**, 319-353 (2021)

[2] Hariri, R.H., Fredericks, E.M. and Bowers,K.M.: Uncertainty in big data analytics: Survey, opportunities, and challenges. J. Big Data **6**, 44 (2019)

[3] Dutt, L.S., Kurian, M.: Handling of uncertainty—A survey. Int. J. Sci. Res. Publ. **3**, 1-4 (2013)

[4] Li, Y., Chen, J., Feng, L.: Dealing with uncertainty: A survey of theories and practices. IEEE Trans. Knowl. Data Eng., **25**, 2463-2482 (2013)

[5] Wang, X., Song, Y.: Uncertainty measure in evidence theory with its applications. Appl. Intell. **48**, 1672-1688 (2018)

[6] Yager, R.R.: On using the Shapley value to approximate the Choquet integral in cases of uncertain arguments, IEEE Trans. Fuzzy Syst. **26**, 1303-1310 (2018)

[7] Zavadskas,E.K., Turskis, Z., Vilutienė, T., Lepkova, N.: Integrated group fuzzy multi-criteria model: Case of facilities management strategy selection. Expert Syst. Appl. **82**, 317-331 (2017)

[8] Zadeh, L.A.: Fuzzy sets. Inf. Control. **8**, 338-353 (1965)

[9] Atanassov, K.T.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**, 87-96 (1986)

[10] Xu, Z., Chen, J., Wu, J.: Clustering algorithm for intuitionistic fuzzy sets. Inf. Sci. **178**, 3775-3790 (2008)

[11] Yager, R.R.: Some aspects of intuitionistic fuzzy sets. Fuzzy Optim. Decis. Mak. **8**, 67-90 (2009)

[12] Aggarwal,M.: Rough information set and its applications in decision making. IEEE Trans. Fuzzy Syst. **25**, 265-276

[13] Ayub, S., Shabir, M., Riaz, M., Mahmood, W., Bozanic, D., Marinkovic, D.: Linear diophantine fuzzy rough sets: A new rough set approach with decision making. Symmetry **14**, 525 (2022)

[14] Wei, W., Liang, J.: Information fusion in rough set theory: An overview, Inf. Fusion. **48**, 107-118 (2019)

[15] Luo, X., Li, W., Zhao, W.: Intuitive distance for intuitionistic fuzzy sets with applications in pattern recognition. Appl. Intell. **48**, 2792-2808 (2018)

[16] Xiao, F.: A distance measure for intuitionistic fuzzy sets and its application to pattern classification problems, IEEE Trans. Syst. Man Cybern. Syst. **51**, 3980-3992 (2021)

[17] Ejegwa, P.A., Ahemen, S.: Enhanced intuitionistic fuzzy similarity operators with applications in emergency management and pattern recognition. Granul. Comput., 1-12 (2022)

[18] Zeng, W., Cui, H., Liu, Y., Yin, Q., Xu, Z.: Novel distance measure between intuitionistic fuzzy sets and its application in pattern recognition. Iran. J. Fuzzy Syst. **19**, 127-137 (2022)

[19] Khatibi, V., Montazer, G.A.: Intuitionistic fuzzy set vs. fuzzy set application in medical pattern recognition. Artif. Intell. Med. **47**, 43-52 (2009) [Google Scholar] [CrossRef]

[20] Ejegwa, P.A., Onyeke, I.C., Adah, V.: An algorithm for an improved intuitionistic fuzzy correlation measure with medical diagnostic application. Ann. Optim. Theory Pract. **3**, 51-66 (2020).

[21] Garg, H., Kaur, G.: Novel distance measures for cubic intuitionistic fuzzy sets and their applications to pattern recognitions and medical diagnosis. Granul. Comput. **5**, 169-184 (2020)

[22] Xu, Z., Zhao, N.: Information fusion for intuitionistic fuzzy decision making: An overview. Inf. Fusion **28**, 10-23 (2016) [Google Scholar] [CrossRef]

[23] Garg, H.: Novel intuitionistic fuzzy decision making method based on an improved operation laws and its application. Eng. Appl. Artif. Intell. **60**, 164-174 (2017) [Google Scholar] [CrossRef]

[24] Yu, D.: Prioritized information fusion method for triangular intuitionistic fuzzy set and its application to teaching quality evaluation. Int. J. Intell. Syst. **28**, 411-435 (2013) [Google Scholar] [CrossRef]

[25] Torra, V.: Hesitant fuzzy sets. Int. J. Intell. Syst. **25**, 529-539 (2010) [CrossRef]

[26] Rodríguez, R.M., Martínez, L., Torra, V., Xu, Z.S., Herrera, F.: Hesitant fuzzy sets: State of arts and future directions. Int. J. Intell. Syst. **29**, 495-524 (2014) [CrossRef]

[27] Xu, Z.S., Xia, M.M.: Distance and similarity measures for hesitant fuzzy sets. Inf. Sci., **181**, 2128-2138 (2011)

[28] Hussian, Z., Yang, M.S.: Entropy for hesitant fuzzy sets based on Hausdorff metric with construction of hesitant fuzzy TOPSIS. Int. J. Fuzzy Syst. **20**, 2517-2533 (2018) [CrossRef]

[29] Yang, M.S., Hussian, Z.: Distance and similarity measures of hesitant fuzzy sets based on Hausdorff metric with applications to multi-criteria decision making and clustering, Soft Comput. **23**, 5835-5848 (2019) [CrossRef]

[30] Zhang, Z., Lin, J., Miao, R., Zhou, L.: Novel distance and similarity measures on Hesitant fuzzy linguistic term sets with application to pattern recognition. J. Intell. Fuzzy Syst. **37**, 2981-2990 (2019) [CrossRef]

[31] Yager, R.R., Abbasov, A.M.: Pythagorean membership grades, complex numbers, and decision making. Int. J. Intell. Syst. **28**, 436-452 (2013)

[32] Yager, R.R.: Pythagorean membership grades in multicriteria decision making. IEEE Trans, Fuzzy Syst. **22**, 958-965 (2013)

[33] Senapati, T., Yager, R. R.: Fermatean fuzzy sets. Journal of Ambient Intelligence and Humanized Computing. **11(2)**, 663-674 (2019) https://doi.org/10.1007/s12652-019-01377-0

[34] Yager, R. R.: Generalized Orthopair Fuzzy Sets. IEEE Transactions on Fuzzy Systems, **25(5)**, 1222- 1230 (2017) https://doi.org/10.1109/tfuzz.2016.2604005

[35] Yager, R. R., Alajlan, N.: Approximate reasoning with generalized orthopair fuzzy sets. Information Fusion **38**, 65-73 (2017) https://doi.org/10.1016/j.inffus.2017.02.005

[36] Hwang, C.L., Yoon, K.: Methods for Multiple Attribute Decision Making. Multiple Attribute Decision Making 58-191 (1981)

[37] Adeel, A., Akram, M., Koam, A. N.: Group Decision-Making Based on m-Polar Fuzzy Linguistic TOPSIS Method. Symmetry**11(6)**, 735 (2019)

[38] Akram, M., Shumaiza, Smarandache, F.: Decision-Making with Bipolar Neutrosophic TOPSIS and Bipolar Neutrosophic ELECTRE-I. Axioms **7(2)**, 33 (2018) https://doi.org/10.3390/axioms7020033

[39] Boran, F. E., Genc, S., Kurt, M., Akay, D.: A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method, Expert Systems with Applications. **36(8)**, 11363-11368 (2009) https://doi.org/10.1016/j.eswa.2009.03.039

[40] Chen, C.T.: Extensions of the TOPSIS for group decision-making under fuzzy environment. Fuzzy Sets and Systems **114(1)**, 1-9 (2000) https://doi.org/10.1016/s0165-0114(97)00377-1

[41] Chu, T.-C.: Selecting Plant Location via a Fuzzy TOPSIS Approach. The International Journal of Advanced Manufacturing Technology **20(11)**, 859-864 (2002) https://doi.org/10.1007/s001700200227

[42] Balioti, V., Tzimopoulos, C., Evangelides, C.: Multi-Criteria Decision Making Using TOPSIS Method Under Fuzzy Environment. Application in Spillway Selection. Proceedings **2(11)**, 637 (2018) https://doi.org/10.3390/proceedings2110637

[43] Li, D.-F., Nan, J.-X.: Extension of the TOPSIS for Multi-Attribute Group Decision Making under Atanassov IFS Environments. International Journal of Fuzzy System Applications **1(4)**, 47-61 (2011) https://doi.org/10.4018/ijfsa.2011100104

[44] Nădăban, S., Dzitac, S., Dzitac, I.: Fuzzy TOPSIS: A General View. Procedia Computer Science 91, 823-831 (2016) https://doi.org/10.1016/j.procs.2016.07.088

[45] Mahdavi, I., Heidarzade, A., Sadeghpour-Gildeh, B., Mahdavi-Amiri, N.: A general fuzzy TOPSIS model in multiple criteria decision making. The International Journal of Advanced Manufacturing Technology **45(3-4)**, 406-420 (2009) https://doi.org/10.1007/s00170-009-1971-5

[46] Vahdani, B., Mousavi, S. M., Tavakkoli-Moghaddam, R.: Group decision making based on novel fuzzy modified TOPSIS method. Applied Mathematical Modelling **35(9)**, 4257-4269 (2011) https://doi.org/10.1016/j.apm.2011.02.040

[47] Peng, X., Liu, L.: Information measures for q-rung orthopair fuzzy sets. Int. J. Intell. Syst. **34**, 1795-1834 ( 2019)

[48] Wang, C., Zhou, X., Tu, H. and Tao, S., Some geometric aggregation operators based on picture fuzzy sets and their application in multiple attribute decision making. Ital. J. Pure Appl. Math **37**, 477-492 (2017)

[49] Dengfeng, L. and Chuntian, C.: New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. Pattern recognition letters **23(1-3)**, 221-225 (2002)

[50] Mitchell, H.B.: On the Dengfeng-Chuntian similarity measure and its application to pattern recognition. Pattern Recognition Letters **24(16)**, 3101-3104 (2003)

[51] Chen, S.M.: Similarity measures between vague sets and between elements. IEEE Transactions on Systems Man and Cybernetics Part B (Cybernetics) **27(1)**, 153-158 (1997).

[52] Hong, D.H., Kim, C.: A note on similarity measures between vague sets and between elements. Information sciences **115(1-4)**, 83-96 (1999)

[53] Fan, L., Zhangyan,X.: Similarity measures between vague sets, J. Software **12(6)**, 922-927 (2001)

[54] Li, Y., Olson, D.L., Qin, Z., Similarity measures between intuitionistic fuzzy (vague) sets: A comparative analysis. Pattern Recognition Letters **28(2)**, 278-285 (2007)

[55] Gupta, R., Kumar, S.: Intuitionistic Fuzzy Similarity-Based Information Measure in the Application of Pattern Recognition and Clustering. International Journal of Fuzzy Systems 1-18 (2022)

[56] Hussian, Z., Yang, M.S.: Distance and similarity measures of Pythagorean fuzzy sets based on the Hausdorff metric with application to fuzzy TOPSIS. International Journal of Intelligent Systems **34(10)**, 2633-2654 (2019)

[57] Yang, M. S., Hussian, Z.: Distance and similarity measures of hesitant fuzzy sets based on Hausdorff metric with applications to multi-criteria decision making and clustering. Soft Computering **23(14)**, 5835-5848 (2019)

[58] Tahani, V.: A conceptual framework for fuzzy query processing—A step toward very intelligent database systems. Inf. Process. Manag. **13**, 289-303 (1977)

[59] Kacprzyk, J., Ziolkowski, A.: Database queries with fuzzy linguistic quantifiers. IEEE Trans. Syst. Man Cybern **16**, 474-479 (1986)

[60] Petry, F.E.: Fuzzy Database: Principles and Applications. Kluwer: Dordrecht, The Netherlands (1996)

[61] Candan, K.S., Li, W.S., Priya, M.L.: Similarity-based ranking and query processing in multimedia databases. Data Knowl. Eng. **35**, 259-298 (2000)

[62] Hussian, Z., Yang, M.S.: Distance and similarity measures of Pythagorean fuzzy sets based on the Hausdorff metric with application to fuzzy TOPSIS. Int. J. Intell. Syst. **34**, 2633-2654 (2019)

[63] Farhadinia, B., Effati, S., Chiclana, F.: A family of similarity measures for q-rung orthopair fuzzy sets and their applications to multiple criteria decision making. Int. J. Intell. Syst. **36**, 1535-1559 (2021)

[64] Gomes, L.F.A.M., Lima, M.M.P.P.: TODIM: Basics and application to multicriteria ranking of projects with environmental impacts. Found. Comput. Decis. Sci. **16**, 113-127 (1992)

[65] Trotta, L.T.F., Nobre, F.F., Gomes, L.F.A.M.: Multi-criteria decision making-An approach to setting priorities in health care, Stat. Med. **18**, 3345-3354 (1999)

[66] Gomes, L.F.A.M., Rangel, L.A.D.: An application of the TODIM method to the multicriteria rental evaluation of residential properties, Eur. J. Oper. Res. **193**, 204-211 (2009)


## Author information


Vanita Rani, Department of Mathematics, Maharishi Markandeshwar (Deemed To Be University), Mullana-Ambala 133207, India.
E-mail: vanitanassa5279@gmail.com

Satish Kumar, Department of Mathematics, Maharishi Markandeshwar (Deemed To Be University), Mullana-Ambala 133207, India.
E-mail: profsatish74@mmumullana.org