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DIFFERENTIAL GEOMETRY IN MODERN TECHNOLOGY AND DESIGNS

S. P. Chauhan, P. K. Pandey and S. K. Chaubey

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Abstract Differential geometry, a classical branch of mathematics, is important in many fields, including physics, engineering, computer science, etc. This article reviews its applications and importance in various fields. In computer aided geometric design (CAGD), it is used to create and manipulate curves and surfaces, thereby improving design accuracy and efficacy. In road engineering, it provides tools to understand the road geometry and helps in optimizing road curvature and elevation for safety and comfort. Geologists model and analyze the earth's surface to understand geological features using methods of differential geometry. In electrical engineering, the interpretation of electrical values using the Frenet frame is examined, as well as the use of the geometric algebra in analyzing the power system and electric circuit. Thus, the paper emphasizes differential geometry's critical role in solving real-world problems and driving technological innovations.

1 Introduction

Differential geometry is a branch of mathematics that studies geometric problems using methods from calculus [15, 36]. The principles of curves, surfaces, planes, and sectional curvature in 3-dimensional Euclidean space were developed in the 18th and 19th centuries, forming the foundation for the field of differential geometry [27]. Prior to the 18th century, many new curves were discovered, but interest in differential geometry declined sharply afterwards. It was the efforts of Clairaut and Euler who revitalized the field by applying calculus to geometric problems [32]. A key concept that emerged during this time was Gaussian curvature, thoroughly investigated by Carl Friedrich Gauss, who demonstrated it as an intrinsic property of surfaces, independent of their isometric embedding in Euclidean space [9].

The concept of a manifold was introduced by Bernhard Riemann in 1854 A.D., marking the notion of Riemannian geometry [33]. Riemann's work extended the scope of differential geometry to include the geometric structure of differentiable manifolds. Extra structures can be added to manifolds to create a variety of geometric objects. Sasakian, Kenmotsu, Finsler, Kahler, and Hermitian manifolds are a few examples; each has special qualities. These topics offer an interesting area of research, as shown in [7, 10, 13, 22, 23, 29]. This new perspective not only provided a unified explanation of a broad class of geometries but also introduced mathematical tools essential for addressing various issues in differential equations, analysis, and mathematical physics [33]. These developments have opened new avenues for applying geometric and topological notions to solve complex problems.

In recent years, there have been significant advancements in differential geometry that we explore in this paper. We also discuss various applications and how modern developments have continued to advance our understanding and capabilities within this field.

The rest of the paper is organized as follows. Section 2 includes some basic concepts of differential geometry. Section 3 is divided into four subsections where applications in various fields have been discussed. Sub-section 3.1 is dedicated to the ruled surfaces, 3.2 has applications in road designing, 3.3 discusses the tools of differential geometry that can be utilized in modeling of geological surfaces and 3.4 investigates applications in electrical circuits. An example is presented in section 4 that discusses the applications of differential geometry to calculate the area of a cylindrical antenna. The conclusion of the article is given in section 5.

2 Preliminaries

In this section, we have briefly discussed the principal concepts from the theory of curves and surfaces in order to fulfill the requirements of the next section.

Theorem 2.1. *([1, 21, 26]) Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s) : I \to \mathbb{R}^3$, $(I \subseteq \mathbb{R})$, be a unit-speed curve parameterized by arc length s, with the non-vanishing curvature, and $\dot{\boldsymbol{\alpha}} = \frac{d\boldsymbol{\alpha}}{ds}$ then,*

$$\begin{aligned} \dot{\boldsymbol{t}} &= \kappa \boldsymbol{n} \\ \dot{\boldsymbol{n}} &= -\kappa \boldsymbol{t} + \tau \boldsymbol{b} \\ \dot{\boldsymbol{b}} &= -\tau \boldsymbol{n} \end{aligned} \tag{1}$$

The above set of formulae is called Frenet formulae, where $\boldsymbol{t}$, $\boldsymbol{n}$, and $\boldsymbol{b}$ denote the tangent, normal, and binormal vectors respectively, of $\boldsymbol{\alpha}$. The set $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}, \kappa, \tau\}$ is called the Frenet apparatus [26]. κ is curvature and τ is the torsion of curve $\boldsymbol{\alpha}$ given by

$$\kappa = \|\ddot{\boldsymbol{\alpha}}\| \qquad \text{and} \qquad \tau = \frac{\dot{\boldsymbol{\alpha}} \cdot (\ddot{\boldsymbol{\alpha}} \times \dddot{\boldsymbol{\alpha}})}{\kappa^2}$$

Theorem 2.2. *([4, 15]) Given any continuous functions κ and τ defined on an interval I, then there exists a unique space curve $\boldsymbol{\alpha}$ whose curvature and torsion correspond to κ and τ, respectively. This is called the fundamental theorem of space curves.*

The definition of ruled surfaces is given below. Moreover, the definitions of the first and second fundamental forms on surface are also given.

Definition 2.3. ([11, 35]) Consider a one-parameter family of lines represented by $\{\boldsymbol{\alpha}(t), \boldsymbol{\zeta}(t)\}$. The corresponding parameterized surface is given by

$$\boldsymbol{\chi}(t, \lambda) = \boldsymbol{\alpha}(t) + \lambda \boldsymbol{\zeta}(t), \quad t \in I,\ \lambda \in \mathbb{R}, \tag{3}$$

which is known as the ruled surface generated by $\{\boldsymbol{\alpha}(t), \boldsymbol{\zeta}(t)\}$. The curve $\boldsymbol{\alpha}(t)$ is referred to as the directrix of $\boldsymbol{\chi}$.
If we reparameterize eq.(3) using the variable s, then

$$\boldsymbol{\chi}(s, \lambda) = \boldsymbol{\alpha}(s) + \lambda \boldsymbol{\zeta}(s), \qquad \lambda \in \mathbb{R}. \tag{4}$$

In simple terms, ruled surfaces are formed by sweeping a vector along a base curve.

Definition 2.4. ([11]) Let $\boldsymbol{\chi}(s, \lambda) = \boldsymbol{\alpha}(s) + \lambda\, \boldsymbol{\zeta}(s)$ be an arbitrary ruled surface generated by a family $\{\boldsymbol{\alpha}(s), \boldsymbol{\zeta}(s)\}$ with $\|\boldsymbol{\zeta}(s)\| \equiv 1$. The surface $\boldsymbol{\chi}(s, \lambda)$ is said to be developable if

$$(\boldsymbol{\zeta},\, \dot{\boldsymbol{\zeta}},\, \dot{\boldsymbol{\alpha}}) \equiv 0 \tag{5}$$

The Gaussian curvature of a developable surface vanishes everywhere [11], i.e.,

$$\kappa_{\mathrm{G}} = \kappa_1 \kappa_2 = 0$$

First Fundamental Form: ([11, 26]) Let $\boldsymbol{\alpha}(t) = \boldsymbol{\chi}(u(t), v(t))$ be a curve on a surface patch $\boldsymbol{\chi}$. The first fundamental form is

$$E\, du^2 + 2F\, du\, dv + G\, dv^2 \tag{6}$$

where

$$E = \|\boldsymbol{\chi}_u\|^2, \quad F = \boldsymbol{\chi}_u \cdot \boldsymbol{\chi}_v, \quad G = \|\boldsymbol{\chi}_v\|^2$$

Second Fundamental Form: ([11, 26]) Let $\boldsymbol{\alpha}(t) = \boldsymbol{\chi}(u(t), v(t))$ be a curve on a surface patch $\boldsymbol{\chi}$. If a unit normal vector to the surface is given by

$$\boldsymbol{N} = \frac{\boldsymbol{\chi}_u \times \boldsymbol{\chi}_v}{\|\boldsymbol{\chi}_u \times \boldsymbol{\chi}_v\|}$$

The second fundamental form is

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 \tag{7}$$

where

$$L = \boldsymbol{\chi}_{uu} \cdot \boldsymbol{N}, \quad M = \boldsymbol{\chi}_{uv} \cdot \boldsymbol{N}, \quad N = \boldsymbol{\chi}_{vv} \cdot \boldsymbol{N}$$

3 Applications of Differential Geometry

3.1 For Developable Ruled Surfaces

Ruled surfaces find applications in conventional manufacturing techniques in many industries, including engineering, clothing, art, design, and sculpture. In [34], the author examines a straightforward technique for creating ruled surfaces, with curvature axis of the curve as generator. The ruled surface of this kind is developable [10], and there is at least one curve on the surface for which the axis of curvature lies entirely on the surface. Additionally, it studies the classification of the developable surfaces associated with space curves having singularities, as these surfaces and curves are not used in real-world applications. This improves our knowledge about the singularities of the developable surfaces by suggesting the use of round pattern environment maps to visualize the singularities and form structures like flowers around them.

3.1.1. Some Definitions

Definition 3.1. ([31]) Let $\boldsymbol{\sigma}$ be a unit speed curve parameterized by arc length s. A circle having 3-point contact with the given space curve $\boldsymbol{\sigma}$ at point s_0 is called an osculating circle and the center of circle at any point s_0 is given by

$$\boldsymbol{C} = \boldsymbol{\sigma}(s_0) + \frac{1}{\kappa(s_0)}\,\boldsymbol{n}(s_0) \tag{8}$$

Definition 3.2. ([31]) A sphere having 4-point contact with the curve $\boldsymbol{\sigma}$ at s_0 is known as osculating sphere. The center of the sphere at any point s_0 is given by

$$\boldsymbol{m}_0 = \boldsymbol{\sigma}(s_0) + \frac{1}{\kappa(s_0)}\,\boldsymbol{n}(s_0) - \frac{\dot{\kappa}(s_0)}{\kappa(s_0)^2\tau(s_0)}\,\boldsymbol{b}(s_0) \tag{9}$$

Definition 3.3. ([31, 34]) Let $\boldsymbol{\sigma}(s)$ be a space curve, for any point s_0 on the curve, the position vector $\boldsymbol{d}$ of the curvature axis is given by

$$\boldsymbol{d} = \boldsymbol{\sigma}(s_0) + \frac{1}{\kappa(s_0)}\,\boldsymbol{n}(s_0) + \lambda\boldsymbol{b}(s_0) \tag{10}$$

The curvature axis is a line that is generated by the locus of center of these spheres, and this line traverses from the center of curvature of the curve [11].

3.1.2. Method to Construct a Developable Ruled Surface

Theorem 3.4. *([34]) Let $\boldsymbol{\sigma}(s)$ be a smooth parameterized curve with $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}, \kappa, \tau\}$ be its Frenet apparatus , then*

$$\boldsymbol{R}(s, \lambda) = \boldsymbol{\sigma}(s) + \frac{1}{\kappa(s)}\boldsymbol{n}(s) + \lambda\boldsymbol{b}(s) \tag{11}$$

defines a developable surface.

Proof. To prove $\boldsymbol{R}(s,\lambda)$ is a developable surface, eq.(5) must be satisfied. i.e.

$$(\boldsymbol{\zeta},\dot{\boldsymbol{\zeta}},\dot{\boldsymbol{\alpha}})\equiv 0$$

To compute $\dot{\boldsymbol{\alpha}}$, $\boldsymbol{\zeta}$ and $\dot{\boldsymbol{\zeta}}$, comparing eq.(4) and eq.(11)

$$\boldsymbol{\alpha}(s)=\boldsymbol{\sigma}(s)+\frac{1}{\kappa(s)}\boldsymbol{n}(s) \qquad \text{and} \qquad \boldsymbol{\zeta}(s)=\lambda\boldsymbol{b}(s)$$

$$\dot{\boldsymbol{\alpha}}(s)=-\frac{\dot{\kappa}(s)}{\kappa(s)^2}\boldsymbol{n}(s)+\frac{1}{\kappa(s)}\tau(s)\boldsymbol{b}(s)$$

and
$$\dot{\boldsymbol{\zeta}}(s)=-\lambda\tau(s)\boldsymbol{n}(s)$$

$\Rightarrow (\boldsymbol{\zeta},\dot{\boldsymbol{\zeta}},\dot{\boldsymbol{\alpha}})=0$, this shows $\boldsymbol{R}(s,\lambda)=\boldsymbol{\sigma}(s)+\frac{1}{\kappa(s)}\boldsymbol{n}(s)+\lambda\boldsymbol{b}(s)$ is a developable surface. □

3.1.3. Finding Curve α from a Given Developable Surface

Theorem 3.5. *([16]) Any developable ruled surface in 3-dimensional Euclidean space is classified as one of the following surfaces*

(i) cylinder

(ii) cone

(iii) tangent ruled surfaces.

(a) Generalized cylindrical developable surfaces are given by

$$\boldsymbol{R}(s,\lambda)=\boldsymbol{\alpha}(s)+\lambda\boldsymbol{e} \tag{12}$$

Comparing eq.(12) with eq.(11),

$$\boldsymbol{\alpha(s)}=\boldsymbol{\sigma}(s)+\frac{1}{\kappa(s)}\boldsymbol{n}(s), \quad \boldsymbol{e}=\boldsymbol{b}(s), \text{where } \|\boldsymbol{e}\|=1$$

The binormal vector of curve $\boldsymbol{\sigma}$ is a constant vector. Therefore, $\boldsymbol{\alpha}$ is a plane curve and torsion $\tau=0$.

(b) For conical developable surfaces

$$\boldsymbol{R}(s,\lambda)=\boldsymbol{\alpha}+\lambda\boldsymbol{e}(s) \tag{13}$$

where $\boldsymbol{\alpha}$ is a point in $\mathbb{R}^3$ and $\boldsymbol{e}(s)$ is smooth unit vector field.
If the surface is conical developable, then from eq.(11) and eq.(13), we get

$$\ddot{\boldsymbol{\sigma}}(s)+\boldsymbol{\sigma}(s)=0$$

On solving the above differential equation, we get the coordinate functions $\boldsymbol{\sigma}_i(s)$ of the curve $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma}_i(s)=c_{1i}\cos s+c_{2i}\sin s$$

where c_{1i}, c_{2i} are constants.

(c) Tangent ruled surfaces

$$\boldsymbol{R}(s,\lambda)=\boldsymbol{\alpha}(s)+\lambda\dot{\boldsymbol{\alpha}}(s) \tag{14}$$

Comparing eq.(14) with eq.(11),

$$\boldsymbol{\alpha(s)}=\boldsymbol{\sigma}(s)+\frac{1}{\kappa(s)}\boldsymbol{n}(s) \quad \text{and} \quad \boldsymbol{b}(s)=\dot{\boldsymbol{\alpha}}(s) \tag{15}$$

$$\Rightarrow \boldsymbol{b}(s)=-\frac{\dot{\kappa}(s)}{\kappa(s)^2}\boldsymbol{n}(s)+\frac{\tau(s)}{\kappa(s)}\boldsymbol{b}(s)$$

$\Rightarrow \dot{\kappa}(s) = 0$, which results that $\kappa(s)$ = constant and also $\tau = \kappa$

Since,
$$\dot{\boldsymbol{\alpha}}(s) = \dot{\boldsymbol{\sigma}}(s) + \frac{1}{\kappa(s)}\dot{\boldsymbol{n}}(s)$$

From eq.(15),
$$\boldsymbol{b}(s) = \dot{\boldsymbol{\sigma}}(s) - \boldsymbol{t}(s) + \boldsymbol{b}(s) \quad \text{and } \kappa(s) = \tau(s) \tag{16}$$
$$\ddot{\boldsymbol{\sigma}}(s) = \dot{\boldsymbol{t}}(s)$$
$$\frac{1}{\kappa(s)^2}\ddot{\boldsymbol{\sigma}}(s) + \boldsymbol{\sigma}(s) = \boldsymbol{\alpha}(s) \tag{17}$$

On solving eq.(17)) we get

$$\boldsymbol{\sigma}_i(s) = c_{i1}\sin(\kappa s) + c_{i2}\cos(\kappa s) + \kappa\left[\sin(\kappa s)\int x_i(s)\cos(\kappa s)\,ds - \cos(\kappa s)\int x_i(s)\sin(\kappa s)\,ds\right]$$

where c_{i1} and c_{i2} are constants.

Corollary 1: ([34]) For a developable surface, there is at least a curve $\boldsymbol{\sigma}$ for which every line on the surface can act as a curvature axis at each corresponding regular point.

3.1.4. Discussion on Singularities

Singular point of a parameterized surface: A point where the differential map is not one-to-one is called a singular point [11].
Using a black background and distributing white circles on it uniformly by using environment map is an effective method for illustrating singularities on the surfaces. In this instance, circles surrounding singularity form a pattern that resembles flowers. In CAD, environment maps is a surface analysis tool.
Singularities on a developable surface are the points where the surface cannot be smoothly flattened. Singularities can be classified into three types [34]

(i) cuspidal edge

(ii) cuspidal cross-cap

(iii) swallowtail

The type of surfaces discussed in this paper have singularities along the base curves [34].

- A tangent developable surface has a cuspidal edge singularity at $\boldsymbol{\sigma}(t)$ if $\tau(t) \neq 0$.
- A tangent developable surface has a cuspidal cross-cap singularity along $\boldsymbol{\sigma}(t_0)$ if $\tau(t_0) = 0$ and $\tau'(t_0) \neq 0$.

Theorem 3.6. *([34]) Let $\boldsymbol{\sigma}$ represent a space curve with singularity at t_0. Then, the developable surface formed by its curvature axis possesses singularity.*

3.2 Application of Differential Geometry in Highway Designing

The author in [6] studied that space curves play a vital role in roadway design. In [6], the author presents new approaches for 3-dimensional road designs, along with a review of two-dimensional road designs from [28] and 3-dimensional designs from [2, 17]. To model and analyze roads, knowledge of surface theory is required; however, curves can be used to approximate the road centerline. A road can be thought of as a ruled surface whose centerline serves as its directrix. Based on the fundamental theorem of the space curves, a curve can be uniquely determined by its curvature and torsion [4]. Some special curves need to be investigated for the same purpose. Additionally, a few techniques from differential equations, function theory, and numerical analysis are used in the modeling of highways.
Two new ideas have been suggested by the author to utilize differential geometry in road designs, and a general case of a parabolic curve has been examined.

3.2.1. Suggested New Ideas

(a) The first fundamental form on a surface patch is given by

$$I = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

If $F = 0$, then the parameters u and v are orthogonal, i.e., the v-parameter curve cuts equal segments from all u-parameters as shown in Figure 1 ([6, 15]).

$$I = E\,du^2 + G\,dv^2$$

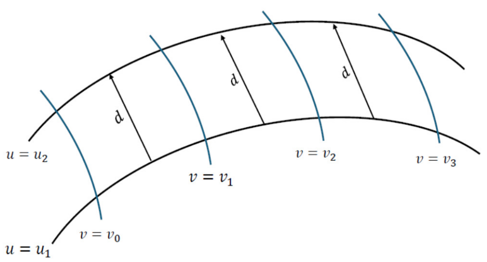

Figure 1. u and v-parameters are orthogonal when $F = 0$

(b) The Frenet frame when κ and τ are kept constant:

$$\begin{aligned} \dot{\boldsymbol{t}}(s) &= \kappa\,\boldsymbol{n}(s) \\ \dot{\boldsymbol{n}}(s) &= -\kappa\,\boldsymbol{t}(s) + \tau\,\boldsymbol{b}(s) \\ \dot{\boldsymbol{b}}(s) &= -\tau\,\boldsymbol{n}(s) \end{aligned} \tag{18}$$

Also, $$\ddot{\boldsymbol{t}}(s) = \kappa(-\kappa\boldsymbol{t}(s) + \tau\boldsymbol{b}(s))$$

and $$\dddot{\boldsymbol{t}}(s) = -\kappa^2\dot{\boldsymbol{t}}(s) - \tau^2\dot{\boldsymbol{t}}(s)$$

If we denote $\dot{\boldsymbol{t}}(s) = y$, then this equation becomes a second order linear differential equation with constant coefficients as

$$\ddot{y} + (\kappa^2 + \tau^2)y = 0$$

This implies, depending on the values of κ and τ, y can be easily computed, then $\dot{t}(s)$ and $t(s)$. After substituting these values into the Frenet formulae, $\boldsymbol{n}(s)$ and $\boldsymbol{b}(s)$ can be obtained, and the set of eq.(18) can be solved.

3.2.2. A General Case for Plane Road

For a certain length, a plane road is thought to have a parabola-shaped profile. A conic section has the general form

$$g(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00},$$

where at least one of the real coefficients a_{11}, a_{12}, a_{22} is non-zero.

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{10} \\ a_{21} & a_{22} & a_{20} \\ a_{01} & a_{02} & a_{00} \end{vmatrix}$ and $\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ where $\Delta \neq 0$ and $\delta = 0$, then $g(x, y)$ is a parabola.

Considering the conic $y = ax^2 + bx + c$, which can be rewritten as $ax^2 + bx - y + c = 0$, then $\Delta = -\frac{a}{4} \neq 0$, $\delta = 0$ and this conic represents a parabola [3].

The geometry of a parabola depends upon the values of a, b, c. For a choice where $a > 0, b \leq 0$, the parabola will be upward and is shifted toward the positive x-axis. To get a positive value of discriminant $D = b^2 - 4ac$, we must have $c > 0$.

For $$y = ax^2 + bx + c$$

$$\frac{dy}{dx} = 2ax + b$$

The coordinate of the local minima is $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$.
The curvature of the parabola is given by

$$\kappa = \frac{\|y''(x_0)\|}{\left(\sqrt{1+(f'(x))^2}\right)^3}$$

$$\kappa = \frac{2a}{\left(\sqrt{1+(2ax+b)^2}\right)^3}$$

$$\text{at} \quad x = -\frac{b}{2a}, \qquad \kappa = 2a$$

The designing and construction of this type of road definitely depends upon the curvature of the parabola.

Case I: Let A and B be points that are symmetric about the axis of the parabola such that $AB = 2l$. If θ is the angle between the tangents drawn at points A and B as shown in Figure 2 [6]. It is evident that this angle θ ($\theta_0 < \theta < 2\pi$), affects how the road is constructed as well. A formula that relates the angle θ to the curvature of the parabola is required.

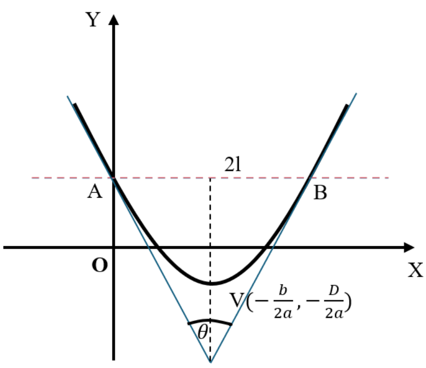

Figure 2. A general case when road section is considered parabolic

The equation of tangent to the parabola $g(x,y) = ax^2 + bx + c - y$ at point A is given by [3]

$$\left.\frac{\partial g(x,y)}{\partial x}\right|_A (x - x_A) + \left.\frac{\partial g(x,y)}{\partial y}\right|_A (y - y_A) = 0$$

$$\frac{\partial g}{\partial x} = 2ax + b$$

$$\frac{\partial g}{\partial y} = -1$$

From Figure 2 coordinates of A and B are

$$A = (0,\, c), \quad B = (2l, 4al^2 + 2bl + c)$$

The tangent at A is

$$y = bx + c$$

The direction vector $\boldsymbol{v}_1$ for the line can be represented as $(1, b)$, where 1 represents change in x and b represents change in y. Tangent at A can be expressed as

$$\frac{x}{1} = \frac{y-c}{b}$$

Tangent at B has equation

$$y = (4al + b)x - 4al^2 + c$$

The direction vector $\boldsymbol{v}_2$ of the line can be represented as $(1, 4al + b)$ where 1 represents change in x and $4la + b$ represents change in y.
The tangent at B is given by

$$\frac{x}{1} = \frac{y + 4l^2a - c}{4la + b}$$

We know that $$\cos\theta = \frac{\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_1\|\|\boldsymbol{v}_2\|} = \frac{1 + b(4al + b)}{\sqrt{1 + b^2}\sqrt{1 + (4al + b)^2}}$$

Substituting $l = -\frac{b}{2a}$ but $2a = \kappa \Rightarrow l = -\frac{b}{\kappa}$, we get:

$$\cos\theta = \frac{1 - b^2}{1 + b^2}$$

This relation indicates that $\cos\theta$ is determined by the coefficient of x only when $b < 0$. Also, $\cos\theta$ is independent of curvature κ of the curve.
Now, we consider the case when $b = 0$, to have a positive value of discriminant, $D = -4ac > 0$ we have to consider $c < 0$.

Case II: Let A' and B' be points that are symmetric about the axis of the parabola, with $A'B' = 2l'$, as shown in Figure 3 [6]. Let the tangents at A' and B' has angle θ' between them.

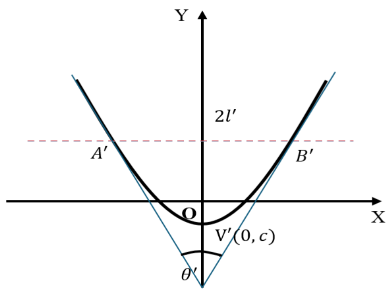

Figure 3. A similar general case when axis of parabola is y-axis

Coordinates at $A' = (-l', al'^2 + c)$ and at $B' = (l', al'^2 + c)$.
A similar computation as in Case I gives

$$\cos\theta' = \frac{1 - \kappa^2 l'^2}{1 + \kappa^2 l'^2}$$

Here, θ' depends on the curvature κ and the distance between A' and B'.

3.3 Applications to Geology

In [20], the author discusses a new tool that proposes applications of differential geometry to characterize geological surfaces, focusing on fold orientation, shape, and cylindricity. It provides aspects of fold shape that are geologically significant by measuring surface curvature and disregarding curvature values below a certain magnitude using a curvature threshold. These comprehensive explanations replace 2-D approximations that relied on fold cylindricity assumptions and traditional analysis methods that provided stereographic projections of strike and dip values across a fold.
Structural geologists have used curve computations to characterize folded surface geometry, measure strain or deformation in deformed strata, and forecast fracture orientations and densities in bent strata. The author builds on this research and presents a novel computer tool to facilitate fold structure investigations.

If the temperature of the rock is warm enough for it to behave like plastic, it is prone to fold when tectonic forces squeeze it from the sides, especially in sedimentary rock. The folds are of different types. The most common are

(a) Anticline: An upward fold is called an anticline [12].

(b) Syncline: A downward fold is called a syncline [12].

The folds have similar geometry to some of the most common surfaces, that is why structural geologists analyze the structure of folds.
It is possible to imagine natural surfaces as continuous functions or sample points on scanned or seismic grids. Consider $z(u, v)$, a single-valued continuous function for the elevation measurements. The surface patch can be described as

$$\boldsymbol{\chi}(u, v) = u\mathbf{e}_1 + v\mathbf{e}_2 + z(u, v)\mathbf{e}_3,$$

where u and v are two independent parameters and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are unit vectors in the Cartesian coordinate system.
At any point on the surface, infinitely many curves can be drawn, resulting in different curvatures at the same point. Therefore, the concept of principal curvatures becomes useful here. The maximum and the minimum values of curvature at any point on the surface are called the principal curvatures. They determine the local geometry of the surface. The magnitudes and orientations of the principal curvatures are computed using the components of the first and the second fundamental forms [25]. At each grid point, matrices are created as follows

$$A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad B = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

The shape operator at any point p on the surface is given as follows

$$S(p) = A^{-1}B.$$

The eigenvalues of $S(p)$ are the magnitudes of principal curvatures at any point p, while the eigenvectors of $S(p)$ correspond to the orientations of the principal curvatures.

3.3.1. Geologic Curvature and Curvature Threshold

Gaussian curvature has been used in the study of geological structures. It is denoted by $\kappa_{\text{G}} = \kappa_1\kappa_2$, where κ_1 and κ_2 denote principal curvatures at the point p [1, 11, 21, 26].

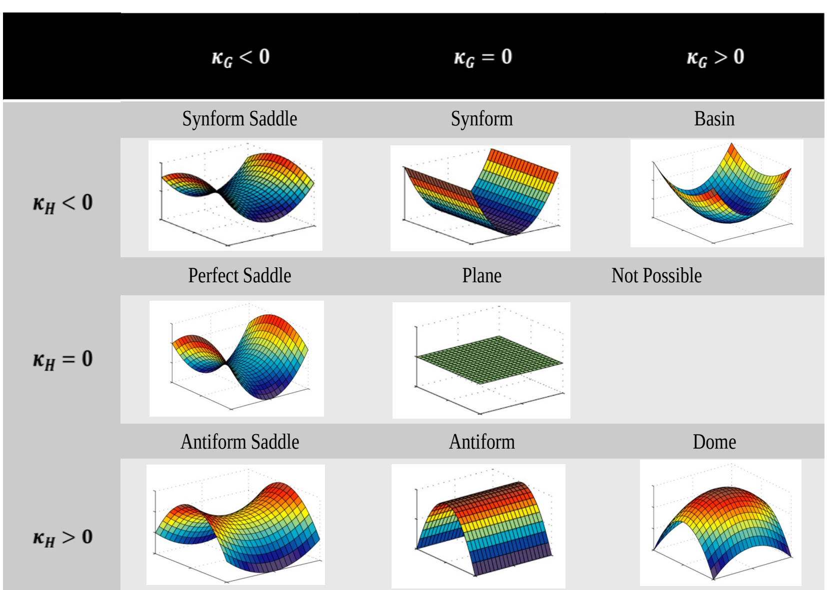

Figure 4. Geological curvature determined using Gaussian curvature κ_G and mean curvature κ_H

If $\kappa_G = 0$, it follows that at least one of the values of principal curvatures is necessarily zero. If either value is zero, the surface is locally shaped like a cylinder. If both values are zero, then the surface behaves locally as a plane. Also, if $\kappa_G < 0$, the principal curvatures are of opposite signs, i.e., the surface is locally saddle. Furthermore, if $\kappa_G > 0$, both the principal curvatures exhibit same sign, then the surface assumes the shape of a basin or dome. Gaussian curvature alone cannot provide a complete description of a surface patch. Therefore, the idea of mean curvature is used. Mean curvature is denoted by $\kappa_H = \frac{1}{2}(\kappa_1 + \kappa_2)$. The mean and Gaussian curvatures at a point collectively define the orientations and shape of a surface, as shown in Figure 4 [5, 20].
Idealized shapes like the plane, cylindrical antiform and synform, and the perfect saddle are included in the categorization of the geologic curvature. Since these shapes require at least one of the values of principal curvatures to be exactly zero, they are not found in geologic datasets. This is not possible due to measurement error and the irregular nature of geological surfaces. Although geologists frequently describe and analyze folds as cylindrical for simplicity, even if they are not perfectly round.
On the other hand, it might be helpful to estimate some geological surfaces as ideal or to measure the deviation of a geologic surface from the ideal. To evaluate these approximations, a curvature threshold κ_t is used. By defining an absolute value of curvature beyond which computed principal curvatures, regardless of sign, are considered zero, this threshold enables the categorization of idealized shapes.
For approximating the geological surfaces as perfect saddles, a very similar technique is used. Saddle surfaces have opposite-sign principal curvatures κ_1 and κ_2, with a perfect saddle having $\kappa_1 = -\kappa_2$ or $\kappa_H = 0$. Although principal curvature values for geologic surfaces are unlikely to be equal, they may be close enough. Idealized perfect saddles may be defined as sites where the absolute sum of $|\kappa_1 + \kappa_2| < \kappa_t$. is less than κ_t.

3.4 Applications in Electronics

The correlations between geometrical quantities, such as curvature and torsion, and electrical values, e.g. current, voltage, and frequency are discussed in [19]. The suggested methods offer a comprehensive framework for defining the time derivative of the electrical values in both stationary and transient settings. It utilizes the Frenet frame used in differential geometry. The given methodology generalizes and unifies the phasor and time domain frameworks. The time domain represents changes in electrical quantities over time, while the phasor domain is used to analyze sinusoidal steady-state behavior [8]. The transient state of an electric system refers to its initial response when switching from one state to another, while the steady state represents the circuit's long-term behavior after the transient state has diminished [8].
Other notable findings include a description of the rate of frequency change, which incorporates the idea of the torsional frequency, and an interesting interpretation of the relationship between time derivatives of current, frequency, and voltage.
From now on, we denote the time variable by t. Also, it must be noted that the derivative with respect to t is denoted by prime and the derivative with respect to arc length s is denoted by dot. Let $\boldsymbol{\alpha} : [0, \infty) \to \mathbb{R}^3$ be a space curve in parameter t given by $\boldsymbol{\alpha}(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$. If $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ constitutes an orthonormal basis in $\mathbb{R}^3$. Similarly, curve $\boldsymbol{\alpha}$ can be represented as

$$\boldsymbol{\alpha} = \alpha_1 \boldsymbol{e}_1 + \alpha_2 \boldsymbol{e}_2 + \alpha_3 \boldsymbol{e}_3$$

The arc length s of curve is given by

$$s = \int_0^t \sqrt{\boldsymbol{\alpha}'(r) \cdot \boldsymbol{\alpha}'(r)}\, dr + s_0,$$

$$s' = \frac{ds}{dt} = \|\boldsymbol{\alpha}'\| \tag{19}$$

where
$$\boldsymbol{\alpha}' = \frac{d}{dt}(\alpha_1 \boldsymbol{e}_1) + \frac{d}{dt}(\alpha_2 \boldsymbol{e}_2) + \frac{d}{dt}(\alpha_3 \boldsymbol{e}_3)$$

also
$$\dot{\boldsymbol{\alpha}} = \frac{d\boldsymbol{\alpha}}{ds} = \frac{d\boldsymbol{\alpha}}{dt} \cdot \frac{dt}{ds} = \frac{\boldsymbol{\alpha}'}{s'} = \frac{\boldsymbol{\alpha}'}{\|\boldsymbol{\alpha}'\|}$$

This shows that $\dot{\boldsymbol{\alpha}}$ is tangent to curve $\boldsymbol{\alpha}$ and it has unit magnitude. The vectors constituting the Frenet frame are given as follows

$$\boldsymbol{t} = \dot{\boldsymbol{\alpha}}, \quad \boldsymbol{n} = \frac{\ddot{\boldsymbol{\alpha}}}{\|\ddot{\boldsymbol{\alpha}}\|}, \quad \boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n} \tag{20}$$

also

$$\kappa = \|\ddot{\boldsymbol{\alpha}}\| = \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3} \tag{21}$$

$$\tau = \frac{\dot{\boldsymbol{\alpha}} \cdot (\ddot{\boldsymbol{\alpha}} \times \dddot{\boldsymbol{\alpha}})}{\|\dot{\boldsymbol{\alpha}} \times \ddot{\boldsymbol{\alpha}}\|} = \frac{\boldsymbol{\alpha}' \cdot (\boldsymbol{\alpha}'' \times \boldsymbol{\alpha}''')}{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|^2} \tag{22}$$

3.4.1. Expression for Voltage, Current and Their Time Derivatives

In this sub-section we give the expressions for voltage, current, frequency, and their time derivatives based on the Frenet frame. The voltage vector $\boldsymbol{\nu}$ is assumed to be the time derivative of the curve $\boldsymbol{\alpha}$. Let $\boldsymbol{\phi}$ denote the magnetic flux vector, defined as

$$\boldsymbol{\phi} = -\boldsymbol{\alpha} \tag{23}$$

By Faraday's law [14]

$$\boldsymbol{\nu} = -\boldsymbol{\phi}' \tag{24}$$

Now, if $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is written in terms of voltage vector and its derivative. The time derivative of arc length s from eq.(19), eq.(23) and eq.(24))

$$s' = \|\boldsymbol{\nu}\| = \nu$$

also,

$$\dot{\boldsymbol{\alpha}} = -\dot{\boldsymbol{\phi}} = -\frac{\boldsymbol{\phi}'}{s'} = \frac{\boldsymbol{\nu}}{\nu} \tag{25}$$

$$\ddot{\boldsymbol{\alpha}} = -\ddot{\boldsymbol{\phi}} = \frac{\boldsymbol{\nu}'}{\nu^2} - \frac{\nu'\boldsymbol{\nu}}{\nu^3}$$

and

$$\dddot{\boldsymbol{\alpha}} = -\dddot{\boldsymbol{\phi}} = \frac{\boldsymbol{\nu}''}{\nu^3} - \frac{3\,\nu'\boldsymbol{\nu}'}{\nu^4} + \frac{3\,(\nu')^2\boldsymbol{\nu}}{\nu^5} - \frac{\boldsymbol{\nu}\nu''}{\nu^4}$$

Since $\dot{\boldsymbol{\alpha}} \cdot \ddot{\boldsymbol{\alpha}} = 0$

$\Rightarrow$

$$\left(\frac{\boldsymbol{\nu}}{\nu}\right) \cdot \left(\frac{\boldsymbol{\nu}'}{\nu^2} - \frac{\nu'\boldsymbol{\nu}}{\nu^3}\right) = 0$$

$$\frac{\boldsymbol{\nu} \cdot \boldsymbol{\nu}'}{\nu^3} = \frac{\nu'\boldsymbol{\nu} \cdot \boldsymbol{\nu}}{\nu^4}$$

which gives

$$\rho = \frac{\nu'}{\nu} = \frac{\boldsymbol{\nu}}{\nu} \cdot \frac{\boldsymbol{\nu}'}{\nu} \tag{26}$$

In time-frequency analysis, ρ is commonly referred as instantaneous bandwidth [8]. However, author has used the interpretation of ρ as provided in [18], i.e., the symmetric component of geometric frequency.

From eq.(26)

$$\rho\nu = \nu'$$

From a geometrical aspect, ν' may be seen as the radial component of the velocity $\boldsymbol{\nu}$ and ρ can be interpreted as the radial frequency.

From eq.(21)

$$\kappa = \frac{\|\boldsymbol{\nu} \times \boldsymbol{\nu}'\|}{\nu^3} = \frac{\omega}{\nu} \tag{27}$$

Here $\boldsymbol{\omega}$ is the vector which is anti-symmetric component of geometric frequency and written as [18]

$$\boldsymbol{\omega} = \frac{\boldsymbol{\nu} \times \boldsymbol{\nu}'}{\nu^2}$$

$\omega\nu$ may be seen as the azimuthal element of velocity $\boldsymbol{\nu}$. Therefore, ω represents azimuthal frequency.

From eq.(22)

$$\tau = \frac{\boldsymbol{\nu} \cdot \boldsymbol{\nu}' \times \boldsymbol{\nu}''}{\omega^2\nu^4}$$

or
$$\tau = \frac{\boldsymbol{\nu}'' \cdot (\boldsymbol{\nu} \times \boldsymbol{\nu}')}{\omega^2 \nu^4} = \frac{\boldsymbol{\nu}'' \cdot \boldsymbol{\omega}}{\kappa^2 \nu^4}$$

Except in the cases when $\boldsymbol{\nu}'' = 0$ or $\boldsymbol{\omega} = 0$, the torsion vanishes when the vectors $\boldsymbol{\nu}''$ and $\boldsymbol{\omega}$ are perpendicular to each other. This happens when the voltage is imbalanced.
Therefore, the Frenet frame vectors can be expressed as

$$\boldsymbol{t} = \frac{\boldsymbol{\nu}}{\nu}, \quad \boldsymbol{n} = \frac{\ddot{\boldsymbol{\alpha}}}{\|\ddot{\boldsymbol{\alpha}}\|}, \quad \boldsymbol{b} = \frac{\boldsymbol{\omega}}{\omega}$$

$$\ddot{\boldsymbol{\alpha}} = (\boldsymbol{\nu}' - \rho\boldsymbol{\nu}) \tag{28}$$

$$\|\ddot{\boldsymbol{\alpha}}\| = \sqrt{\|\boldsymbol{\nu}'\|^2 - \|\rho\boldsymbol{\nu}\|^2}$$

From $\boldsymbol{n} = \boldsymbol{b} \times \boldsymbol{t}$, we get
$$\frac{\ddot{\boldsymbol{\alpha}}}{\|\ddot{\boldsymbol{\alpha}}\|} = \frac{\boldsymbol{\omega}}{\omega} \times \frac{\boldsymbol{\nu}}{\nu}$$

Also, $\|\ddot{\boldsymbol{\alpha}}\| = \omega\nu$, gives
$$\ddot{\boldsymbol{\alpha}} = \boldsymbol{\omega} \times \boldsymbol{\nu}$$

From eq.(28)
$$\boldsymbol{\nu}' = \rho\boldsymbol{\nu} + \boldsymbol{\omega} \times \boldsymbol{\nu} \tag{29}$$

or
$$\left(\frac{d}{dt} - [\rho + \boldsymbol{\omega}\times]\right)\boldsymbol{\nu} = 0$$

which gives operator
$$D_t^{\boldsymbol{\nu}} : = \frac{d}{dt} - [\rho + \boldsymbol{\omega}\times] \tag{30}$$

The upper symbol in the $D_t^{\boldsymbol{\nu}}$ shows the vector on which operator $D_t^{\boldsymbol{\nu}}$ is applied. Also, the derivative w.r.t. time of a vector is divided into two components in eq.(29) and eq.(30) namely, the symmetric component ρ and the anti-symmetric component $(\boldsymbol{\omega}\times)$.

3.4.2. Derivative of Frequency in Frenet Frame

In order to derive the expression for $\boldsymbol{\omega}'$, using eq.(20) and eq.(25), we have

$$\boldsymbol{t}' = \nu\dot{\boldsymbol{t}}, \quad \boldsymbol{n}' = \nu\dot{\boldsymbol{n}}, \quad \boldsymbol{b}' = \nu\dot{\boldsymbol{b}}$$

Then the Frenet frame can be written as

$$\boldsymbol{t}' = \omega\boldsymbol{n}, \quad \boldsymbol{n}' = -\omega\boldsymbol{t} + \xi\boldsymbol{b}, \quad \boldsymbol{b}' = -\xi\boldsymbol{n}$$

where $\omega = \nu\kappa$ from eq.(27) and $\xi = \nu\tau$ and ξ may be described as torsional frequency.

$$\frac{d}{dt}\frac{\boldsymbol{\omega}}{\omega} = \frac{\boldsymbol{\omega}'}{\omega} - \frac{\boldsymbol{\omega}\omega'}{\omega^2} = -\frac{\xi\boldsymbol{n}}{n}$$

$$\boldsymbol{\omega}' = \frac{\omega'\boldsymbol{\omega}}{\omega} - \omega\frac{\xi\boldsymbol{n}}{n}$$

or
$$\boldsymbol{\omega}' = \eta\boldsymbol{\omega} + \tau\boldsymbol{\nu} \times \boldsymbol{\omega} \tag{31}$$

where $\eta = \frac{\omega'}{\omega}$ is symmetric component and $[\tau\boldsymbol{\nu}\times]$ is the anti-symmetric component of the time derivative of $\boldsymbol{\omega}$.
eq.(31) represents the generalized form of the rate of change of frequency. The rate of change of frequency is crucial for evaluating energy network stability and inferring power system safety and control [30]. The $\tau = 0$ implies the system is balanced, thus it leads to

$$\|\boldsymbol{\omega}'\| = \omega',$$

At last, the time derivative of the symmetric component of geometric frequency is described in [18] as
$$\rho' = \frac{\boldsymbol{\nu} \cdot \boldsymbol{\nu}''}{\nu^2} + \omega^2 - \rho^2 = \frac{\nu''}{\nu} - \rho^2.$$

4 Example

Let's consider a real-life example. The surface area of an antenna affects its bandwidth [24]. Therefore, we are interested in calculating the area of a cylindrical antenna given by the equation

$$\chi(u, v) = (r\,\cos u, \quad r\,\sin u, \quad v) \qquad u \in [0,\, \frac{2\pi}{3}],\ v \in [0,\, 0.1]$$

Area is given by

$$A = \int_0^{0.1} \int_0^{\frac{2\pi}{3}} \sqrt{EG - F^2}\, du\, dv$$

$$A = \frac{2}{15}\,\pi\, r$$

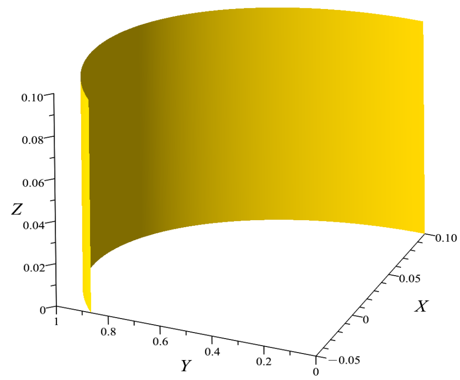

Figure 5. Cylindrical antenna design at $r = 0.3$

5 Conclusion and Future Directions

In this article, we explored wide and impactful applications of differential geometry in current science and engineering, highlighting its practical and versatile nature. Our investigation highlights the ways in which the fundamental theorem of space curves and the Frenet frame can be applied in road design, as well as the ways in which the study of curvature axis of a curve can produce ruled surfaces. We also discuss how differential geometry techniques can be applied in electronics to analyze electrical values like current, voltage, and frequency, and how fundamental forms can be utilized to characterize geological folds. It is clear that differential geometry applications are not limited to these areas only, demonstrating its significant impact in many other fields. Future work includes developing more efficient algorithms for representing and manipulating ruled surfaces in CAGD, as well as using differential geometry to study the shape optimization of main suspension cables of suspension bridges. Additionally, differential geometric tools can be applied to characterize fractures and analyze the curvature of reservoir layers.

References

[1] E. Abbena, S. Salamon and A. Gray, *Modern differential geometry of curves and surfaces with Mathematica*, Chapman and Hall/CRC, 2006.

[2] K. Amiridis and B. Psarianos, *Applications of differential geometry to solve the highway alignment location problem,* Orenburg Univ. J., (2014), 262-270.

[3] V. Balan and C. Udriste, *Analytic and differential geometry*, Bucharest: Geometry Balkan Press, 2000.

[4] T. F. Banchoff and S. T. Lovett, *Differential geometry of curves and surfaces*, AK Peters/CRC Press, 2010.

[5] S. Bergbauer, T. Mukerji and P. Hennings, *Improving curvature analyses of deformed horizons using scale-dependent filtering techniques,* AAPG Bulletin, **87** (8) (2003), 1255-1272.

[6] A. Burlacu and A. Mihai, *Applications of differential geometry of curves in road design,* Rom. J. Transp. Infrastruct., **12** (2024), 1-13.

[7] F. M. Cabrera and A. Swann, *Curvature of special almost Hermitian manifolds,* Pac. J. Math., **28** (1) (2006), 165-184.

[8] L. Cohen, *Time frequency analysis: theory and applications*, Upper Saddle River, NJ: Prentice-Hall Signal Processing, 1995.

[9] J. T. Cremer, *Neutron and X-ray optics in general relativity and cosmology,* Neutron and X-ray Optics, Elsevier, (2013), 889-1014.

[10] E. Damar, N. Yuksel, and M. K. Karacan, *Ruled surfaces according to parallel transport frame in* $\mathbb{E}^4$*,* Math. Combin., **1** (2020), 20-32.

[11] M. P. do Carmo, *Differential geometry of curves and surfaces: revised and updated second edition*, New York: Dover Publication, Inc., 2016.

[12] S. Earle, *Folding,* in Physical Geology, BC Campus, (2019), 304-307.

[13] J. B. Jun, U. C. De and G. Pathak, *On Kenmotsu manifolds,* J. Korean Math. Soc., **43** (3) (2005), 435-445.

[14] P. Kinsler, *Faraday's law and magnetic induction: cause and effect, experiment and theory,* Physics, **2** (2) (2020), 148-161.

[15] M. M. Lipschutz, *Schaum's outline of differential geometry*, McGraw Hill Professional, 1969.

[16] H. Liu, Y. Liu and S. D. Jung, *Ruled invariants and total classification of non-developable ruled surfaces,* J. Geom., **113** (1) (2022).

[17] R. Mesnil and O. Baverel, *Pseudo-geodesic gridshells,* Eng. Struct., **279** (2023).

[18] F. Milano, *A geometrical interpretation of frequency,* IEEE Trans. Power Syst., **37** (2021), 816-819.

[19] F. Milano, G. Tzounas, I. Dassios and T. Kerci, *Applications of the Frenet frame to electric circuits,* IEEE Trans. Circuits Syst., 69 (4) (2022) 1668-1680.

[20] I. Mynatt, S. Bergbauer and D. D. Pollard, *Using differential geometry to describe 3-D folds,* J. Struct. Geol., **29** (2007) 1256-1266.

[21] B. O'Neill, *Elementry differential geometry*, Cambridge: Academic Press, 1966.

[22] P. K. Pandey and R. S. Gupta, *Existence and uniqueness theorem for slant immersion in Kenmotsu space forms,* Turk. J. Math., **33** (2009), 409-425.

[23] P. K. Pandey and Sameer, *Magnetic and slant curves in Kenmotsu manifolds,* Surv. Math. Appl., **15** (2020), 139-151.

[24] S. Panwar and B. S. Sharma, *Microstrip patch antenna: analysis of surface area for bandwidth imrpovement,* Int. J. Eng. Res. Appl., **3** (5) (2013), 996-999.

[25] M. A. Pearce, R. R. Jones, S.A.F. Smith, K. J. W. McCaffrey and P. Clegg, *"Numerical analysis of fold curvature using data acquired by high-precision GPS,"* J. Struct. Geol., **28** (2006), 1640-1646.

[26] A. Pressley, *Elementry differential geometry*, London: Springer, 2001.

[27] R. A. Rahman, A. I. Gary and A. E. Ageeb, *Some applications of sectional curvature in differential geometry,* Int. J. Math. Comput. Res., **11** (02) (2023), 3236-3242.

[28] J. F. Reinoso, M. Moncayo, M. Pasadas, F. J. Ariza and J. L. Garcia, *The Frenet frame beyond classical differential geometry: Application to Cartographic Generalization of Roads,* Math. Comput. Simul., **79** (12) (2009), 3556-3566.

[29] C. Sayar, M. M. Triphathi, F. Ozdemir and H. M. Tastan, *Generic submersion from Kaehler manifolds,* Bull. Malays. Math. Sci. Soc., **43** (2020), 809-831.

[30] A. K. Singh and B. C. Pal, *Rate of change of frequency estimation for power systems using interpolated DFT and Kalman filter,* IEEE Trans. Power Syst., **34** (4) (2019), 2509-2517.

[31] D. Somasundaram, *Differential geometry: a first course*, Oxford: Alpha Science Int'l Ltd, 2005.

[32] D. J. Struik, *Outline of a history of differential geometry (I),* Chicago, The University of Chicago Press on behalf of The History of Science, (1933), 92-120.

[33] D. J. Struik, *Outline of a history of differential geometry (II),* Chicago, The University of Chicago Press on behalf of The History of Science, (1933), 161-191.

[34] F. Tas and R. Ziatdinov, *Developable ruled surfaces generated by the curvature axis of a curve,* Axioms, **12** (1090) (2023), 1-14.

[35] V. A. Toponogov and V. Rovenski, *Differential geometry of curves and surfaces: a concise guide*, Boston: Birkhäuser, 2006.

[36] L. W. Tu, *Differential geometry: connections, curvature, and characteristic classes*, Springer, 2017.

Author information

S. P. Chauhan, Department of Mathematics, Jaypee University of Information Technology, India.
E-mail: suyash968@gmail.com

P. K. Pandey, Department of Mathematics, Jaypee University of Information Technology, India.
E-mail: pandeypkdelhi@gmail.com

S. K. Chaubey, Department of Information Technology, University of Technology and Applied Sciences, Oman.
E-mail: sudhakar.chaubey@utas.edu.om