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Exploring the Soliton Solutions of the Time Fractional Kadomtsev-Petviashvili Equation using $(G'/G, 1/G)$-Expansion Method

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Abstract Across the various disciplines such as fluid dynamics and non-linear optics, the Time Fractional Kadomtsev-Petviashvili (TF-KP) equation is very crucial in understanding the two-dimensional wave phenomenon. In this study, we are going to observe the various solutions of this equation, derived using $(\frac{G'}{G}, \frac{1}{G})$-expansion method. This method, which is considered to be an extension of $(\frac{G'}{G})$-expansion method, is used to construct soliton solutions for the non-linear partial differential equations. We have shown that it is quite effective in the case of the TF-KP equation. Additionally, the effect of fractional derivative on various values was illustrated using graphs obtained by plotting the solutions.

1 Introduction

Fractional calculus is the branch of calculus that deals with nth order derivative and integral, where $n \in \mathbb{R}$. Even though it may seem like a new concept, its foundation was laid back in the 17th century when in 1695, Guillaume de l'Hópital asked Gottfried Wilhelm Leibniz in a letter what the $\frac{1}{2}$-th derivative of a function, would be [5]. Since then, many mathematicians like Riemann, Liouville, Joseph Fourier, and others have played their roles in the development of fractional calculus.

During the 20th century, various fractional-order derivatives were proposed such as the Riemann-Liouville fractional derivative , Grünwald-Letnikov fractional derivative, Caputo fractional derivative, Weyl fractional derivative, Marchaud fractional derivative, and Hadamard fractional derivative to name a few. Each of the fractional derivatives mentioned above has its own pros and cons. Some of the applications of fractional derivatives are modeling memory and hereditary properties of the systems involved in everyday life, enhancing simulations and predictions, and improving already existing mathematical systems as shown in [31, 32]. In this paper, we are interested in the use of the conformable fractional derivative (CFD), given in [12, 29], which is known for its simplicity and resemblance to integer-ordered derivative. We will use it in the process of finding solutions to non-linear evolution equations (NLEEs), where the NLEE is defined as the non-linear time-dependent differential equation used to model real-world phenomena [28]. NLEEs are used in various fields such as ocean engineering [6, 9], optical fiber communication [16, 18], condensed matter physics [17, 23], plasma physics [10, 19, 20], and many other fields. The NLEE, we will be concerned with, is the Time Fractional Kadomtsev-Petviashvili (TF-KP) equation

$$u^{\sigma}_{tx} - uu_{xx} - u^2_x - u_{xxxx} = u_{yy} \tag{1.1}$$

which is obtained from the Kadomtsev-Petviashvili equation [14], used for modeling shallow water waves, plasma waves and non-linear optics, given below

$$u_{tx} - uu_{xx} - u^2_x - u_{xxxx} = u_{yy}$$

by replacing partial derivative with respect to (w.r.t.) time t with corresponding fractional derivative of order σ, where $0 < \sigma \leq 1$, $D_t^\sigma u(x, y, t) = u_t^\sigma$ and D_t^σ represents the fractional derivative w.r.t. time t of order σ. There are various types of solutions to an NLEE, such as exact solutions, analytic solutions, numerical solutions, approximate solutions, etc. The preferable type of solutions for the TF-KP equation are soliton solutions, which are exact solutions, as they are stable and can be observed and analysed with ease.

Solitons are special types of waves that maintain their structure and velocity upon collision with each other. The solitons are also known as solitary waves, as a result of their nature [4]. John Scott Russell, a Scottish naval architect, was the first person who noticed these waves and tried to reconstruct them [2, 21]. Later on, personalities like Joseph Boussinesq, Lord Rayleigh, and Diederik Korteweg contributed in this area. There are various types of solitons such as kink, bright, breather, dark, and more. These are preferable solutions for NLEEs due to their stable nature. Many methods have been developed to find these types of solutions to NLEEs. Some of these are Hirota's direct method [8], $tanh$ method [25], novel Kudryashov method [30], fractional sub-equation method [3], extended $tanh$ method [26], $\left(\frac{G'}{G}\right)$-expansion method [7, 24], $\left(\frac{G'}{G^2}\right)$-expansion method [1, 11], and $\left(\frac{G'}{G}, \frac{1}{G}\right)$-expansion method [15, 22, 27]. The method we are going to use for solving TF-KP is the one mentioned last.

This paper is divided into five sections. The first section provides an introduction, while the second and third sections cover the preliminaries and the description of the method, respectively. The fourth section demonstrates the application of the mentioned method to the TF-KP equation, which is followed by the conclusion.

2 Preliminaries

In this section, we provide the definition of the Conformable Fractional Derivative (CFD) with some properties and derivative of some basic functions.

Definition 2.1. (CFD) For a continuous real valued function $h(z)$ defined over the interval $[0, \infty)$ and $\sigma \in (0, 1]$, CFD of $h(z)$, given by Khalil et al. [13], is as follows:

$$D_z^\sigma[h(z)] = \lim_{\delta \to 0} \frac{h(z + \delta z^{1-\sigma}) - h(z)}{\delta},$$

where $0 < \delta \in \mathbb{R}$ and σ denotes the order of CFD.

Similarly, the CFD of order σ for the function $H(v, w, z) : [0, \infty) \times [0, \infty) \times [0, \infty) \to \mathbb{R}$ w.r.t. z is defined as

$$D_z^\sigma[H(v, w, z)] = \lim_{\delta \to 0} \frac{H(v, w, z + \delta z^{1-\sigma}) - H(v, w, z)}{\delta},$$

where $\delta > 0$ is a real number.

Properties of CFD

Let $H(z)$ and $R(z)$ be two continuous real valued functions defined over the interval $[0, \infty)$, then

- For a constant P, if $H(z) = P$ then its CFD of order σ is zero, that is

$$D_z^\sigma[H(z)] = 0.$$

- If $R(z) = z^t$, then its CFD is given by

$$D_z^\sigma[R(z)] = tz^{t-\sigma}.$$

- For arbitrary constants ρ and τ, the CFD of linear combination of $H(z)$ and $R(z)$ is linear combination of their CFDs, that is

$$D_z^\sigma[\rho H(z) + \tau R(z)] = \rho D_z^\sigma[H(z)] + \tau D_z^\sigma[R(z)].$$

- CFD of the product of two continuous real valued functions is provided by

$$D_z^\sigma[H(z)R(z)] = R(z)D_z^\sigma[H(z)] + H(z)D_z^\sigma[R(z)].$$

Here are the CFDs of some basic functions

(a) $D_z^\sigma[z^\sigma] = \sigma$,

(b) $D_z^\sigma[\sin z^\sigma] = \sigma \cos z^\sigma$,

(c) $D_z^\sigma[\exp z^\sigma] = \sigma \exp z^\sigma$,

(d) $D_z^\sigma[\cos z^\sigma] = \sigma(-\sin z^\sigma)$,

(e) $D_z^\sigma[a^{z^\sigma}] = \sigma a^{z^\sigma} \log a$.

Since we have defined CFD and outlined its properties, we now turn to describing the concerned method.

3 Methodology

This section describes the method to be used for solving the NLEE. Suppose an NLEE, of the form

$$F\left(u, u_t^\sigma, u_x, u_y, u_{tx}^\sigma, u_{xx}, u_{yy}, u_{xy}, ...\right) = 0, \tag{3.1}$$

where u is function of x, y, t and u_t^σ represents the fractional derivative of u w.r.t. time t. Then the concerned method can be summarized into the following steps:

Step 1: First, we will convert $u(x, y, t)$ into a single variable function by taking

$$u(x, y, t) = \mathcal{U}(\zeta), \tag{3.2}$$

where $\zeta = x + y - \frac{ct^\sigma}{\sigma}$ is a travelling wave transformation and c is an arbitrary constant. Then by using (3.2) in (3.1) with conformable derivative, we obtain an ordinary differential equation (ODE)

$$H(\mathcal{U},\mathcal{U}',\mathcal{U}'',\mathcal{U}''',\dots) = 0, \tag{3.3}$$

where H is a function of $\mathcal{U}(\zeta)$ and its derivatives represented by $\mathcal{U}',\mathcal{U}'',\mathcal{U}'''$ and so on.

Step 2: Let the solution for (3.3) be

$$\mathcal{U}(\zeta) = \sum_{i=0}^{P} p_i \left(\frac{G'}{G}\right)^i + \sum_{j=1}^{P} q_j \left(\frac{G'}{G}\right)^{j-1} \left(\frac{1}{G}\right), \tag{3.4}$$

where $P \in \mathbb{N}$, and G' denotes the derivative of G, which is a solution of the Riccati equation given below

$$G''(\zeta) + \kappa G(\zeta) = \rho, \tag{3.5}$$

where κ and ρ are constants. The value of P can be obtained by comparing the degree of the highest order derivative of $\left(\frac{G'}{G}\right)$ with the degree of non-linear term $\left(\frac{G'}{G}\right)$ in (3.4), after substituting the value of $\mathcal{U}(\zeta)$. This principle is known as the homogeneous balance principle. The equation (3.5) has three different types of solutions based upon the values of κ, which are discussed as follows:

For $\kappa < 0$

$$G(\zeta) = M_1 \sinh\left(\sqrt{-\kappa}\zeta\right) + M_2 \cosh\left(\sqrt{-\kappa}\zeta\right) + \frac{\rho}{\kappa} \tag{3.6}$$

with

$$\left(\frac{1}{G}\right)^2 = \frac{-\kappa}{\kappa^2\eta + \rho^2}\left(\left(\frac{G'}{G}\right)^2 - 2\rho\left(\frac{1}{G}\right) + \kappa\right),$$

where M_1 and M_2 are two arbitrary constants and $\eta = M_1^2 - M_2^2$.

For $\kappa > 0$

$$G(\zeta) = M_1 \sin\left(\sqrt{\kappa}\zeta\right) + M_2 \cos\left(\sqrt{\kappa}\zeta\right) + \frac{\rho}{\kappa} \tag{3.7}$$

with

$$\left(\frac{1}{G}\right)^2 = \frac{\kappa}{\kappa^2\omega - \rho^2}\left(\left(\frac{G'}{G}\right)^2 - 2\rho\left(\frac{1}{G}\right) + \kappa\right),$$

where M_1 and M_2 are two arbitrary constants and $\omega = M_1^2 + M_2^2$.

For $\kappa = 0$

$$G(\zeta) = \frac{\rho}{2}\zeta^2 + M_1\zeta + M_2 \tag{3.8}$$

with

$$\left(\frac{1}{G}\right)^2 = \frac{1}{M_1^2 - 2\rho M_2}\left(\left(\frac{G'}{G}\right)^2 - 2\rho\left(\frac{1}{G}\right)\right),$$

where M_1 and M_2 are two arbitrary constants.

Step 3: Let

$$L\left(\left(\frac{G'}{G}\right), \left(\frac{1}{G}\right), \left(\frac{G'}{G}\right)\left(\frac{1}{G}\right), \left(\frac{G'}{G}\right)^2\left(\frac{1}{G}\right), \ldots\right) = 0 \tag{3.9}$$

be the equation formed by substituting the value of $\mathcal{U}$ from (3.4) in (3.3) and using the value of $\left(\frac{1}{G}\right)^2$ for required number of times to eliminate $\left(\frac{1}{G}\right)^2$. Then on equating the constant and non-constant terms of (3.9) to zero, we get a system of equations.

Step 4. In this step, we will determine the values of p_i and q_j, by solving the system obtained in step 3, using different mathematical computing software such as Mathematica or Maple.

Step 5. The solution $\mathcal{U}$ of (3.1) is determined by substituting values p_i, q_j, $G(\zeta)$ and ζ. And based on the values of $G(\zeta)$, we shall get three different types of solutions, namely trigonometric, hyperbolic, and rational solutions.

4 Application of the method on TF-KP equation

This section is dedicated to using the method described above to obtain exact solutions to the TF-KP equation given as equation (1.1). Using the transformation in (3.2) on (1.1) and integrating twice, we obtain the following ODE

$$2\mathcal{U}'' + 2(1+c)\mathcal{U} + (\mathcal{U})^2 = 0, \tag{4.1}$$

where $\mathcal{U}''$ denotes the double derivative of $\mathcal{U}$ w.r.t. ζ.
Using HBP in (4.1), we get $P = 2$. Now, substituting the value of P in equation (3.4), gives us

$$\mathcal{U}(\zeta) = p_0 + p_1\left(\frac{G'}{G}\right) + p_2\left(\frac{G'}{G}\right)^2 + q_1\left(\frac{1}{G}\right) + q_2\left(\frac{G'}{G}\right)\left(\frac{1}{G}\right) \tag{4.2}$$

As the previously mentioned method suggests, we have three cases depending upon the value of κ which are given below. It should be noted that it is not possible to plot the graphs in four dimensions, so the graphs are plotted by assigning a constant value to y, for demonstrating the effect of σ.

Case 1. (When $\kappa < 0$)
Using Step 3, mentioned in the section above, we have obtained the following system of algebraic equations:

$$\left(\frac{G'}{G}\right): 2p_1 + 2p_0p_1 + 2p_1c + 4p_1\kappa - \frac{2q_1q_2\kappa^2}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} + \frac{12q_2\kappa^2\rho}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} = 0;$$

$$\left(\frac{G'}{G}\right)^2: p_1^2 + 2p_2 + 2p_0p_2 + 2p_2c + 16p_2\kappa - \frac{q_1^2\kappa}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} - \frac{q_2^2\kappa^2}{(M_1^2 - M_2^2)\kappa^2 + \rho^2}$$
$$+ \frac{2q_1\kappa\rho}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} - \frac{4p_2\kappa\rho^2}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} = 0;$$

$$\left(\frac{G'}{G}\right)^3: 4p_1 + 2p_1p_2 - \frac{2q_1q_2\kappa}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} + \frac{12q_2\kappa\rho}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} = 0;$$

$$\left(\frac{G'}{G}\right)^4: 12p_2 + p_2^2 - \frac{q_2^2\kappa}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} = 0;$$

$$\left(\frac{1}{G}\right): 2q_1 + 2p_0q_1 + 2q_1c + 2q_1\kappa - 8p_2\kappa\rho + \frac{2q_1^2\kappa\rho}{(M_1^2 - M_2^2)\kappa^2 + \rho^2}$$
$$- \frac{4q_1\kappa\rho^2}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} + \frac{8p_2\kappa\rho^3}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} = 0;$$

$$\left(\frac{G'}{G}\right)\left(\frac{1}{G}\right): 2p_1q_1 + 2q_2 + 2p_0q_2 + 2q_2c + 10q_2\kappa - 6p_1\rho + \frac{4q_1q_2\kappa\rho}{(M_1^2 - M_2^2)\kappa^2 + \rho^2}$$
$$- \frac{24q_2\kappa\rho^2}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} = 0;$$

$$\left(\frac{G'}{G}\right)^2\left(\frac{1}{G}\right): 4q_1 + 2p_2q_1 + 2p_1q_2 - 20p_2\rho + \frac{2q_2^2\kappa\rho}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} = 0;$$

$$\left(\frac{G'}{G}\right)^3\left(\frac{1}{G}\right): 12q_2 + 2p_2q_2 = 0;$$

$$\left(\frac{G'}{G}\right)^0\left(\frac{1}{G}\right)^0: 2p_0 + p_0^2 + 2p_0c + 4p_2\kappa^2 - \frac{q_1^2\kappa^2}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} + \frac{2q_1\kappa^2\rho}{(M_1^2 - M_2^2)\kappa^2 + \rho^2}$$
$$- \frac{4p_2\kappa^2\rho^2}{(M_1^2 - M_2^2)\kappa^2 + \rho^2} = 0.$$

Using Step 4, we have obtained the solutions represented by S_{1i}, where $1 \leq i \leq 8$, for the equations above with the help of Mathematica software, which are given below:

$$S_{11} = \{p_0 \to 4\kappa, p_1 \to 0, p_2 \to -6, q_1 \to 6\rho, q_2 \to -\frac{6\sqrt{M_1^2\kappa^2 - M_2^2\kappa^2 + \rho^2}}{\sqrt{\kappa}}, c \to -1 + \kappa\},$$

$$S_{12} = \{p_0 \to 6\kappa, p_1 \to 0, p_2 \to -6, q_1 \to 6\rho, q_2 \to -\frac{6\sqrt{M_1^2\kappa^2 - M_2^2\kappa^2 + \rho^2}}{\sqrt{\kappa}}, c \to -1 - \kappa\},$$

$$S_{13} = \{p_0 \to 4\kappa, p_1 \to 0, p_2 \to -6, q_1 \to 6\rho, q_2 \to \frac{6\sqrt{M_1^2\kappa^2 - M_2^2\kappa^2 + \rho^2}}{\sqrt{\kappa}}, c \to -1 + \kappa\},$$

$$S_{14} = \{p_0 \to 10\kappa, p_1 \to 0, p_2 \to -12, q_1 \to 12\rho, q_2 \to 0, c \to -1 - \kappa, M_2 \to -M_1\},$$

$$S_{15} = \{p_0 \to 6\kappa, p_1 \to 0, p_2 \to -6, q_1 \to 6\rho, q_2 \to \frac{6\sqrt{M_1^2\kappa^2 - M_2^2\kappa^2 + \rho^2}}{\sqrt{\kappa}}, c \to -1 - \kappa\},$$

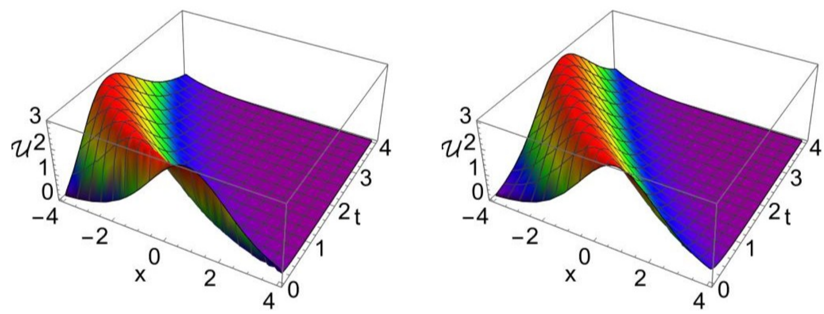
$$S_{16} = \{p_0 \to 12\kappa, p_1 \to 0, p_2 \to -12, q_1 \to 12\rho, q_2 \to 0, c \to -1+\kappa, M_2 \to -M_1\},$$
$$S_{17} = \{p_0 \to 10\kappa, p_1 \to 0, p_2 \to -12, q_1 \to 12\rho, q_2 \to 0, c \to -1-\kappa, M_2 \to M_1\},$$
$$S_{18} = \{p_0 \to 12\kappa, p_1 \to 0, p_2 \to -12, q_1 \to 12\rho, q_2 \to 0, c \to -1+\kappa, M_2 \to M_1\}.$$

The corresponding values of $\mathcal{U}$ with the graphs, after using (3.2) and (3.6) in (4.2) are given below:

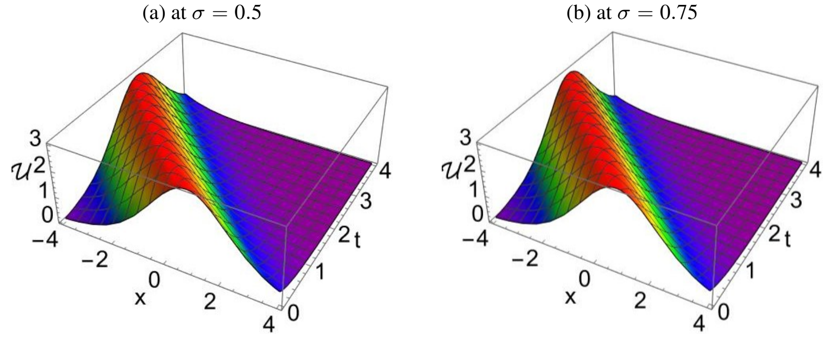
$$\mathcal{U}_{11} = -\frac{6M_1\sqrt{M_1^2\kappa^2 - M_2^2\kappa^2 + \rho^2}\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}\right)^2}$$
$$-\frac{6M_2\sqrt{M_1^2\kappa^2 - M_2^2\kappa^2 + \rho^2}\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6M_1^2\kappa\cosh^2\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6M_2^2\kappa\sinh^2\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}\right)^2}$$
$$+\frac{12M_1M_2\kappa\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right)\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6\rho}{M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}} - 4\kappa,$$

$$\mathcal{U}_{12} = -\frac{6M_1\sqrt{M_1^2\kappa^2 - M_2^2\kappa^2 + \rho^2}\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}\right)^2}$$
$$-\frac{6M_2\sqrt{M_1^2\kappa^2 - M_2^2\kappa^2 + \rho^2}\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6M_1^2\kappa\cosh^2\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6M_2^2\kappa\sinh^2\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}\right)^2}$$
$$+\frac{12M_1M_2\kappa\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right)\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6\rho}{M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}} - 6\kappa,$$

Note: Since σ has no observable effect on $\mathcal{U}_{12}$, as depicted by the graphs in Figure 2, only two graphs are displayed.

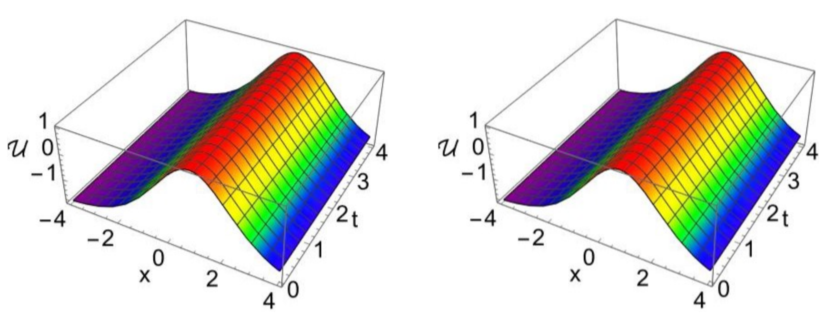


(a) at $\sigma = 0.5$ (b) at $\sigma = 0.75$



(c) at $\sigma = 0.9$ (d) at $\sigma = 1$

Figure 1: 3D graphs of $\mathcal{U}_{11}$ for different σ, when $y = 1$, $M_1 = -20$, $M_2 = -12$, $\rho = 100$, and $\kappa = -1$.



(a) at $\sigma = 0.5$ (b) at $\sigma = 0.1$

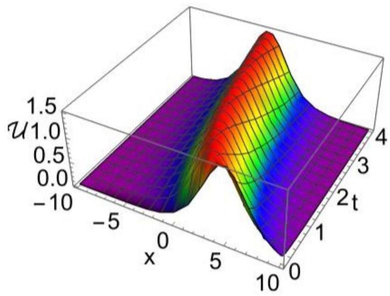
Figure 2: 3D graphs of $\mathcal{U}_{12}$ for different σ, when $y = 1$, $M_1 = -20$, $M_2 = -12$, $\rho = 100$, and $\kappa = -1$.

$$\mathcal{U}_{13}=\frac{6M_1\sqrt{M_1^2\kappa^2-M_2^2\kappa^2+\rho^2}\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6M_2\sqrt{M_1^2\kappa^2-M_2^2\kappa^2+\rho^2}\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6M_1^2\kappa\cosh^2\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6M_2^2\kappa\sinh^2\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{12M_1M_2\kappa\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6\rho}{M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}}-4\kappa,$$

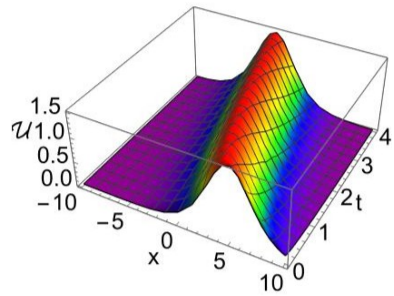
$$\mathcal{U}_{14}=\frac{12M_1^2\kappa\cosh^2\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{12M_1^2\kappa\sinh^2\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$-\frac{24M_1^2\kappa\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{12\rho}{M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}}-12\kappa,$$

Note: Only two graphs are shown in Figure 4, due to the lack of observable effects of σ in $\mathcal{U}_{14}$.

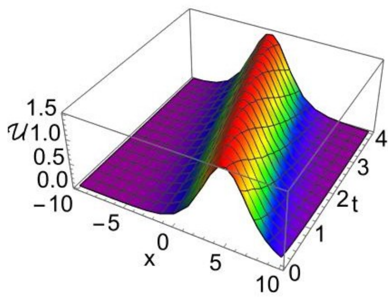
$$\mathcal{U}_{15}=-\frac{6M_1\sqrt{M_1^2\kappa^2-M_2^2\kappa^2+\rho^2}\cosh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cosh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-\frac{\rho}{\kappa}\right)^2}$$
$$-\frac{6M_2\sqrt{M_1^2\kappa^2-M_2^2\kappa^2+\rho^2}\sinh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cosh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-\frac{\rho}{\kappa}\right)^2}$$



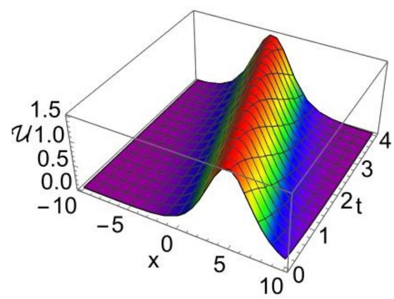
(a) at $\sigma = 0.5$



(b) at $\sigma = 0.75$

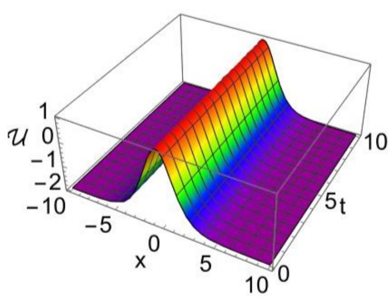


(c) at $\sigma = 0.9$

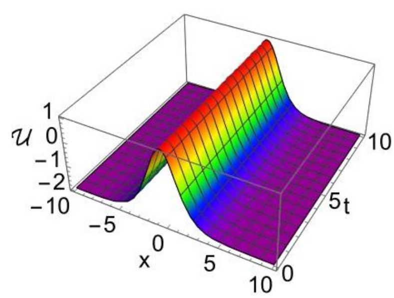


(d) at $\sigma = 1$

Figure 3: 3D graphs of $\mathcal{U}_{13}$ for different σ, when $y = -11.5$, $M_1 = 15$, $M_2 = 8$, $\rho = 200$, and $\kappa = -0.5$.



(a) at $\sigma = 0.5$



(b) at $\sigma = 0.1$

Figure 4: 3D graphs of $\mathcal{U}_{14}$ for different σ, when $y = 2$, $M_1 = 15$, $\rho = 1$, and $\kappa = -1$.

$$-\frac{6M_1^2\kappa\cosh^2\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cosh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-\frac{\rho}{\kappa}\right)^2}$$
$$-\frac{6M_2^2\kappa\sinh^2\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cosh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-\frac{\rho}{\kappa}\right)^2}$$
$$-\frac{12M_1M_2\kappa\sinh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)\cosh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cosh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6\rho}{M_1\sinh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cosh\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-\frac{\rho}{\kappa}}+6\kappa,$$

$$\mathcal{U}_{16}=\frac{12M_1^2\kappa\cosh^2\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{12M_1^2\kappa\sinh^2\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$-\frac{24M_1^2\kappa\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{12\rho}{M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)-M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}}-10\kappa,$$

$$\mathcal{U}_{17}=\frac{12M_1^2\kappa\cosh^2\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$

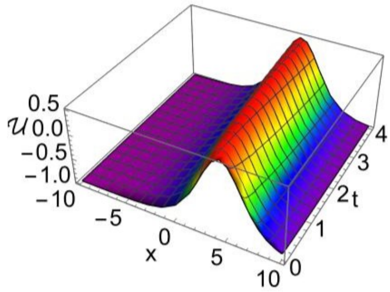
$$+\frac{12M_1^2\kappa\sinh^2\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$

$$+\frac{24M_1^2\kappa\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$

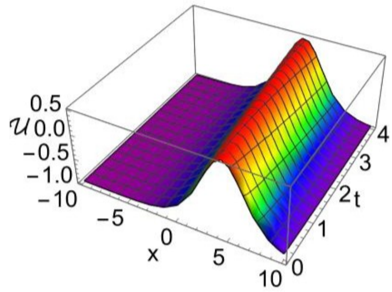
$$+\frac{12\rho}{M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}}-12\kappa,$$

Note: Only two graphs are shown in Figure 7, due to the lack of observable effects of σ in $\mathcal{U}_{17}$.

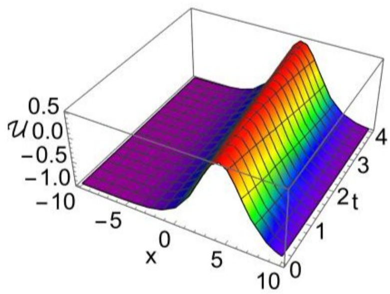
$$\mathcal{U}_{18}=\frac{12M_1^2\kappa\cosh^2\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$



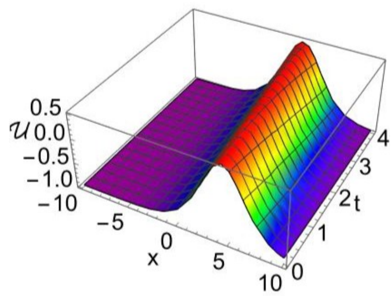
(a) at $\sigma = 0.5$



(b) at $\sigma = 0.75$

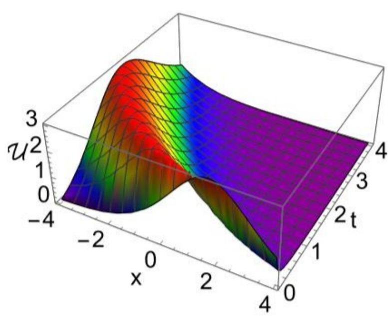


(c) at $\sigma = 0.9$

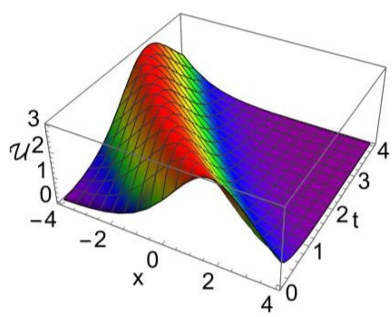


(d) at $\sigma = 1$

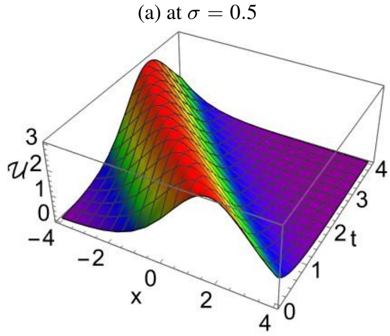
Figure 5: 3D graphs of $\mathcal{U}_{15}$ for different σ, when $y = -10$, $M_1 = 15$, $M_2 = 8$, $\rho = 100$, and $\kappa = -0.5$.



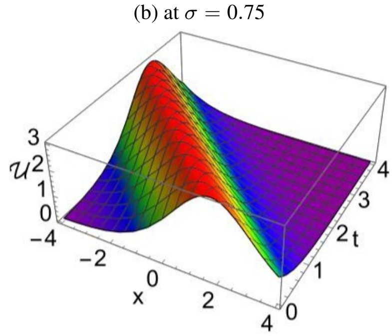
(a) at $\sigma = 0.5$



(b) at $\sigma = 0.75$

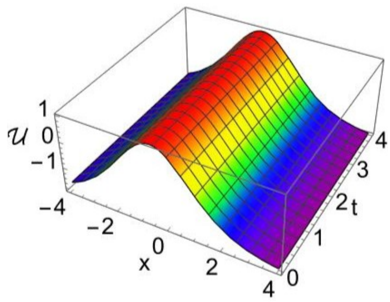


(c) at $\sigma = 0.9$

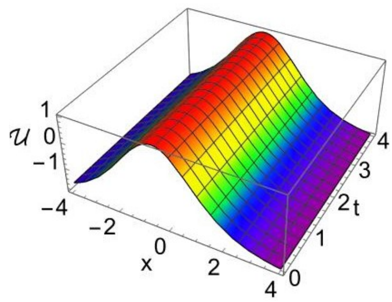


(d) at $\sigma = 1$

Figure 6: 3D graphs of $\mathcal{U}_{16}$ for different σ, when $y = 1$, $M_1 = 15$, $\rho = 1$, and $\kappa = -1$.



(a) at $\sigma = 0.5$



(b) at $\sigma = 0.1$

Figure 7: 3D graphs of $\mathcal{U}_{17}$ for different σ, when $y = -1$, $M_1 = -15$, $\rho = 3$, and $\kappa = -1$.

$$+\frac{12M_1^2\kappa\sinh^2\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$

$$+\frac{24M_1^2\kappa\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$

$$+\frac{12\rho}{M_1\sinh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_1\cosh\left(\sqrt{-\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}}-10\kappa,$$

Case 2. (When $\kappa = 0$)
Following the same procedure as Case 1, we get a system of equations, which is given below:

$$\left(\frac{G'}{G}\right): 2p_1+2p_0p_1+2p_1c=0;$$

$$\left(\frac{G'}{G}\right)^2: p_1^2+2p_2+2p_0p_2+2p_2c+\frac{q_1^2}{M_1^2-2M_2\rho}-\frac{2q_1\rho}{M_1^2-2M_2\rho}+\frac{4p_2\rho^2}{M_1^2-2M_2\rho}=0;$$

$$\left(\frac{G'}{G}\right)^3: 4p_1+2p_1p_2+\frac{2q_1q_2}{M_1^2-2M_2\rho}-\frac{12q_2\rho}{M_1^2-2M_2\rho}=0;$$

$$\left(\frac{G'}{G}\right)^4: 12p_2+p_2^2+\frac{q_2^2}{M_1^2-2M_2\rho}=0;$$

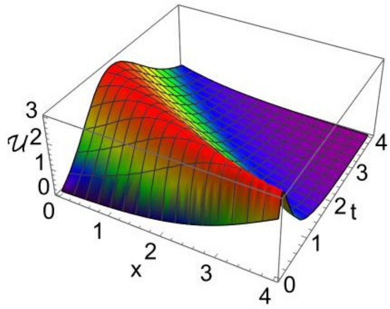
$$\left(\frac{1}{G}\right): 2q_1+2p_0q_1+2q_1c-\frac{2q_1^2\rho}{M_1^2-2M_2\rho}+\frac{4q_1\rho^2}{M_1^2-2M_2\rho}-\frac{8p_2\rho^3}{M_1^2-2M_2\rho}=0;$$

$$\left(\frac{G'}{G}\right)\left(\frac{1}{G}\right): 2p_1q_1+2q_2+2p_0q_2+2q_2c-6p_1\rho-\frac{4q_1q_2\rho}{M_1^2-2M_2\rho}+\frac{24q_2\rho^2}{M_1^2-2M_2\rho}=0;$$

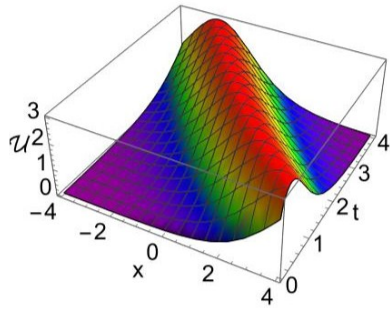
$$\left(\frac{G'}{G}\right)^2\left(\frac{1}{G}\right): 4q_1+2p_2q_1+2p_1q_2-20p_2\rho-\frac{2q_2^2\rho}{M_1^2-2M_2\rho}=0;$$

$$\left(\frac{G'}{G}\right)^3\left(\frac{1}{G}\right): 12q_2+2p_2q_2=0;$$

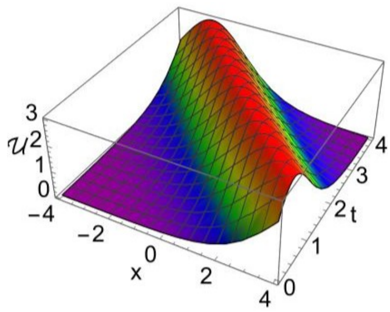
$$\left(\frac{G'}{G}\right)^0\left(\frac{1}{G}\right)^0: 2p_0+p_0^2+2p_0c=0.$$



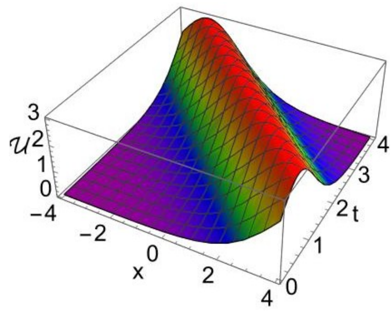
(a) at $\sigma = 0.5$



(b) at $\sigma = 0.75$



(c) at $\sigma = 0.9$



(d) at $\sigma = 1$

Figure 8: 3D graphs of $\mathcal{U}_{18}$ for different σ, when $y = -5$, $M_1 = -15$, $\rho = 18$, and $\kappa = -1$.

On solving these equations by using Mathematica, we have got the solutions given below, denoted by S_{0j}, where $j = 1, 2$:

$$S_{01} = \{p_0 \to 0, p_1 \to 0, p_2 \to -6, q_1 \to 6\rho, q_2 \to -6\sqrt{M_1^2 - 2M_2\rho}, c \to -1\},$$

$$S_{02} = \{p_0 \to 0, p_1 \to 0, p_2 \to -6, q_1 \to 6\rho, q_2 \to 6\sqrt{M_1^2 - 2M_2\rho}, c \to -1\}.$$

On substitution of these solutions with (3.2) and (3.8) in (4.2), we get the following results:

$$\begin{aligned}
\mathcal{U}_{01} = & \frac{6x^2\rho^2}{\left(M_2 + M_1\left(x+y+\frac{t^\sigma}{\sigma}\right) + \frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2} - \frac{12xy\rho^2}{\left(M_2 + M_1\left(x+y+\frac{t^\sigma}{\sigma}\right) + \frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2} \\
& - \frac{6y^2\rho^2}{\left(M_2 + M_1\left(x+y+\frac{t^\sigma}{\sigma}\right) + \frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2} - \frac{12x\rho M_1}{\left(M_2 + M_1\left(x+y+\frac{t^\sigma}{\sigma}\right) + \frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2} \\
& - \frac{12y\rho M_1}{\left(M_2 + M_1\left(x+y+\frac{t^\sigma}{\sigma}\right) + \frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2} - \frac{6M_1^2}{\left(M_2 + M_1\left(x+y+\frac{t^\sigma}{\sigma}\right) + \frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2} \\
& - \frac{6x\rho\sqrt{M_1^2 - 2\rho M_2}}{\left(M_2 + M_1\left(x+y+\frac{t^\sigma}{\sigma}\right) + \frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2} - \frac{6y\rho\sqrt{M_1^2 - 2\rho M_2}}{\left(M_2 + M_1\left(x+y+\frac{t^\sigma}{\sigma}\right) + \frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2} \\
& - \frac{6M_1\sqrt{M_1^2 - 2\rho M_2}}{\left(M_2 + M_1\left(x+y+\frac{t^\sigma}{\sigma}\right) + \frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2} + \frac{6\rho}{M_2 + M_1\left(x+y+\frac{t^\sigma}{\sigma}\right) + \frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2} \\
& - \frac{6\rho^2 t^{2\sigma}}{\left(M_2 + M_1\left(x+y+\frac{t^\sigma}{\sigma}\right) + \frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2\sigma^2} - \frac{12x\rho^2 t^\sigma}{\left(M_2 + M_1\left(x+y+\frac{t^\sigma}{\sigma}\right) + \frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2\sigma}
\end{aligned}$$

$$-\frac{12y\rho^2 t^\sigma}{\left(M_2+M_1\left(x+y+\frac{t^\sigma}{\sigma}\right)+\frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2\sigma}-\frac{12\rho M_1 t^\sigma}{\left(M_2+M_1\left(x+y+\frac{t^\sigma}{\sigma}\right)+\frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2\sigma}$$

$$-\frac{6\rho\sqrt{M_1^2-2\rho M_2}t^\sigma}{\left(M_2+M_1\left(x+y+\frac{t^\sigma}{\sigma}\right)+\frac{1}{2}\rho\left(x+y+\frac{t^\sigma}{\sigma}\right)^2\right)^2\sigma},$$

$$\mathcal{U}_{02}=\frac{6\rho t^\sigma\sqrt{M_1^2-2M_2\rho}}{\sigma\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}+\frac{6\mathrm{x}\rho\sqrt{M_1^2-2M_2\rho}}{\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}$$

$$+\frac{6\mathrm{y}\rho\sqrt{M_1^2-2M_2\rho}}{\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}+\frac{6M_1\sqrt{M_1^2-2M_2\rho}}{\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}$$

$$-\frac{6M_1^2}{\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}-\frac{6\mathrm{x}^2\rho^2}{\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}$$

$$-\frac{6\mathrm{y}^2\rho^2}{\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}-\frac{12\mathrm{x}\rho^2 t^\sigma}{\sigma\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}$$

$$-\frac{12\mathrm{y}\rho^2 t^\sigma}{\sigma\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}-\frac{12\mathrm{x}\mathrm{y}\rho^2}{\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}$$

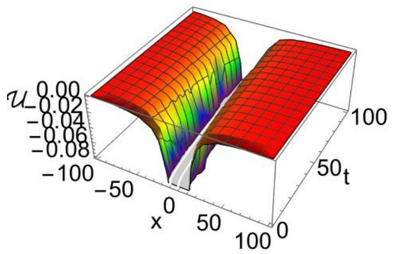
$$-\frac{12M_1\rho t^\sigma}{\sigma\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}+\frac{6\rho}{M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2}$$

$$-\frac{12M_1\mathrm{x}\rho}{\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}-\frac{12M_1\mathrm{y}\rho}{\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}$$

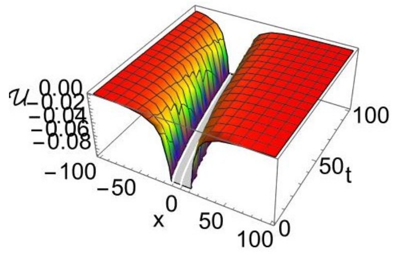
$$-\frac{6\rho^2 t^{2\sigma}}{\sigma^2\left(M_1\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)+M_2+\frac{1}{2}\rho\left(\frac{t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)^2\right)^2}.$$

Note: The graphs of $\mathcal{U}_{02}$ have not been displayed due to their similarity to the graphs of $\mathcal{U}_{01}$, which are shown in Figure 9.

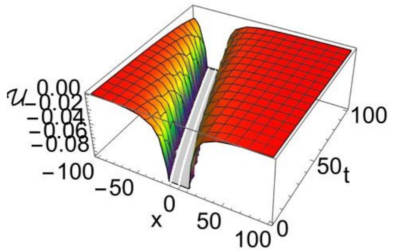
Case 3. (When $\kappa>0$)



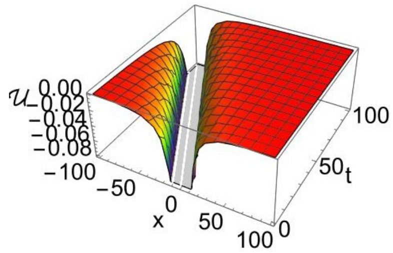
(a) at $\sigma = 0.5$



(b) at $\sigma = 0.75$



(c) at $\sigma = 0.9$



(d) at $\sigma = 1$

Figure 9: 3D graphs of $\mathcal{U}_{01}$ for different σ, when $y = 1$, $M_1 = 5$, $M_2 = 1$, $\rho = 1$, and $c = 2$.

Following Step 3 of the method illustrated above, we have obtained the following equations:

$$\left(\frac{G'}{G}\right) : 2p_1 + 2p_0p_1 + 2p_1c + 4p_1\kappa + \frac{2q_1q_2\kappa^2}{(M_1^2+M_2^2)\kappa^2-\rho^2} - \frac{12q_2\kappa^2\rho}{(M_1^2+M_2^2)\kappa^2-\rho^2} = 0;$$

$$\left(\frac{G'}{G}\right)^2 : p_1^2 + 2p_2 + 2p_0p_2 + 2p_2c - 16p_2\kappa - \frac{q_1^2\kappa}{(M_1^2+M_2^2)\kappa^2-\rho^2} + \frac{q_2^2\kappa^2}{(M_1^2+M_2^2)\kappa^2-\rho^2} + \frac{2q_1\kappa\rho}{(M_1^2+M_2^2)\kappa^2-\rho^2} - \frac{4p_2\kappa\rho^2}{(M_1^2+M_2^2)\kappa^2-\rho^2} = 0;$$

$$\left(\frac{G'}{G}\right)^3 : 4p_1 + 2p_1p_2 - \frac{2q_1q_2\kappa}{(M_1^2+M_2^2)\kappa^2-\rho^2} + \frac{12q_2\kappa\rho}{(M_1^2+M_2^2)\kappa^2-\rho^2} = 0;$$

$$\left(\frac{G'}{G}\right)^4 : 12p_2 + p_2^2 - \frac{q_2^2\kappa}{(M_1^2+M_2^2)\kappa^2-\rho^2} = 0;$$

$$\left(\frac{1}{G}\right) : 2q_1 + 2p_0q_1 + 2q_1c - 2q_1\kappa + 8p_2\kappa\rho + \frac{2q_1^2\kappa\rho}{(M_1^2+M_2^2)\kappa^2-\rho^2} - \frac{4q_1\kappa\rho^2}{(M_1^2+M_2^2)\kappa^2-\rho^2} + \frac{8p_2\kappa\rho^3}{(M_1^2+M_2^2)\kappa^2-\rho^2} = 0;$$

$$\left(\frac{G'}{G}\right)\left(\frac{1}{G}\right)) : 2p_1q_1 + 2q_2 + 2p_0q_2 + 2q_2c - 10q_2\kappa - 6p_1\rho + \frac{4q_1q_2\kappa\rho}{(M_1^2+M_2^2)\kappa^2-\rho^2} - \frac{24q_2\kappa\rho^2}{(M_1^2+M_2^2)\kappa^2-\rho^2} = 0;$$

$$\left(\frac{G'}{G}\right)^2\left(\frac{1}{G}\right) : 4q_1 + 2p_2q_1 + 2p_1q_2 - 20p_2\rho + \frac{2q_2^2\kappa\rho}{(M_1^2+M_2^2)\kappa^2-\rho^2} = 0;$$

$$\left(\frac{G'}{G}\right)^3\left(\frac{1}{G}\right) : 12q_2 + 2p_2q_2 = 0;$$

$$\left(\frac{G'}{G}\right)^0\left(\frac{1}{G}\right)^0 : 2p_0 + p_0^2 + 2p_0c + 4p_2\kappa^2 + \frac{q_1^2\kappa^2}{(M_1^2+M_2^2)\kappa^2-\rho^2} - \frac{2q_1\kappa^2\rho}{(M_1^2+M_2^2)\kappa^2-\rho^2} + \frac{4p_2\kappa^2\rho^2}{(M_1^2+M_2^2)\kappa^2-\rho^2} = 0.$$

The following are the solutions of the system of equations above, computed by Mathematica, we represent the solutions for this case as S_{2k}, where $1 \leq k \leq 4$:

$$S_{21} = \{p_0 \to -6\kappa, p_1 \to 0, p_2 \to -6, q_1 \to 6\rho, q_2 \to -\frac{6\sqrt{M_1^2\kappa^2 + M_2^2\kappa^2 - \rho^2}}{\sqrt{\kappa}}, c \to -1+\kappa\},$$

$$S_{22} = \{p_0 \to -6\kappa, p_1 \to 0, p_2 \to -6, q_1 \to 6\rho, q_2 \to \frac{6\sqrt{M_1^2\kappa^2 + M_2^2\kappa^2 - \rho^2}}{\sqrt{\kappa}}, c \to -1+\kappa\},$$

$$S_{23} = \{p_0 \to -4\kappa, p_1 \to 0, p_2 \to -6, q_1 \to 6\rho, q_2 \to -\frac{6\sqrt{M_1^2\kappa^2 + M_2^2\kappa^2 - \rho^2}}{\sqrt{\kappa}}, c \to -1-\kappa\},$$

$$S_{24} = \{p_0 \to -4\kappa, p_1 \to 0, p_2 \to -6, q_1 \to 6\rho, q_2 \to \frac{6\sqrt{M_1^2\kappa^2 + M_2^2\kappa^2 - \rho^2}}{\sqrt{\kappa}}, c \to -1-\kappa\}.$$

Corresponding values of $\mathcal{U}$, after using (3.2) and (3.7) in (4.2) are provided below:

$$\mathcal{U}_{21} = -\frac{6M_1\sqrt{M_1^2\kappa^2+M_2^2\kappa^2-\rho^2}\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6M_2\sqrt{M_1^2\kappa^2+M_2^2\kappa^2-\rho^2}\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$-\frac{6M_1^2\kappa\cos^2\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$-\frac{6M_2^2\kappa\sin^2\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{12M_1M_2\kappa\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$+\frac{6\rho}{M_1\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}}-6\kappa,$$

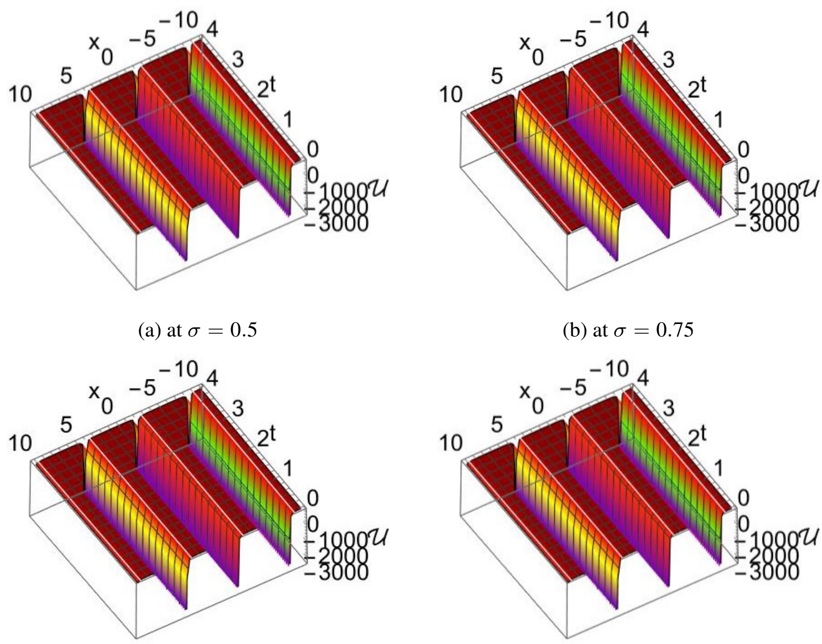
$$\mathcal{U}_{22} = \frac{6M_1\sqrt{M_1^2\kappa^2+M_2^2\kappa^2-\rho^2}\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$-\frac{6M_2\sqrt{M_1^2\kappa^2+M_2^2\kappa^2-\rho^2}\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$
$$-\frac{6M_1^2\kappa\cos^2\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}$$

$$
\begin{aligned}
&-\frac{6M_2^2\kappa\sin^2\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}\\
&+\frac{12M_1M_2\kappa\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}\\
&+\frac{6\rho}{M_1\sin\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}}-6\kappa,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{U}_{23}=-4\kappa&-\frac{6M_1\sqrt{M_1^2\kappa^2+M_2^2\kappa^2-\rho^2}\cos\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)}{\left(\frac{\rho}{\kappa}+M_2\cos\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)+M_1\sin\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)\right)^2}\\
&-\frac{6M_1^2\kappa\cos^2\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)}{\left(\frac{\rho}{\kappa}+M_2\cos\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)+M_1\sin\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)\right)^2}\\
&+\frac{6M_2\sqrt{M_1^2\kappa^2+M_2^2\kappa^2-\rho^2}\sin\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)}{\left(\frac{\rho}{\kappa}+M_2\cos\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)+M_1\sin\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)\right)^2}\\
&+\frac{12M_1M_2\kappa\cos\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)\sin\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)}{\left(\frac{\rho}{\kappa}+M_2\cos\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)+M_1\sin\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)\right)^2}\\
&-\frac{6M_2^2\kappa\sin^2\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)}{\left(\frac{\rho}{\kappa}+M_2\cos\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)+M_1\sin\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)\right)^2}\\
&+\frac{6\rho}{\frac{\rho}{\kappa}+M_2\cos\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)+M_1\sin\left(\sqrt{\kappa}\left(x+y-\frac{(-1-\kappa)t^\sigma}{\sigma}\right)\right)},
\end{aligned}
$$

Note: The graphs corresponding to other solutions have not been showcased because they are similar to those shown above.

$$
\begin{aligned}
\mathcal{U}_{24}=&\frac{6M_1\sqrt{M_1^2\kappa^2+M_2^2\kappa^2-\rho^2}\cos\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}\\
&-\frac{6M_2\sqrt{M_1^2\kappa^2+M_2^2\kappa^2-\rho^2}\sin\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}\\
&-\frac{6M_1^2\kappa\cos^2\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}\\
&-\frac{6M_2^2\kappa\sin^2\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}\\
&+\frac{12M_1M_2\kappa\sin\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)\cos\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)}{\left(M_1\sin\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+M_2\cos\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^\sigma}{\sigma}+\mathrm{x}+\mathrm{y}\right)\right)+\frac{\rho}{\kappa}\right)^2}
\end{aligned}
$$



(a) at $\sigma = 0.5$

(b) at $\sigma = 0.75$

(c) at $\sigma = 0.9$

(d) at $\sigma = 1$

Figure 10: 3D graphs of $\mathcal{U}_{21}$ for different σ, when $y = 1000$, $M_1 = 42$, $M_2 = 1$, $\rho = 40$, and $\kappa = 1$.

$$+\frac{6\rho}{M_1 \sin\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^{\sigma}}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + M_2 \cos\left(\sqrt{\kappa}\left(-\frac{(-\kappa-1)t^{\sigma}}{\sigma} + \mathrm{x} + \mathrm{y}\right)\right) + \frac{\rho}{\kappa}} - 4\kappa.$$

5 Results and Discussion

In this paper, we have implemented the above mentioned method, explained in section 3, on the TF-KP equation and derived a bunch of exact solutions in the form of hyperbolic functions, trigonometric functions, and rational functions. The hyperbolic solutions are the bright solitons as shown in Figures 1, 2, 3, 4, 5, 6, 7, and 8, and rational function solutions are dark solitons with singularity as depicted in Figure 9. The Figures 1, 6, 8 shows the formation of a bright soliton, providing us with a better understanding of wave formation in real-world systems. The Figures 3, 5 depict slight deviation due to variation in σ. Thus, Figures 1, 3, 5, 6, 8 showcase the usefulness of fractional derivative in improving real-world modeling mathematical systems. In addition, we have also derived some periodic singular solutions in the form of trigonometric functions.

6 Conclusion

The implementation of the selected method on the TF-KP equation has resulted in various types of solutions, namely bright soliton solutions, dark soliton solutions with singularity, and periodic singular solutions in the form of hyperbolic functions, rational functions, and trigonometric functions, respectively. The findings discussed in the previous section indicate that the method mentioned above is an effective tool for solving the TF-KP equation and can be useful in showcasing the potential of fractional derivatives in modeling real-world phenomena in a better way.

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