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A STUDY OF k-GONAL NUMBERS

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Abstract The study explores k**-gonal numbers** (denoted ${}_{k}^{n}P$) and their algebraic structures, extending their definition to all integers n and introducing specialized binary operations for cryptographic applications. This paper focused on generating n^{th}, k-gonal numbers, denoted ${}_{k}^{n}P$, for all integers as follows:

$$ {}_{k}^{n}P \text{ is } \begin{cases} \frac{n}{2}[(k-3)(n-1)+(n+1)] & \text{for } k>2, n\geq 0 \\ \frac{n}{2}[(k-3)(n-1)+(n-1)] & \text{for } k>2, n<0 \end{cases} \tag{1} $$

- for $n \geq 0$, this aligns with the standard formula for k-gonal numbers.
- for $n < 0$, the formula generalizes k-gonal numbers to negative indices, enabling symmetry in algebraic operations.

Also, it introduces to study of the Algebraic structure of some sets of k-gonal numbers, which can form a Monoid, under the binary operations,

$$ {}_{k}^{n}P \oplus_{k}^{m} P = \begin{cases} {}_{k}^{n}P +_{k}^{m} P + (k-2)mn, & \text{if } m,n \text{ both are the same sign} \\ {}_{k}^{n}P +_{k}^{m} P - (k-2)mn, & \text{if } m,n \text{ is an opposite sign} \end{cases}, \text{ and} $$

$$ {}_{k}^{n}P \odot_{k}^{m} P = \begin{cases} {}_{k}^{n}P \cdot_{k}^{m} P - \frac{(k-4)(k-2)mn}{4}(m-1)(n-1), & \text{if } m\geq 0, n\geq 0 \\ {}_{k}^{n}P \cdot_{k}^{m} P - \frac{(k-4)(k-2)mn}{4}(m-1)(n+1), & \text{if } m\geq 0, n<0 \\ {}_{k}^{n}P \cdot_{k}^{m} P - \frac{(k-4)(k-2)mn}{4}(m+1)(n-1), & \text{if } m<0, n\geq 0 \\ {}_{k}^{n}P \cdot_{k}^{m} P - \frac{(k-4)(k-2)mn}{4}(m+1)(n+1), & \text{if } m<0, n<0 \end{cases}, $$

i.e., the paper generalizes k-gonal numbers to a monoid with specialized operations, offering a novel framework for cryptographic systems. By leveraging the geometric and algebraic properties of polygonal numbers, it provides a foundation for secure key generation and exchange.
Notations: Successively replacing $k = 3, 4, 5, 6, 7, 8, \ldots$, etc., in above k-gonal numbers.

${}_{3}^{n}P =$ Set of 3-gonal (Triangular) numbers; ${}_{4}^{n}P =$ Set of 4-gonal (Square) numbers;
${}_{5}^{n}P =$ Set of 5-gonal (Pentagonal) numbers; ${}_{6}^{n}P =$ Set of 6-gonal (Hexagonal) numbers;
${}_{7}^{n}P =$ Set of 7-gonal (Heptagonal) numbers; ${}_{8}^{n}P =$ Set of 8-gonal (Octagonal) numbers;

1 Introduction

Number theory, the “Queen of Mathematics" is a broad and diverse part of pure and applied Mathematics that developed from the study of the integers with origins reaching back as far as the ancient Greeks and continuing into modern times.
The study of Number Theory is critical because all other branches depend on their final results on this branch. It is widespread in cryptography.
In this present topic, we are discussing some inherent properties of k-gonal numbers.

The 3-gonal numbers are represented as a triangular array of dots [4, 5, 9, 10]. The oblong numbers are represented as rectangular arrays of dots. The 4-gonal numbers are represented as a square array of dots. Hence k-gonal numbers are represented as regular arrangements of dots in geometric patterns. From a literature review point of view, k-gonal numbers are regular polygons as shown below in Figure 1. One of the Greek mathematicians, Hypsicles of Alexandria

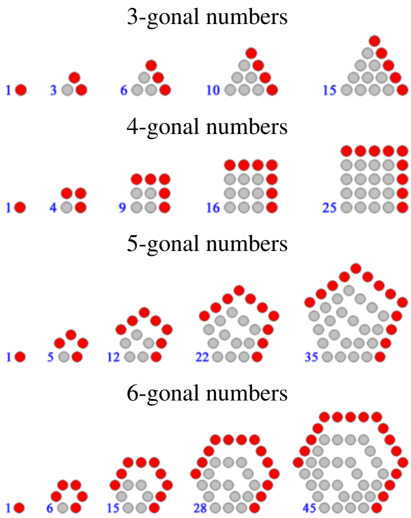

Figure 1: Geometric Representations of k-gonal numbers

gave the first general definition of k-gonal number in 170 BC. Diophantus, generate them as $P_k = \frac{(k-2)n^2-(k-4)n}{2}$. Also, Fermat introduces Polygonal number theorem.

Cryptographic Applications
The algebraic properties of k-gonal numbers under $\oplus$ and $\odot$ make them suitable for **symmetric key generation** [1, 2, 3, 4, 5, 6, 7, 8]:

(i) **Non-linearity**: The operations $\oplus$ and $\odot$ introduce complexity, resisting linear cryptanalysis.

(ii) **Scalability**: The parameter k allows customization of the cryptographic system.

(iii) **Unique Factorization**: The monoid structure may enable secure key exchange protocols by leveraging the difficulty of reversing operations.

Key Properties

- **Closure**: $\oplus$ and $\odot$ ensure results remain k-gonal numbers.
- **Associativity**: Critical for monoid structure.
- **Symmetry**: Negative indices extend applicability, enabling bidirectional operations

2 Main Work

In this paper, I have proposed to redefine to generate k-gonal' numbers not only for Positive integers but also Negative integers.
Also, introduces to study Algebraic structure of k-gonal numbers by defining additive and multiplicative binary operations.
Also, discuss some inherent properties of some sets of k-gonal numbers.

3-GONAL NUMBERS (Triangular Numbers)

Handshake Puzzle problems, Full Mesh Network problems, and Number strip Puzzles are solvable with using of 3-gonal numbers. Now we can introduce a set of 3-gonal numbers as follows, replace $k = 3$ in Equation (1),

$$ {}^{n}_{3}P = \begin{cases} \frac{n}{2}(n+1), & \text{if } n \geq 0 \\ \frac{n}{2}(n-1), & \text{if } n < 0 \end{cases} = \{0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, \cdots, \text{ etc.}\} $$

Case 1: Now introduces Additive Binary operation ($\oplus$) on above set ${}^{n}_{3}P$ as follows:

$$ {}^{n}_{3}P \oplus {}^{m}_{3}P = \begin{cases} {}^{m}_{3}P + {}^{n}_{3}P + mn, & \text{if } m, n \text{ are same sign} \\ {}^{m}_{k}P + {}^{n}_{k}P - mn, & \text{if } m, n \text{ are opposite sign} \end{cases} $$

Proof. **Case 1.1**: Let m, and n both are the same sign. Without loss of generality, let $m \geq 0, n \geq 0$. Consider

$$ \begin{aligned} {}^{m}_{3}P + {}^{n}_{3}P + mn &= \frac{m}{2}(m+1) + \frac{n}{2}(n+1) + mn \\ &= \frac{m+n}{2}((m+n)+1) = {}^{m+n}_{3}P. \end{aligned} $$

Let m, and n both are the same sign. Without loss of generality, let $m < 0, n < 0$. Consider

$$ \begin{aligned} {}^{m}_{3}P + {}^{n}_{3}P + mn &= \frac{m}{2}(m-1) + \frac{n}{2}(n-1) + mn \\ &= \frac{m+n}{2}((m+n)-1) = {}^{m+n}_{3}P. \end{aligned} $$

Hence, I conclude that, if m and n are the same sign, the binary operation $\oplus$, on a set of 3-gonal numbers are defined as ${}^{n}_{3}P \oplus {}^{m}_{3}P = {}^{m}_{3}P + {}^{n}_{3}P + mn \cong {}^{m+n}_{3}P$.
Case 1.2: Let m, n be the opposite sign. Without loss of generality, let $m \geq 0, n < 0$. Consider

$$ \begin{aligned} {}^{m}_{3}P + {}^{n}_{3}P - mn &= \frac{m}{2}(m+1) + \frac{n}{2}(n-1) - mn \\ &= \frac{m-n}{2}((m-n)+1) = {}^{m-n}_{3}P. \end{aligned} $$

Hence, I conclude that, if m, n is an opposite sign, additive binary operation $\oplus$, on a set of 3-gonal numbers are defined as

$$ {}^{n}_{3}P \oplus {}^{m}_{3}P = {}^{m}_{3}P + {}^{n}_{3}P + mn \cong {}^{m-n}_{3}P. $$

Case 1.3: Now introduces **another Additive Binary Operation** on ${}^{n}_{3}P$, whenever m, n has an opposite sign, which is defined as follows:

$$ {}^{n}_{3}P \oplus {}^{m}_{3}P = \begin{cases} {}^{m}_{3}P + {}^{n}_{3}P + (m+1)n, & \text{if } m, n \text{ are opposite sign}, m+n \geq 0 \\ {}^{m}_{k}P + {}^{n}_{k}P + m(n-1), & \text{if } m, n \text{ are opposite sign}, m+n < 0 \end{cases} $$

If m, n are opposite sign, $m + n \geq 0$. Let $m \geq 0, n < 0$, consider

$$ \begin{aligned} {}^{m}_{3}P + {}^{n}_{3}P + (m+1)n &= \frac{m}{2}(m+1) + \frac{n}{2}(n-1) + (m+1)n \\ &= \frac{m+n}{2}((m+n)+1) \cong {}^{m+n}_{3}P. \end{aligned} $$

Hence, I conclude that, if m and n are an opposite sign, with $m + n \geq 0$, binary operation $\oplus$, on a set of 3-gonal numbers is defined as

$$ {}^{n}_{3}P \oplus {}^{m}_{3}P = {}^{m}_{3}P + {}^{n}_{3}P + (m+1)n \cong {}^{m+n}_{3}P, $$

if m, n are opposite sign, $m + n < 0$.
Let $m \geq 0, n < 0$, consider

$$ \begin{aligned} {}^{m}_{k}P + {}^{n}_{k}P + m(n-1) &= \frac{m}{2}(m+1) + \frac{n}{2}(n-1) + m(n-1) \\ &= \frac{m+n}{2}((m+n)-1) \cong {}^{m+n}_{3}P. \end{aligned} $$

Hence, I conclude that, if m and n are an opposite sign, with $m+n<0$, binary operation $\oplus$, on a set of 3-gonal numbers is defined as

$$ {}^{n}_{3}P \oplus {}^{m}_{3}P = {}^{m}_{3}P + {}^{n}_{3}P + m(n-1) = {}^{m+n}_{3}P, $$

□

Case 2: Now introduces multiplicative binary operations ($\odot$) on the above set 3-gonal numbers as follows.

$$ {}^{n}_{3}P \odot {}^{m}_{3}P = \begin{cases} {}^{n}_{3}P{}^{m}_{3}P + \frac{mn}{4}(m-1)(n-1), & \text{if } m \geq 0, n \geq 0 \\ {}^{n}_{3}P{}^{m}_{3}P + \frac{mn}{4}(m-1)(n+1), & \text{if } m \geq 0, n < 0 \\ {}^{n}_{3}P{}^{m}_{3}P + \frac{mn}{4}(m+1)(n-1), & \text{if } m < 0, n \geq 0 \\ {}^{n}_{3}P{}^{m}_{3}P + \frac{mn}{4}(m+1)(n+1), & \text{if } m < 0, n < 0 \end{cases} $$

Proof. **Case 2.1**: Let $m \geq 0, n \geq 0$. Consider

$$ \begin{aligned} {}^{n}_{3}P \cdot {}^{m}_{3}P + \frac{mn}{4}(m-1)(n-1) &= \left(\frac{m}{2}(m+1)\right)\left(\frac{n}{2}(n+1)\right) + \frac{mn}{4}(m-1)(n-1) \\ &= \frac{mn}{4}[(m+1)(n+1) + (m-1)(n-1)] \\ &= \frac{mn}{2}(mn+1) \cong {}^{mn}_{3}P. \end{aligned} $$

Hence, I conclude that, if $m \geq 0$ and $n \geq 0$, another binary operation ($\odot$), on a set of 3-gonal numbers is defined as

$$ {}^{n}_{3}P \odot {}^{m}_{3}P = {}^{m}_{3}P{}^{n}_{3}P + \frac{mn}{4}(m-1)(n-1) \cong {}^{mn}_{3}P. $$

Case 2.2: Let $m \geq 0, n < 0$. Consider

$$ \begin{aligned} {}^{n}_{3}P \cdot {}^{m}_{3}P + \frac{mn}{4}(m-1)(n+1) &= \left(\frac{m}{2}(m+1)\right)\left(\frac{n}{2}(n-1)\right) + \frac{mn}{4}(m-1)(n+1) \\ &= \frac{mn}{4}[(m+1)(n-1) + (m-1)(n+1)] \\ &= \frac{mn}{2}(mn-1) \cong {}^{mn}_{3}P. \end{aligned} $$

Hence, I conclude that, if $m \geq 0$ and $n < 0$, binary operation ($\odot$), on a set of 3-gonal numbers is defined as

$$ {}^{n}_{3}P \odot {}^{m}_{3}P = {}^{m}_{3}P{}^{n}_{3}P + \frac{mn}{4}(m-1)(n+1) \cong {}^{mn}_{3}P. $$

Case 2.3: Let $m < 0, n \geq 0$. Consider

$$ \begin{aligned} {}^{n}_{3}P \cdot {}^{m}_{3}P + \frac{mn}{4}(m+1)(n-1) &= \left(\frac{m}{2}(m-1)\right)\left(\frac{n}{2}(n+1)\right) + \frac{mn}{4}(m+1)(n-1) \\ &= \frac{mn}{4}[(m-1)(n+1) + (m+1)(n-1)] \\ &= \frac{mn}{2}(mn-1) \cong {}^{mn}_{3}P. \end{aligned} $$

Hence, I conclude that, if $m < 0$ and $n \geq 0$, Multiplicative binary operation ($\odot$), on a set of 3-gonal numbers is defined as

$$ {}^{n}_{3}P \odot {}^{m}_{3}P = {}^{m}_{3}P{}^{n}_{3}P + \frac{mn}{4}(m+1)(n-1) \cong {}^{mn}_{3}P. $$

Case 2.4: Let $m < 0, n < 0$. Consider

$$ \begin{aligned} {}^{n}_{3}P \cdot {}^{m}_{3}P + \frac{mn}{4}(m+1)(n+1) &= \left(\frac{m}{2}(m-1)\right)\left(\frac{n}{2}(n-1)\right) + \frac{mn}{4}(m+1)(n+1) \\ &= \frac{mn}{4}[(m-1)(n-1) + (m+1)(n+1)] \\ &= \frac{mn}{2}(mn+1) \cong {}^{mn}_{3}P. \end{aligned} $$

Hence, I conclude that, if $m < 0$ and $n < 0$, Multiplicative binary operation $(\odot)$, on a set of 3-gonal numbers is defined as

$$ {}^{n}_{3}P \odot {}^{m}_{3}P = {}^{m}_{3}P\,{}^{n}_{3}P + \frac{mn}{4}(m+1)(n+1) \cong {}^{mn}_{3}P. $$

□

Case 3: Now we can go to introduce Algebraic Structure on the set of ${}^{n}_{3}P$.
From case 1 and case 2, the Binary operation of Addition and Multiplication is well defined on a set of 3-gonal numbers. Hence Closure Axiom exists on ${}^{n}_{3}P$.
Also, concerning Additive Binary Operation $\oplus$, the Associative Axiom is well-defined

$$ ({}^{n}_{3}P \oplus {}^{m}_{3}P) \oplus {}^{l}_{3}P = {}^{n}_{3}P \oplus ({}^{m}_{3}P \oplus {}^{l}_{3}P) $$

on the following subcases.
Case 3.1: If all are the same sign. Suppose, if $m \geq 0, n \geq 0, l \geq 0$ then

$$ ({}^{n}_{3}P \oplus {}^{m}_{3}P) \oplus {}^{l}_{3}P = {}^{m+n}_{3}P \oplus {}^{l}_{3}P = {}^{m+n+l}_{3}P. $$

Also,

$$ {}^{n}_{3}P \oplus ({}^{m}_{3}P \oplus {}^{l}_{3}P) = {}^{n}_{3}P \oplus {}^{m+l}_{3}P = {}^{m+n+l}_{3}P. $$

Hence $({}^{n}_{3}P \oplus {}^{m}_{3}P) \oplus {}^{l}_{3}P = {}^{n}_{3}P \oplus ({}^{m}_{3}P \oplus {}^{l}_{3}P)$.
Case 3.2: If one of them is a negative integer. Suppose, if $m \geq 0, n \geq 0, l < 0$ then

$$ ({}^{n}_{3}P \oplus {}^{m}_{3}P) \oplus {}^{l}_{3}P = {}^{m+n}_{3}P \oplus {}^{l}_{3}P = {}^{m+n-l}_{3}P. $$

Also,

$$ {}^{n}_{3}P \oplus ({}^{m}_{3}P \oplus {}^{l}_{3}P) = {}^{n}_{3}P \oplus {}^{m-l}_{3}P = \begin{cases} {}^{n+m-l}_{3}P, & \text{if } m-l \geq 0 \\ {}^{n-m+l}_{3}P, & \text{if } m-l < 0 \end{cases}. $$

Hence, $({}^{n}_{3}P \oplus {}^{m}_{3}P) \oplus {}^{l}_{3}P = {}^{n}_{3}P \oplus ({}^{m}_{3}P \oplus {}^{l}_{3}P)$.
Case 3.2: If two of them is a negative integer. Suppose, if $m < 0, n < 0, l \geq 0$ then

$$ ({}^{n}_{3}P \oplus {}^{m}_{3}P) \oplus {}^{l}_{3}P = {}^{n+m}_{3}P \oplus {}^{l}_{3}P = \begin{cases} {}^{l-n-m}_{3}P, & \text{if } l-n-m \geq 0 \\ {}^{-n-m-l}_{3}P, & \text{if } l-n-m < 0 \end{cases}. $$

Also,

$$ {}^{n}_{3}P \oplus ({}^{m}_{3}P) \oplus {}^{l}_{3}P = {}^{n}_{3}P \oplus {}^{l-m}_{3}P = \begin{cases} {}^{l-n-m}_{3}P, & \text{if } l-m \geq 0 \\ {}^{n+l-m}_{3}P, & \text{if } l-m < 0 \end{cases}. $$

Hence,

$$ ({}^{n}_{3}P \oplus {}^{m}_{3}P) \oplus {}^{l}_{3}P = {}^{n}_{3}P \oplus ({}^{m}_{3}P \oplus {}^{l}_{3}P). $$

Also, the existence of an Additive Identity element on a set of 3-gonal numbers is ${}^{0}_{3}P = 0 \in {}^{n}_{3}P$. With ${}^{n}_{3}P \oplus {}^{0}_{3}P = {}^{0}_{3}P \oplus {}^{n}_{3}P = {}^{n}_{3}P$. Hence we conclude that $({}^{n}_{3}P\oplus)$ is a Monoid.
Since under this binary operation '$\oplus$', a set of 3-gonal numbers ${}^{n}_{3}P$ satisfies the Closure Axiom, Associative Axiom, and Existence of Identity Axiom.
Case 4: Similarly, concerning Binary operation '$\odot$', the set of 3-gonal numbers ${}^{n}_{3}P$ satisfies the Associative Axiom,

$$ ({}^{n}_{3}P \odot {}^{m}_{3}P) \odot {}^{l}_{3}P = {}^{n}_{3}P \odot ({}^{m}_{3}P \odot {}^{l}_{3}P). $$

Hence

$$ ({}^{n}_{3}P \odot {}^{m}_{3}P) \odot {}^{l}_{3}P = {}^{n}_{3}P \odot ({}^{m}_{3}P \odot {}^{l}_{3}P) \cong {}^{mnl}_{3}P. $$

Also, the existence of a Multiplicative Identity element on a set of 3-gonal numbers is ${}^{1}_{3}P = 1 \in {}^{n}_{3}P$.
With ${}^{n}_{3}P \odot {}^{1}_{3}P = {}^{1}_{3}P \odot {}^{n}_{3}P = {}^{n}_{3}P$. Hence we conclude that $({}^{n}_{3}P, \odot)$ is a Monoid.
Since under this Multiplicative binary operation '$\odot$', Set of 3-gonal numbers ${}^{n}_{3}P$ satisfies the Closure Axiom, Associative Axiom, and Existence of Identity Axiom.

2.1 Properties of 3-gonal numbers

Property 1: Sum of First $n(n>0)$ 3-gonal numbers $\sum_{i=1}^{n} {}^{i}_{3}P$ is $\left(\frac{n+2}{3}\right) {}^{n}_{3}P$.

Proof. Consider

$$\sum_{i=1}^{n} {}^{i}_{3}P = \sum_{i=1}^{n} \frac{n+1}{2} = \frac{1}{2}\left(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right)$$
$$= \frac{n+1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right) = \frac{n(n+1)}{4}\left(\frac{(2n+1)}{3}+1\right)$$
$$= \frac{n(n+1)}{4}\left(\frac{(2n+1)}{3}+1\right) = \frac{n(n+1)(2n+4)}{12} = \left(\frac{n+2}{3}\right) {}^{n}_{3}P$$

□

Property 2: Residues of a set of ${}^{n}_{3}P$, mod m (for each positive integer m), repeats every m step if m is odd, and every $2m$ step if m is even.

Proof. By replacing each positive integer m, we can easily verify the residues of a set of ${}^{n}_{3}P$, mod m repeats every m step if m is odd, and every $2m$ step if m is even. □

Property 3: $({}^{n}_{3}P)^2 + \left({}^{n-1}_{3}P\right)^2 = \begin{cases} {}^{n^2}_{3}P, & \text{if } n \geq 0 \\ {}^{(n-1)^2}_{3}P, & \text{if } n < 0 \end{cases}$

Proof. If $n \geq 0$, consider

$$({}^{n}_{3}P)^2 + \left({}^{n-1}_{3}P\right)^2 = \left(\frac{n(n+1)}{2}\right)^2 + \left(\frac{n(n-1)}{2}\right)^2 = \frac{n^2(2n^2+2)}{4} = {}^{n^2}_{3}P.$$

If $n < 0$, consider

$$({}^{n}_{3}P)^2 + \left({}^{n-1}_{3}P\right)^2 = \left(\frac{n(n-1)}{2}\right)^2 + \left(\frac{(n-2)(n-1)}{2}\right)^2 = \frac{(n-1)^2(n^2+(n-2)^2)}{4}$$
$$= \frac{(n-1)^2(2n^2+4-4n)}{4} = {}^{(n-1)^2}_{3}P.$$

It is also one of the additive binary operations $\oplus$, between successive 3-gonal numbers.

$${}^{n}_{3}P \oplus {}^{n-1}_{3}P = ({}^{n}_{3}P)^2 + \left({}^{n-1}_{3}P\right)^2.$$

□

Property 4: The sum of Reciprocal's of 3-gonal numbers (for $n > 0$) is 2.

Proof. Consider

$$\sum_{i=1}^{n} \frac{1}{{}^{n}_{3}P} = \sum_{i=1}^{n} \frac{2}{n(n+1)} = 2\sum_{i=1}^{n} \frac{1}{n(n+1)} = 2\sum_{i=1}^{n}\left[\frac{1}{n} - \frac{1}{1+n}\right] = 2.$$

□

Property 5: If n, m, and l are three positive integers then ${}^{n}_{3}P = lm$ if and only if

$${}^{n+m+l}_{3}P = {}^{n+m}_{3}P + {}^{n+l}_{3}P.$$

Property 6: Let $n > 0$, define a mapping $f : {}^{n}_{3}P \rightarrow {}^{n}_{3}P$ with $f(x) = 9x+1, g(x) = 25x+3, f(x) = 49x+6$ all are 3-gonal numbers.
It follows that recursive function, for $n > 0, f : (z^+, {}^{n}_{3}P) \rightarrow {}^{n}_{3}P$, is

$$f(n,x) = (2n+1)^2 x + \frac{n}{2}(n+1) = \frac{n^2(8x+1)+n(8x+1)+2x}{2}$$

for all $x \in {}^{n}_{3}P$.

Proof. $f(x) = 9x + 1 = 9\left(\frac{n}{2}(n+1)\right) + 1 = \frac{9n^2 + 9n + 2}{2} = \frac{3n+1}{2}(3n+2) = {}^{3n+1}_{3}P.$
Similarly,

$$g(x) = 25x + 3 = 25\left(\frac{n}{2}(n+1)\right) + 3 = \frac{25n^2 + 25n + 6}{2} = \frac{5n+2}{2}(5n+3) = {}^{5n+2}_{3}P.$$

Similarly,

$$h(x) = 49x + 6 = 49\left(\frac{n}{2}(n+1)\right) + 6 = \frac{49n^2 + 49n + 12}{2} = \frac{7n+3}{2}(7n+4) = {}^{7n+3}_{3}P.$$

Now, I can extend to generate composite functions using of above functions.
Let

$$\begin{aligned} P(x) &= f \circ g(x) = f(g(x)) = 9\left({}^{5n+2}_{3}P\right) + 1 \\ &= 9\left(\frac{5n+2}{2}(5n+3)\right) + 1 = 9\left(\frac{25n^2 + 25n + 6}{2}\right) + 1 = \frac{225n^2 + 225n + 56}{2} \\ &= \frac{15n+7}{2}(15n+8) = {}^{15n+7}_{3}P. \end{aligned}$$

Similarly, $f \circ h(x), g \circ h(x), f \circ p(x), \ldots$, etc. All are becoming 3-gonal numbers. □

Also, $f(x) = 9x + 1, f(x) = 25x + 3, f(x) = 49x + 6$ are maps ${}^{3n+1}_{3}P, {}^{5n+2}_{3}P, {}^{7n+3}_{3}P$ respectively.
It follows that the next sequence of 3-goal number ${}^{9n+4}_{3}P$ maps $81x+10$, ${}^{11n+5}_{3}P$ maps $121x+15$, ${}^{13n+6}_{3}P$ maps $169x + 21, \ldots$, etc.
Now I can go to introduce that recursive function, for $n > 0, f : (z^+, {}^{n}_{3}P) \to {}^{n}_{3}P$, is

$$f(n, x) = (2n+1)^2 x + \frac{n}{2}(n+1) = \frac{n^2(8x+1) + n(8x+1) + 2x}{2} \text{ for all } x \in {}^{n}_{3}P.$$

Also, it is easy to verify that for fixed values of x. In particular,
When $x = 1\left(\cong {}^{1}_{3}P\right), f(n, 1) = {}^{3n+1}_{3}P$;
When $x = 3\left(\cong {}^{2}_{3}P\right), f(n, 3) = {}^{5n+2}_{3}P$;
When $x = 6\left(\cong {}^{3}_{3}P\right), f(n, 6) = {}^{7n+3}_{3}P$;
if follows that at $x = 10\left(\cong {}^{4}_{3}P\right), f(n, 10) = {}^{9n+4}_{3}P$;
Hence, for generalisation for some $x = {}^{k}_{3}P, f(n, {}^{k}_{3}P) = {}^{((2k+1)n+k)}_{3}P$ and
$f(n, x) = (2n+1)^2 x + \frac{n}{2}(n+1)$.
Also, from above property 5, it is possible to generate at least k number of countable subsets ${}^{((2k+1)n+k)}_{3}P$ on Set of ${}^{n}_{3}P$.

4-GONAL NUMBERS (SQUARE NUMBERS)

Now I can go to introduce square numbers, by replace $k = 4$, in Equation 1, for all integers as follows ${}^{n}_{4}P = n^2 = \{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots, \text{etc.}\}$.
Case 1: Now introduces two types of additive binary operations on Set of ${}^{n}_{4}P$ as follows:

$${}^{n}_{4}P \oplus {}^{m}_{4}P = \begin{Bmatrix} {}^{n}_{4}P + {}^{m}_{4}P + 2mn \\ {}^{n}_{4}P + {}^{m}_{4}P - 2mn \end{Bmatrix}.$$

Consider ${}^{n}_{4}P + {}^{m}_{4}P + 2mn = m^2 + n^2 + 2mn = (n+m)^2 = {}^{n}_{4}P + m$.
Again consider ${}^{n}_{4}P + {}^{m}_{4}P - 2mn = m^2 + n^2 - 2mn = (n-m)^2 = {}^{n-m}_{4}P$.
Hence, conclude that the above additive binary operation is well-defined on the set of ${}^{n}_{4}P$.
Case 2: Similarly, introduces multiplicative binary operation on Set of ${}^{n}_{4}P$ as follows:

$${}^{n}_{4}P \odot {}^{m}_{4}P = {}^{n}_{4}P \cdot {}^{m}_{4}P$$

Consider ${}_4^nP \cdot {}_4^mP = m^2 \cdot n^2 = (mn)^2 = {}_4^{mn}P$.
Hence, conclude that the above Multiplicative binary operation is well-defined on the set of ${}_4^nP$.
Property 1: Under these binary operations set of 4-gonal numbers can satisfy the Associative Axiom.

$$({}_4^nP \oplus {}_4^mP) \oplus {}_4^lP = {}_4^nP \oplus ({}_4^mP \oplus {}_4^lP),$$
$$({}_4^nP \odot {}_4^mP) \odot {}_4^lP = {}_4^nP \odot ({}_4^mP \odot {}_4^lP)$$

Concerning the Additive binary operation, the existence of the identity element is ${}_4^0P \in {}_4^nP$.
Since ${}_4^nP \oplus {}_4^0P = {}_4^0P \oplus {}_4^nP = {}_4^nP$.
Concerning Multiplicative binary operation, the existence of the identity element is ${}_4^1P \in {}_4^nP$.
Since ${}_4^nP \odot {}_4^1P = {}_4^1P \oplus {}_4^nP = {}_4^nP$.
Hence, we can conclude that a set of 4-gonal numbers, under these additive and multiplicative binary operations can form a Monoid.
Hence, $({}_4^nP, \oplus), ({}_4^nP, \odot)$ are Monoid.
Property 2: Let p be an odd prime. Then p can be expressed as the sum of two squares if and only if $p \cong 1(\text{mod } 4)$.

Proof. It is easily with using of induction method.
Suppose, choose any two squares, Let 4, 9 be squares, and their sum is 13, which obviously satisfies $p \cong 1(\text{mod } 4)$. Conversely, also if $p \cong 1(\text{mod } 4)$ then p can be expressed as the sum of two squares. □

Property 3: If ${}_3^mP$ is a 3-gonal number then $8({}_3^mP) + 1$ is a 4-gonal number.
Also, ${}_3^mP$ is a perfect square then $4({}_3^mP)(8({}_3^mP) + 1)$ is a 3-gonal number.
There are some sets of 3-gonal numbers as well as 4-gonal numbers. The following sequence is a set of 3-gonal as well as 4-gonal numbers.
$\{1, 36, 1225, 41616, 1413721, 48024900, 1631432881, 55420693056, \ldots, \text{etc.}\}$.

Proof. If ${}_3^mP$ is a Triangular number, consider

$$8({}_3^mP) + 1 = 8\left(\frac{m}{2}(m+1)\right) + 1 = (2m+1)^2 = {}_4^{2m+1}P,$$

which is a 4-gonal number.
Hence $f : {}_3^nP \to {}_4^nP$ with $f(x) = 8x + 1$ is well-defined.
If ${}_3^mP$ is a perfect square then

$$4({}_3^mP)(8({}_3^mP) + 1) = \frac{(8({}_3^mP))(8({}_3^mP) + 1)}{2} = {}_3^{8({}_3^mP)}P,$$

which is a 3-gonal number. □

Property 4: The Sum of two consecutive 3-gonal numbers is always 4-gonal number.

Proof. If $m \geq 0$, consider

$$\begin{aligned} {}_3^mP + {}_3^{m+1}P &= \frac{m}{2}(m+1) + \frac{m+1}{2}((m+1)+1) \\ &= \frac{m+1}{2}(m + (m+1) + 1) = (m+1)^2 = {}_4^{m+1}P. \end{aligned}$$

If $m < -1$ and $m + 1 < 0$, then consider

$$\begin{aligned} {}_3^mP + {}_3^{m+1}P &= \frac{m}{2}(m-1) + \frac{m+1}{2}((m+1)-1) \\ &= \frac{m}{2}(m - 1 + (m+1)) = (m)^2 = {}_4^mP. \end{aligned}$$

If $m = -1$ or $m + 1 = 0$, then consider ${}_3^mP + {}_3^{m+1}P = {}_3^{-1}P + {}_3^0P = 1 + 0 = 1 = (\pm1)^2$. □

5-GONAL NUMBERS (Pentagonal numbers)

Now I can go to introduce 5-gonal numbers by replacing $k = 5$, in Equation (1)

$$ {}^{n}_{5}P = \begin{cases} \frac{n}{2}(3n-1), & \text{if } n \geq 0 \\ \frac{n}{2}(3n+1), & \text{if } n < 0 \end{cases} = \{0, 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, \ldots, etc.\} $$

Case 1: Now I can go to introducing binary operation on a set of 5-gonal numbers as follows:

$$ {}^{n}_{5}P \oplus {}^{m}_{5}P = \begin{cases} {}^{n}_{5}P + {}^{m}_{5}P + 3mn, & \text{if } m, n \text{ both are same sign} \\ {}^{n}_{5}P + {}^{m}_{5}P - 3mn, & \text{if } m, n \text{ are opposite sign} \end{cases} $$

Case 1.1: Let m, and n both are the same sign. Choose $m \geq 0, n \geq 0$. Consider

$$ \begin{aligned} {}^{n}_{5}P + {}^{m}_{5}P + 3mn &= \frac{n}{2}(3n-1) + \frac{m}{2}(3m-1) + 3mn \\ &= \frac{m+n}{2}(3(m+n)-1) = {}^{m+n}_{5}P. \end{aligned} $$

Case 1.2: Let m, and n both be the same sign. Choose $m < 0, n < 0$. Consider

$$ \begin{aligned} {}^{n}_{5}P + {}^{m}_{5}P + 3mn &= \frac{n}{2}(3n+1) + \frac{m}{2}(3m+1) + 3mn \\ &= \frac{m+n}{2}(3(m+n)+1) = {}^{m+n}_{5}P. \end{aligned} $$

Hence from Case 1.1 and Case 1.2, I conclude that if m, and n both are the same sign, then the additive binary operation on Set of 5-gonal numbers is

$$ {}^{n}_{5}P \oplus {}^{m}_{5}P = {}^{n}_{5}P + {}^{m}_{5}P + 3mn. $$

Case 1.3: Let m, and n are opposite signs. Choose $m < 0, n \geq 0$. Consider

$$ \begin{aligned} {}^{n}_{5}P + {}^{m}_{5}P - 3mn &= \frac{n}{2}(3n+1) + \frac{m}{2}(3m-1) - 3mn \\ &= \frac{m-n}{2}(3(m-n)+1) = {}^{m-n}_{5}P. \end{aligned} $$

Hence I conclude that if m, and n both are opposite signs, then the binary operation on a set of 5-gonal numbers is ${}^{n}_{5}P \oplus {}^{m}_{5}P = {}^{n}_{5}P + {}^{m}_{5}P - 3mn$.
Case 2: Now introducing binary operation on a set of 5-gonal numbers as follows:

$$ {}^{n}_{5}P \odot {}^{m}_{5}P = \begin{cases} {}^{n}_{5}P \cdot {}^{m}_{5}P - \frac{3mn}{4}(m-1)(n-1), & \text{if } m \geq 0, n \geq 0 \\ {}^{n}_{5}P \cdot {}^{m}_{5}P - \frac{3mn}{4}(m-1)(n+1), & \text{if } m \geq 0, n < 0 \\ {}^{n}_{5}P \cdot {}^{m}_{5}P - \frac{3mn}{4}(m+1)(n-1), & \text{if } m < 0, n \geq 0 \\ {}^{n}_{5}P \cdot {}^{m}_{5}P - \frac{3mn}{4}(m+1)(n+1), & \text{if } m < 0, n < 0 \end{cases} $$

Case 2.1: if $m \geq 0, n \geq 0$. Consider

$$ \begin{aligned} {}^{n}_{5}P \cdot {}^{m}_{5}P - \frac{3mn}{4}(m-1)(n-1) &= \left(\frac{n}{2}(3n-1)\right)\left(\frac{m}{2}(3m-1)\right) - \frac{3mn}{4}(m-1)(n-1) \\ &= \frac{mn}{4}[(3n-1)(3m-1) - 3(m-1)(n-1)] \\ &= \left(\frac{3mn}{2}(3nm-1)\right) = {}^{mn}_{5}P \end{aligned} $$

Case 2.2: if $m \geq 0, n < 0$. Consider

$$ \begin{aligned} {}^{n}_{5}P \cdot {}^{m}_{5}P - \frac{3mn}{4}(m-1)(n+1) &= \left(\frac{n}{2}(3n+1)\right)\left(\frac{m}{2}(3m-1)\right) - \frac{3mn}{4}(m-1)(n+1) \\ &= \frac{mn}{4}[(3n+1)(3m-1) - 3(m-1)(n+1)] \\ &= \left(\frac{3mn}{2}(3nm+1)\right) = {}^{mn}_{5}P \end{aligned} $$

Case 2.3: if $m < 0, n \geq 0$. Consider

$$
\begin{aligned}
{}_5^nP \cdot {}_5^mP - \frac{3mn}{4}(m+1)(n-1) &= \left(\frac{n}{2}(3n-1)\right)\left(\frac{m}{2}(3m+1)\right) - \frac{3mn}{4}(m+1)(n-1) \\
&= \frac{mn}{4}[(3n-1)(3m+1) - 3(m+1)(n-1)] \\
&= \left(\frac{3mn}{2}(3nm+1)\right) = {}_5^{mn}P
\end{aligned}
$$

Case 2.4: if $m < 0, n < 0$. Consider

$$
\begin{aligned}
{}_5^nP \cdot {}_5^mP - \frac{3mn}{4}(m+1)(n+1) &= \left(\frac{n}{2}(3n+1)\right)\left(\frac{m}{2}(3m+1)\right) - \frac{3mn}{4}(m+1)(n+1) \\
&= \frac{mn}{4}[(3n+1)(3m+1) - 3(m+1)(n+1)] \\
&= \left(\frac{3mn}{2}(3nm-1)\right) = {}_5^{mn}P.
\end{aligned}
$$

Hence, conclude that the above Multiplicative binary operation is well-defined on the set of ${}_5^nP$.
Case 3: Now I can go to introduce Algebraic Structure on the set of 5-gonal numbers.
From case 1 and case 2, the Binary operations are well defined on a set of 5-gonal numbers. Hence Closure Axiom exists on ${}_5^nP$.
Also, concerning Additive Binary Operation $\oplus$, the Associative Axiom is well-defined

$$({}_5^nP \oplus {}_5^mP) \oplus {}_5^lP = {}_5^nP \oplus ({}_5^mP \oplus {}_5^lP)$$

Also, the existence of an Additive Identity element on a set of 5-gonal numbers is ${}_5^0P = 0 \in {}_5^nP$. With ${}_5^nP \oplus {}_5^0P = {}_5^0P \oplus {}_5^nP = {}_5^nP$. Hence we conclude that $({}_5^nP, \oplus)$ is a Monoid.
Since under this additive binary operation '$\oplus$', a set of 5-gonal numbers ${}_5^nP$ satisfies the Closure Axiom, Associative Axiom, and Existence of Identity Axiom.
Similarly, concerning Multiplicative Binary operation '$\odot$', the set of 5-gonal numbers ${}_5^nP$ satisfies the Associative Axiom, $({}_5^nP \odot {}_5^mP) \odot {}_5^lP = {}_5^nP \odot ({}_5^mP \odot {}_5^lP)$.
Suppose choose $n = 1, m = 2, l = -1$. Consider

$$
\begin{aligned}
({}_5^1P \odot {}_5^2P) \odot {}_5^{-1}P &= \left({}_5^nP \cdot {}_5^mP - \frac{3mn}{4}(m-1)(n-1)\right) \odot {}_5^{-1}P = (1 \cdot 5 - 0) \odot {}_5^{-1}P \\
&= 5 \odot {}_5^{-1}P = {}_5^2P \odot {}_5^{-1}P = \left({}_5^nP \cdot {}_5^mP - \frac{3mn}{4}(m+1)(n-1)\right) = 5 \cdot 1 + 0 = 5.
\end{aligned}
$$

Again consider

$${}_5^1P \odot ({}_5^2P \odot {}_5^{-1}P) = {}_5^1P \odot {}_5^2P = 1 \cdot 5 + 0 = 5.$$

Hence $({}_5^nP \odot {}_5^mP) \odot {}_5^lP = {}_5^nP \odot ({}_5^mP \odot {}_5^lP)$. Also, the existence of a Multiplicative Identity element on a set of 5-gonal numbers is ${}_5^1P = 1 \in {}_5^nP$.
With ${}_5^nP \odot {}_5^1P = {}_5^1P \odot {}_5^nP = {}_5^nP$. Hence we conclude that $({}_5^nP, \odot)$ is a Monoid.
Since under this Multiplicative binary operation'$\odot$', Set of 5-gonal numbers ${}_5^nP$ satisfies the Closure Axiom, Associative Axiom, and Existence of Identity Axiom.
Property 1: Sum of first $n (n > 0)$ pentagonal numbers ${}_5^nP = n({}_3^nP)$.
The average value (Expected value) of 5-gonal numbers (for $n > 0$) is always a 3-gonal number.

6-GONAL NUMBERS (Hexagonal numbers)

Now I can go to introduce 6-gonal numbers by replacing $k = 6$ in Equation (1)

$$
{}_5^nP = \begin{cases} n(2n-1), & \text{if } n \geq 0 \\ n(2n+1), & \text{if } n < 0 \end{cases} = \{0, 1, 6, 15, 28, 45, 66, 91, 120, 153, 190, \ldots, \text{ etc.}\}
$$

Case 1: Now I can go to introduce additive binary operation on a set of 6-gonal numbers as follows:

$$ {}_{6}^{n}P \oplus {}_{6}^{n}P = \begin{cases} {}_{6}^{n}P + {}_{6}^{n}P + 4mn, & \text{if } m, n \text{ both are same sign} \\ {}_{5}^{n}P + {}_{5}^{m}P - 4mn, & \text{if } m, n \text{ are opposite sign} \end{cases}. $$

Case 1.1: if m, and n both have the same sign. Let $m \geq 0, n \geq 0$. Consider

$$ \begin{aligned} {}_{6}^{n}P + {}_{6}^{n}P + 4mn &= n(2n-1) + m(2m-1) + 4mn \\ &= (m+n)(2(m+n)-1) = {}_{6}^{n+m}P. \end{aligned} $$

Case 1.2: if m, and n both are the same sign. Let $m < 0, n < 0$. Consider

$$ \begin{aligned} {}_{6}^{n}P + {}_{6}^{n}P + 4mn &= n(2n+1) + m(2m+1) + 4mn \\ &= (m+n)(2(m+n)+1) = {}_{6}^{n+m}P. \end{aligned} $$

Case 1.3: if m, and n both are opposite signs. Let $m \geq 0, n < 0$. Consider

$$ \begin{aligned} {}_{6}^{n}P + {}_{6}^{n}P - 4mn &= n(2n-1) + m(2m+1) - 4mn \\ &= (m-n)(2(m-n)+1) = {}_{6}^{n+m}P. \end{aligned} $$

Hence, conclude that the above additive binary operation is well-defined on the set of ${}_{6}^{n}P$.

Case 2: Now we can introduce Multiplicative binary operation on a set of 6-gonal numbers as follows:

$$ {}_{6}^{n}P \odot {}_{6}^{n}P = \begin{cases} {}_{6}^{n}P \cdot {}_{6}^{n}P - 2mn(m-1)(n-1), & \text{if } m \geq 0, n \geq 0 \\ {}_{6}^{n}P \cdot {}_{6}^{n}P - 2mn(m-1)(n+1), & \text{if } m \geq 0, n < 0 \\ {}_{6}^{n}P \cdot {}_{6}^{n}P - 2mn(m+1)(n-1), & \text{if } m < 0, n \geq 0 \\ {}_{6}^{n}P \cdot {}_{6}^{n}P - 2mn(m+1)(n+1), & \text{if } m < 0, n < 0 \end{cases}. $$

Case 2.1: if $m \geq 0, n \geq 0$. Consider

$$ \begin{aligned} {}_{6}^{n}P \cdot {}_{6}^{n}P - 2mn(m-1)(n-1) &= (n(2n-1))(m(2m-1)) - 2mn(m-1)(n-1) \\ &= nm[(2n-1)(2m-1) - 2(m-1)(n-1)] \\ &= nm[4nm - 2n - 2m + 1 - 2mn + 2m + 2n - 2] \\ &= nm[(2nm-1)] = {}_{6}^{nm}P \end{aligned} $$

Case 2.2: if $m \geq 0, n < 0$. Consider

$$ \begin{aligned} {}_{6}^{n}P \cdot {}_{6}^{n}P - 2mn(m-1)(n+1) &= (n(2n+1))(m(2m-1)) - 2mn(m-1)(n+1) \\ &= nm[(2n+1)(2m-1) - 2(m-1)(n+1)] \\ &= nm[4nm - 2n + 2m - 1 - 2mn - 2m + 2n + 2] \\ &= nm[(2nm-1)] = {}_{6}^{nm}P \end{aligned} $$

Case 2.3: if $m \geq 0, n < 0$. Consider

$$ \begin{aligned} {}_{6}^{n}P \cdot {}_{6}^{n}P - 2mn(m-1)(n+1) &= (n(2n-1))(m(2m+1)) - 2mn(m-1)(n+1) \\ &= nm[(2n+1)(2m-1) - 2(m-1)(n+1)] \\ &= nm[4nm - 2n + 2m - 1 - 2mn - 2m + 2n + 2] \\ &= nm[(2nm-1)] = {}_{6}^{nm}P \end{aligned} $$

Case 2.4: if $m < 0, n < 0$. Consider

$$ \begin{aligned} {}_{6}^{n}P \cdot {}_{6}^{n}P - 2mn(m+1)(n+1) &= (n(2n+1))(m(2m+1)) - 2mn(m+1)(n+1) \\ &= nm[(2n+1)(2m+1) - 2(m+1)(n+1)] \\ &= nm[4nm + 2n + 2m + 1 - 2mn - 2m - 2n - 2] \\ &= nm[(2nm-1)] = {}_{6}^{nm}P. \end{aligned} $$

Hence, conclude that the above Multiplicative binary operation is well-defined on the set of ${}^{n}_{6}P$.
Case 3: Now I can introduce Algebraic Structure on the set of 6-gonal numbers.
From case 1 and case 2, Binary operations are well defined on a set of hexagonal numbers.
Hence Closure Axiom exists on ${}^{n}_{6}P$.
Also, concerning Additive Binary Operation $\oplus$, the Associative Axiom is well-defined

$$({}^{n}_{6}P \oplus {}^{n}_{6}P) \oplus {}^{l}_{6}P = {}^{n}_{6}P \oplus ({}^{n}_{6}P \oplus {}^{l}_{6}P)$$

Also, the existence of an Additive Identity element on a set of 6-gonal numbers is ${}^{0}_{6}P = 0 \in {}^{n}_{6}P$.
With ${}^{n}_{6}P \oplus {}^{0}_{6}P = {}^{0}_{6}P \oplus {}^{n}_{6}P = {}^{n}_{6}P$. Hence we conclude that $({}^{n}_{6}P, , \oplus)$ is a Monoid.
Since under this additive binary operation '$\oplus$', a set of 6-gonal numbers ${}^{n}_{6}P$ satisfies the Closure Axiom, Associative Axiom, and Existence of Identity Axiom.
Similarly, concerning Multiplicative Binary operation '$\odot$', the set of 6-gonal numbers ${}^{n}_{6}P$ satisfies the Associative Axiom, $({}^{n}_{6}P \odot {}^{n}_{6}P) \odot {}^{l}_{6}P = {}^{n}_{6}P \odot ({}^{n}_{6}P \odot {}^{l}_{6}P)$.
Also, the existence of a Multiplicative Identity element on a set of 6-gonal numbers is ${}^{1}_{6}P = 1 \in {}^{n}_{6}P$.
With ${}^{n}_{6}P \odot {}^{1}_{6}P = {}^{1}_{6}P \odot {}^{n}_{6}P = {}^{n}_{6}P$. Hence we conclude that $({}^{n}_{6}P, , \odot)$ is a Monoid.
Since under this Multiplicative binary operation'$\odot$', Set of 6-gonal numbers ${}^{n}_{6}P$ satisfies the Closure Axiom, Associative Axiom, and Existence of Identity Axiom.
Property 1: Sum of the first $n(n > 0)$ hexagonal numbers is $\dfrac{n(n+1)(4n-1)}{6}$.

7-GONAL NUMBERS (Heptagonal numbers)

Now I can go to introduce 7-gonal numbers, by replace $k = 7$ in Equation 1

$$ {}^{n}_{7}P = \begin{cases} \frac{n}{2}(5n-3), & \text{if } n \geq 0 \\ \frac{n}{2}(5n+3), & \text{if } n < 0 \end{cases} = \{0, 1, 7, 18, 34, 55, 81, 112, 148, 199, 235, \ldots, \text{ etc.}\}.$$

Case 1: Now I can introduce binary operation on a set of 7-goal numbers as follows:

$$ {}^{n}_{7}P \oplus {}^{m}_{7}P = \begin{cases} {}^{n}_{7}P + {}^{m}_{7}P + 5mn, & \text{if } m, n \text{ both are the same sign} \\ {}^{n}_{7}P + {}^{m}_{7}P - 5mn, & \text{if } m, n \text{ are an opposite sign} \end{cases}.$$

Case 1.1: if m, and n both are the same sign, let $m \geq 0, n \geq 0$. Consider

$$\begin{aligned} {}^{n}_{7}P + {}^{m}_{7}P + 5mn &= \frac{n}{2}(5n-3) + \frac{m}{2}(5m-3) + 5mn \\ &= \frac{m+n}{2}(5(m+n)-3) = {}^{n+m}_{7}P. \end{aligned}$$

Case 1.2: if m, and n both are the same sign, let $m < 0, n < 0$. Consider

$$\begin{aligned} {}^{n}_{7}P + {}^{m}_{7}P + 5mn &= \frac{n}{2}(5n+3) + \frac{m}{2}(5m+3) + 5mn \\ &= \frac{m+n}{2}(5(m+n)+3) = {}^{n+m}_{7}P. \end{aligned}$$

Case 1.3: if m and n are an opposite sign, let $m \geq 0, n < 0$. Consider

$$\begin{aligned} {}^{n}_{7}P + {}^{m}_{7}P - 5mn &= \frac{n}{2}(5n-3) + \frac{m}{2}(5m+3) - 5mn \\ &= \frac{m-n}{2}(5(m+n)-3) = {}^{m-n}_{7}P \end{aligned}$$

Case 1.4: Another additive binary operation on a set of 7-gonal numbers is as follows:

$$ {}^{n}_{7}P \oplus {}^{m}_{7}P = {}^{n}_{7}P + {}^{m}_{7}P - 5mn + 3n, \text{ if } m \geq 0, n \geq 0$$

Consider

$$\begin{aligned}
{}^{n}_{7}P + {}^{m}_{7}P - 5mn + 3n &= \frac{n}{2}(5n-3) + \frac{m}{2}(5m-3) - 5mn + 3n \\
&= \frac{5n^2 - 3n + 5m^2 - 3m - 10mn + 6n}{2} = \frac{5(m-n)^2 - 3(m-n)}{2} \\
&= \frac{m-n}{2}(5(m-n)-3) = {}^{m-n}_{7}P
\end{aligned}$$

Hence, conclude that the above additive binary operation is well-defined on the set of ${}^{n}_{7}P$.
Case 2: Now I can introduce another binary operation on a set of 7-gonal numbers as follows:

$$ {}^{n}_{7}P \odot {}^{m}_{7}P = \begin{cases} {}^{n}_{7}P \cdot {}^{m}_{7}P - \frac{15mn}{4}(m-1)(n-1), & \text{if } m \geq 0, n \geq 0 \\ {}^{n}_{7}P \cdot {}^{m}_{7}P - \frac{15mn}{4}(m-1)(n+1), & \text{if } m \geq 0, n < 0 \\ {}^{n}_{7}P \cdot {}^{m}_{7}P - \frac{15mn}{4}(m+1)(n-1), & \text{if } m < 0, n \geq 0 \\ {}^{n}_{7}P \cdot {}^{m}_{7}P - \frac{15mn}{4}(m+1)(n+1), & \text{if } m < 0, n < 0. \end{cases} $$

Case 2.1: if $m \geq 0, n \geq 0$. Consider

$$\begin{aligned}
&{}^{n}_{7}P \cdot {}^{m}_{7}P - \frac{15mn}{4}(m-1)(n-1) \\
&= \left(\frac{n}{2}(5n-3)\right)\left(\frac{m}{2}(5m-3)\right) - \frac{15mn}{4}(m-1)(n-1) \\
&= \frac{mn}{4}[(5n-3)(5m-3) - 15(m-1)(n-1)] \\
&= \left(\frac{mn}{4}(25mn - 15n - 15m + 9 - 15mn + 15m + 15n - 15)\right) \\
&= \frac{mn}{4}(10mn - 6) = \frac{mn}{2}(5mn-3) = {}^{mn}_{7}P
\end{aligned}$$

Case 2.2: if $m \geq 0, n < 0$. Consider

$$\begin{aligned}
&{}^{n}_{7}P \cdot {}^{m}_{7}P - \frac{15mn}{4}(m-1)(n+1) \\
&= \left(\frac{n}{2}(5n+3)\right)\left(\frac{m}{2}(5m-3)\right) - \frac{15mn}{4}(m-1)(n+1) \\
&= \frac{mn}{4}[(5n+3)(5m-3) - 15(m-1)(n+1)] \\
&= \left(\frac{mn}{4}(25mn - 15n + 15m - 9 - 15mn - 15m + 15n + 15)\right) \\
&= \frac{mn}{4}(10mn + 6) = \frac{mn}{2}(5mn+3) = {}^{mn}_{7}P
\end{aligned}$$

Case 2.3: if $m < 0, n \geq 0$. Consider

$$\begin{aligned}
&{}^{n}_{7}P \cdot {}^{m}_{7}P - \frac{15mn}{4}(m+1)(n-1) \\
&= \left(\frac{n}{2}(5n-3)\right)\left(\frac{m}{2}(5m+3)\right) - \frac{15mn}{4}(m+1)(n-1) \\
&= \frac{mn}{4}[(5n-3)(5m+3) - 15(m+1)(n-1)] \\
&= \left(\frac{mn}{4}(25mn + 15n - 15m - 9 - 15mn + 15m - 15n + 15)\right) \\
&= \frac{mn}{4}(10mn + 6) = \frac{mn}{2}(5mn+3) = {}^{mn}_{7}P
\end{aligned}$$

Case 2.4: if $m < 0, n < 0$. Consider

$$\begin{aligned} {}_7^nP \cdot {}_7^mP - \frac{15mn}{4}(m+1)(n+1) \\ &= \left(\frac{n}{2}(5n-3)\right)\left(\frac{m}{2}(5m-3)\right) - \frac{15mn}{4}(m+1)(n+1) \\ &= \frac{mn}{4}[(5n-3)(5m-3) - 15(m+1)(n+1)] \\ &= \left(\frac{mn}{4}(25mn - 15n - 15m + 9 - 15mn + 15m + 15n - 15)\right) \\ &= \frac{mn}{4}(10mn-6) = \frac{mn}{2}(5mn-3) = {}_7^{mn}P. \end{aligned}$$

Hence, conclude that the above Multiplicative binary operation is well-defined on the set of ${}_7^nP$.
Case 3: Now we can go to introduce Algebraic Structure on the set of 7-gonal numbers.
Hence, from the above cases, the Binary operations are well-defined on a set of 7-gonal numbers.
Hence Closure Axiom exists on ${}_7^nP$.
Also, concerning Additive Binary Operation $\oplus$, the Associative Axiom is well-defined

$$({}_7^nP \oplus {}_7^mP) \oplus {}_7^lP = {}_7^nP \oplus ({}_7^mP \oplus {}_7^lP)$$

Also, the existence of an Additive Identity element on a set of 7-gonal numbers is ${}_7^0P = 0 \in {}_7^nP$.
With ${}_7^nP \oplus {}_7^0P = {}_7^0P \oplus {}_7^nP = {}_7^nP$. Hence we conclude that $({}_7^nP, \oplus)$ is a Monoid.
Since under this additive binary operation'$\oplus$', a set of 7-gonal numbers ${}_7^nP$ satisfies the Closure Axiom, Associative Axiom, and Existence of Identity Axiom.
Similarly, concerning Multiplicative Binary operation '$\odot$', the set of 7-gonal numbers ${}_7^nP$ satisfies the Associative Axiom,

$$({}_7^nP \odot {}_7^mP) \odot {}_7^lP = {}_7^nP \odot ({}_7^mP \odot {}_7^lP).$$

Also, the existence of a Multiplicative Identity element on a set of 7-gonal numbers is ${}_7^1P = 1 \in {}_7^nP$.
With ${}_7^nP \odot {}_7^1P = {}_7^1P \odot {}_7^nP = {}_7^nP$. Hence we conclude that $({}_7^nP, \odot)$ is a Monoid.
Since under this Multiplicative binary operation'$\odot$', Set of 7-gonal numbers ${}_7^nP$ satisfies the Closure Axiom, Associative Axiom, and Existence of Identity Axiom.
Property 1: Relation between 7-gonal numbers and 4-gonal numbers.
For each 7-gonal number, there is a 4-gonal number in the form of $40({}_7^nP) + 9$.
Property 2: The sum of Reciprocals of positive 7-gonal numbers is 1.32.

Proof. Consider

$$\sum_{i=1}^{n} \frac{1}{{}_7^iP} = \sum_{i=1}^{n} \frac{2}{i(5i-3)} = 2\sum_{i=1}^{n} \frac{1}{i(5i-3)} = \frac{2}{3}\sum_{i=1}^{n}\left[\frac{5}{5n-3} - \frac{1}{n}\right] = 1.32$$

□

8-GONAL NUMBERS (Octagonal numbers)

Now we can introduce 8-gonal numbers, by replace $k = 8$ in Equation (1)

$${}_8^nP = \begin{cases} n(3n-2), & \text{if } n \geq 0 \\ n(3n+2), & \text{if } n < 0 \end{cases} = \{0, 1, 8, 21, 40, 65, 96, \ldots, \text{ etc.}\}.$$

Case 1: Now introduces additive binary operation on a set of 8-gonal numbers as follows;

$${}_8^nP \oplus {}_8^mP = \begin{cases} {}_8^nP + {}_8^mP + 6mn, & \text{if } m, n \text{ both are same sign} \\ {}_8^nP + {}_8^mP - 6mn & \text{if } m, n \text{ are opposite sign} \end{cases}.$$

Case 1.1: if m, and n both are the same sign, let $m \geq 0, n \geq 0$. Consider

$$\begin{aligned} {}_{8}^{n}P + {}_{8}^{m}P + 6mn &= n(3n-2) + m(3m-2) + 6mn \\ &= (m+n)(3(n+m)-2) = {}_{8}^{n+m}P. \end{aligned}$$

Case 1.2: if m, and n both are the same sign, let $m < 0, n < 0$. Consider

$$\begin{aligned} {}_{8}^{n}P + {}_{8}^{m}P + 6mn &= n(3n+2) + m(3m+2) + 6mn \\ &= (m+n)(3(n+m)+2) = {}_{8}^{n+m}P. \end{aligned}$$

Case 1.3: if m and n are opposite signs, let $m \geq 0, n < 0$. Consider

$$\begin{aligned} {}_{8}^{n}P + {}_{8}^{m}P - 6mn &= n(3n+2) + m(3m-2) - 6mn \\ &= (m-n)(3(m-n)-2) = {}_{8}^{n+m}P. \end{aligned}$$

Hence, conclude that the above additive binary operation is well-defined on the set of ${}_{8}^{n}P$.
Case 2: Now introduces multiplicative binary operation on a set of 8-gonal numbers as follows:

$${}_{8}^{n}P \odot {}_{8}^{m}P = \begin{cases} {}_{8}^{n}P \cdot {}_{8}^{m}P - 6mn(m-1)(n-1), & \text{if } m \geq 0, n \geq 0 \\ {}_{8}^{n}P \cdot {}_{8}^{m}P - 6mn(m-1)(n-1), & \text{if } m \geq 0, n < 0 \\ {}_{8}^{n}P \cdot {}_{8}^{m}P - 6mn(m-1)(n-1), & \text{if } m < 0, n \geq 0 \\ {}_{8}^{n}P \cdot {}_{8}^{m}P - 6mn(m-1)(n-1), & \text{if } m < 0, n < 0 \end{cases}.$$

Case 2.1: if $m \geq 0, n \geq 0$. Consider

$$\begin{aligned} {}_{8}^{n}P \cdot {}_{8}^{m}P - 6mn(m-1)(n-1) &= (n(3n-2))(m(3m-2)) - 6mn(m-1)(n-1) \\ &= nm((3n-2)(3m-2)) - 6(m-1)(n-1) \\ &= nm(9mn - 6n - 6m + 4 - 6mn + 6m + 6n - 6) \\ &= nm(3nm-2) = {}_{8}^{nm}P \end{aligned}$$

Case 2.2: if $m \geq 0, n < 0$. Consider

$$\begin{aligned} {}_{8}^{n}P \cdot {}_{8}^{m}P - 6mn(m-1)(n+1) &= (n(3n+2))(m(3m-2)) - 6mn(m-1)(n+1) \\ &= nm((3n+2)(3m-2)) - 6(m-1)(n+1) \\ &= nm(9mn - 6n + 6m - 4 - 6mn - 6m + 6n + 6) \\ &= nm(3nm+2) = {}_{8}^{nm}P \end{aligned}$$

Case 2.3: if $m < 0, n \geq 0$. Consider

$$\begin{aligned} {}_{8}^{n}P \cdot {}_{8}^{m}P - 6mn(m+1)(n-1) &= (n(3n-2))(m(3m+2)) - 6mn(m+1)(n-1) \\ &= nm((3n-2)(3m+2)) - 6(m+1)(n-1) \\ &= nm(9mn + 6n - 6m - 4 - 6mn + 6m - 6n + 6) \\ &= nm(3nm+2) = {}_{8}^{nm}P \end{aligned}$$

Case 2.4: if $m < 0, n < 0$. Consider

$$\begin{aligned} {}_{8}^{n}P \cdot {}_{8}^{m}P - 6mn(m+1)(n+1) &= (n(3n+2))(m(3m+2)) - 6mn(m+1)(n+1) \\ &= nm((3n+2)(3m+2)) - 6(m+1)(n+1) \\ &= nm(9mn + 6n + 6m + 4 - 6mn - 6m - 6n - 6) \\ &= nm(3nm-2) = {}_{8}^{nm}P. \end{aligned}$$

Hence, conclude that the above Multiplicative binary operation is well-defined on the set of ${}_{8}^{n}P$.
Case 3: Now I can introduce Algebraic Structure on the set of 8-gonal numbers.
From case 1 and case 2, the Binary operations are well defined on a set of 8-gonal numbers.

Hence Closure Axiom exists on ${}_8^nP$.
Also, concerning Additive Binary Operation $\oplus$, the Associative Axiom is well-defined

$$({}_8^nP \oplus {}_8^mP) \oplus {}_8^lP = {}_8^nP \oplus ({}_8^mP \oplus {}_8^lP)$$

Also, the existence of the Additive Identity element on a set of 8-gonal numbers is ${}_8^0P = 0 \in {}_8^nP$.
With ${}_8^nP \oplus {}_8^0P = {}_8^0P \oplus {}_8^nP = {}_8^nP$. Hence we conclude that $({}_8^nP, \oplus)$ is a Monoid.
Since under this additive binary operation '$\oplus$', a set of 8-gonal numbers ${}_8^nP$ satisfies the Closure Axiom, Associative Axiom, and Existence of Identity Axiom.
Similarly, concerning the Multiplicative Binary operation '$\odot$', a set of 8-gonal numbers ${}_8^nP$ satisfies Associative Axiom,

$$({}_8^nP \odot {}_8^mP) \odot {}_8^lP = {}_8^nP \odot ({}_8^mP \odot {}_8^lP).$$

Also, the existence of a Multiplicative Identity element on a set of 8-gonal numbers is ${}_8^1P = 1 \in {}_8^nP$.
With ${}_8^nP \odot {}_8^1P = {}_8^1P \odot {}_8^nP = {}_8^nP$. Hence we conclude that $({}_8^nP, \odot)$ is a Monoid.
Since under this Multiplicative binary operation '$\odot$', Set of 8-gonal numbers ${}_8^nP$ satisfies the Closure Axiom, Associative Axiom, and Existence of Identity Axiom.

3 Conclusion

This paper focused on studying the Algebraic structure of k-gonal numbers as a Monoid, concerning the following Binary operations. In particular, concerning Addition, I conclude that

$${}_k^nP \oplus_k^m P = \begin{cases} {}_k^nP +_k^m P + (k-2)mn, & \text{if } m, n \text{ both are the same sign} \\ {}_k^nP +_k^m P - (k-2)mn, & \text{if } m, n \text{ are an opposite sign} \end{cases}.$$

And concerning multiplication,

$${}_k^nP \odot_k^m P = \begin{cases} {}_k^nP \cdot_k^m P - \frac{(k-4)(k-2)mn}{4}(m-1)(n-1), & \text{if } m \geq 0, n \geq 0 \\ {}_k^nP \cdot_k^m P - \frac{(k-4)(k-2)mn}{4}(m-1)(n+1), & \text{if } m \geq 0, n < 0 \\ {}_k^nP \cdot_k^m P - \frac{(k-4)(k-2)mn}{4}(m+1)(n-1), & \text{if } m < 0, n \geq 0 \\ {}_k^nP \cdot_k^m P - \frac{(k-4)(k-2)mn}{4}(m+1)(n+1), & \text{if } m < 0, n < 0 \end{cases}.$$

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