

Soliton Solutions of Space-Time Fractional Coupled Korteweg-de-Vries (KdV) Equation using Residual Power Series Method

Baljinder Kour

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Abstract *This work examines the soliton solution for the space-time fractional coupled Korteweg-de-Vries equation using the residual approximation power series method, which is an extension of Taylor's series methodology. The structure of a rapidly converging series is used to study the solution of the governing equation. A few graphical representations that demonstrate the correctness and quick convergence of the answer are used to explain the method's consistency.*

1 Introduction

The prevalent progress in the fields of fractional differential equations and fractional calculus has been assumed in the recent past. Electromagnetic waves, ion-acoustic waves, bio-informatics, nanotechnology, viscoelasticity, chemical engineering, mechanical engineering, electrode-electrolyte polarisation, heat conduction, diffusion equations, and nearly every other area of science and technology are among the prominent examples of numerous projects that are shaped by fractional order differential equations [1, 2]. Because of its wide range of applications in many different fields, approximate and exact solutions of differential equations involving fractional order derivatives have received a lot of attention. The solutions and their dependability are more significant factors than the modelling aspects of these kinds of differential equations. Critical factors that generate emergent divergence, bifurcation, and convergence of the solution from that model are crucial. Many methods have been developed to find the solution of the differential equations with fractional order derivatives in order to achieve the goal of high accuracy and reliability of solutions. Numerous unconventional techniques, such as the Lie group analysis method [3, 4, 5], homotopy perturbation techniques (HPM) [6], Extended tanh-function method [7], Jacobi elliptic function method [8], Exp-function method [9], Sine-cosine method [11], Fractional sub-equation method [12], Residual power series method [13, 14, 15] and other iterative techniques, can be used to find analytical or numerical solutions of fractional differential equations. A technique for determining the approximate series solution of a particular physical event is the residual power series approach [13, 14, 15, 16, 17, 10]. In this paper, we study the mathematical model of waves on shallow water surfaces the space-time fractional coupled Korteweg-de Vries equation (KdV) [5].

The following is how the paper is organized: The Riemann-Liouville derivative and fractional order power series are the subjects of significant elementary results in section (2). Section (3) completes the Residual power series method, which is designed to solve the Space-time fractional coupled Korteweg-de-Vries (KdV) equation. The following section (4) includes a few graphical penalties that show how stable and effective the method and solution are. The closing observations are in the final section (5).

2 Preliminary

The fundamental notation and definition of fractional study are presented in this section that will be used for further study.

Definition 2.1. Fractional Power Series

For $0 \leq n - 1 < \alpha \leq n$, $m \geq m_0$, the series expansion is defined as

$$\sum_{k=0}^{\infty} b^n (m - m_0)^{n\alpha} = b_0 + b_1 (m - m_0)^\alpha + b_2 (m - m_0)^{2\alpha} + \dots, \quad (2.1)$$

is said to be fractional power series at $m = m_0$ including constant coefficients.

Definition 2.2. For $0 \leq n - 1 < \alpha \leq n$, $m \geq m_0$, the shape of power series defined as

$$\sum_{k=0}^{\infty} f_n(x) (m - m_0)^\alpha = f_0(x) + f_1(x) (m - m_0)^\alpha + f_2(x) (m - m_0)^{2\alpha} + \dots, \quad (2.2)$$

is known as fractional power series at $m = m_0$, where $f_n(x)$, $n = 0, 1, 2, \dots$ are function of x which is known to be coefficient of the series.

Remark 2.3. The power series at $z = z_0$

$$h(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^{m\alpha}, \quad 0 \leq n - 1 < \alpha \leq n, \quad z_0 \leq z < z_0 + R. \quad (2.3)$$

If $D^{m\alpha} h(z)$ is continuous in $(z_0, z_0 + R)$, for $m = 0, 1, 2, \dots$, then a_m 's can be calculated by

$$a_m = \frac{D^{m\alpha} f(z_0)}{\Gamma(m\alpha + 1)},$$

where $D^{m\alpha} = D^\alpha D^\alpha \dots D^\alpha$ (m - times).

3 Residual power series method for Space-time fractional coupled Korteweg-de-Vries (KdV) equation

Consider the system of space-time fractional coupled Korteweg-de Vries equation (KdV) [5]

$$\begin{aligned} \frac{\partial^\gamma u}{\partial t^\gamma} &= a_1 u \frac{\partial^\beta u}{\partial x^\beta} + a_2 v \frac{\partial^\beta v}{\partial x^\beta} + a_3 \frac{\partial^3 u}{\partial x^3}, \\ \frac{\partial^\gamma v}{\partial t^\gamma} &= b_1 u \frac{\partial^\beta v}{\partial x^\beta} + b_2 \frac{\partial^3 v}{\partial x^3}. \end{aligned} \quad (3.1)$$

is a mathematical model of waves on shallow water surfaces, where u, v are velocity and height respectively, and a_1, a_2, a_3, b_1, b_2 are constants.

In this section, we will construct the residual power series (RPS) solution of [12, 13]. Let us consider system (3.1) subject to the initial condition (at $t=0$)

$$\begin{aligned} u(x, 0) &= f(x), \\ v(x, 0) &= g(x). \end{aligned} \quad (3.2)$$

Let the solution of system (3.1) is written in the form of fractional power series expansion about initial point $t = 0$ as written below

$$\begin{aligned} u(x, t) &= \sum_{m=0}^{\infty} f_m(x) \frac{t^{m\rho}}{\Gamma(m\rho + 1)}, \\ v(x, t) &= \sum_{m=0}^{\infty} g_m(x) \frac{t^{m\rho}}{\Gamma(m\rho + 1)}, \quad x \in I, t > 0 \end{aligned} \quad (3.3)$$

With initial condition (3.2) of the analytic approximate solution for the system (3.1) is in the form of an infinite fractional power series given by RPSM.

In order to obtain the numerical values from above series, $u_n(x, t)$ and $v_n(x, t)$ denote the n^{th} truncated series of $u(x, t)$ and $v(x, t)$ respectively. That is,

$$\begin{aligned} u_n(x, t) &= \sum_{m=0}^n f_m(x) \frac{t^{m\rho}}{\Gamma(m\rho + 1)}, \\ v_n(x, t) &= \sum_{m=0}^n g_m(x) \frac{t^{m\rho}}{\Gamma(m\rho + 1)}, \quad x \in I, t > 0. \end{aligned} \tag{3.4}$$

For $m = 0$ on applying initial condition, the 0^{th} residual power series approximate solutions of $u(x, t)$ and $v(x, t)$ are written in the following form:

$$\begin{aligned} u_0(x, t) &= f_0(x) = u(x, 0) = f(x), \\ v_0(x, t) &= g_0(x) = v(x, 0) = g(x). \end{aligned} \tag{3.5}$$

Now (3.4) becomes

$$\begin{aligned} u_n(x, t) &= f(x) + \sum_{m=1}^n f_m(x) \frac{t^{m\rho}}{\Gamma(m\rho + 1)}, \\ v_n(x, t) &= g(x) + \sum_{m=1}^n g_m(x) \frac{t^{m\rho}}{\Gamma(m\rho + 1)}, \quad x \in I, t > 0. \end{aligned} \tag{3.6}$$

In this way, the m^{th} residual power series approximate solution $u_n(x, t)$ and $v_n(x, t)$ can be derived if $f_m(x)$ and $g_m(x)$ are known for $m = 1, 2, \dots, n$.

Now let us define the residual function for system (3.1) as written below:

$$\begin{aligned} Res_u(x, t) &= \partial_t^\gamma u - a_1 u \partial_x^\beta u - a_2 v \partial_x^\beta v - a_3 \partial_x^3 u, \\ Res_v(x, t) &= \partial_t^\gamma v - b_1 u \partial_x^\beta v - b_2 \partial_x^3 v. \end{aligned} \tag{3.7}$$

Also, the n^{th} residual function can be expressed as

$$\begin{aligned} Res_{u,n}(x, t) &= \partial_t^\gamma u_n - a_1 u_n \partial_x^\beta u_n - a_2 v_n \partial_x^\beta v_n - a_3 \partial_x^3 u_n, \\ Res_{v,n}(x, t) &= \partial_t^\gamma v_n - b_1 u_n \partial_x^\beta v_n - b_2 \partial_x^3 v_n. \quad n = 1, 2, 3, \dots \end{aligned} \tag{3.8}$$

Following are fundamental results for residual function

$$Res_u(x, t) = 0, \quad Res_v(x, t) = 0. \tag{3.9}$$

$$\begin{aligned} \lim_{m \rightarrow \infty} Res_{u,m}(x, t) &= Res_u(x, t), \\ \lim_{m \rightarrow \infty} Res_{v,m}(x, t) &= Res_v(x, t). \end{aligned} \tag{3.10}$$

$$\begin{aligned} D_t^{r\alpha} Res_u(x, 0) &= D_t^{r\alpha} Res_{u,m}(x, 0) = 0, \\ D_t^{r\alpha} Res_v(x, 0) &= D_t^{r\alpha} Res_{v,m}(x, 0) = 0, \quad r = 0, 1, 2, \dots, m. \end{aligned} \tag{3.11}$$

On Substituting the n^{th} truncated series (3.6) into (3.8) and by calculating the fractional derivative $D_t^{(n-1)\gamma}$ of $Res_{v,n}(x, t)$, for $n = 1, 2, 3, \dots$, at $t = 0$ together with (3.11), we obtain the following algebraic system

$$D_t^{(n-1)\gamma} Res_{v,n}(x, 0) = 0, \quad 0 < \gamma \leq 1, \quad n = 1, 2, 3, \dots \tag{3.12}$$

The values of $f_n(x)$ and $g_n(x)$ is obtain using (3.12). The n^{th} RPS approximate solution is derived in this way.

For $n=1$, the first RPS solution can be expressed in the following form:

$$\begin{aligned} u_1(x, t) &= f(x) + f_1(x) \frac{t^\rho}{\Gamma(\rho + 1)}, \\ v_1(x, t) &= g(x) + g_1(x) \frac{t^\rho}{\Gamma(\rho + 1)}. \end{aligned} \quad (3.13)$$

Following is the residual function for $n = 1$

$$\begin{aligned} Res_{u,1}(x, t) &= \partial_t^\rho u_1 - a_1 u_1 \partial_x^\beta u_1 - a_2 v_1 \partial^\beta v_1 - a_3 \partial_x^3 u_1, \\ Res_{v,1}(x, t) &= \partial_t^\rho v_1 - b_1 u_1 \partial_x^\beta v_1 - b_2 \partial_x^3 v_1. \end{aligned} \quad (3.14)$$

By putting (3.13) into (3.14) at $t = 0$, we obtain

$$\begin{aligned} Res_{u,1}(x, t) &= \partial_t^\rho \left(f(x) + f_1(x) \frac{t^\rho}{\Gamma(\rho+1)} \right) - a_1 \left(f(x) + f_1(x) \frac{t^\rho}{\Gamma(\rho+1)} \right) \\ &\quad \times \partial_x^\beta \left(f(x) + f_1(x) \frac{t^\rho}{\Gamma(\rho+1)} \right) - a_2 \left(g(x) + g_1(x) \frac{t^\rho}{\Gamma(\rho+1)} \right) \\ &\quad \times \partial_x^\beta \left(g(x) + g_1(x) \frac{t^\rho}{\Gamma(\rho+1)} \right) - a_3 \partial_x^3 \left(f(x) + f_1(x) \frac{t^\rho}{\Gamma(\rho+1)} \right) \end{aligned} \quad (3.15)$$

$$\begin{aligned} Res_{v,1}(x, t) &= \partial_t^\rho \left(g(x) + g_1(x) \frac{t^\rho}{\Gamma(\rho+1)} \right) - b_1 \left(f(x) + f_1(x) \frac{t^\rho}{\Gamma(\rho+1)} \right) \\ &\quad \times \partial_x^\beta \left(g(x) + g_1(x) \frac{t^\rho}{\Gamma(\rho+1)} \right) - b_2 \partial_x^3 \left(g(x) + g_1(x) \frac{t^\rho}{\Gamma(\rho+1)} \right) \end{aligned} \quad (3.16)$$

(3.15), (3.16) and by using result (3.12), we obtain

$$f_1(x) = a_1 f(x) \partial_x^\beta f(x) - a_2 g(x) \partial_x^\beta g(x) - a_3 \partial_x^3 f(x), \quad (3.17)$$

$$g_1(x) = b_1 f(x) \partial_x^\beta g(x) - b_2 \partial_x^3 g(x), \quad (3.18)$$

Therefore, the 1st approximate solution of system (3.1) can be written as

$$u_1(x, t) = f(x) + (a_1 f(x) \partial_x^\beta f(x) - a_2 g(x) \partial_x^\beta g(x) - a_3 \partial_x^3 f(x)) \frac{t^\rho}{\Gamma(\rho + 1)}, \quad (3.19)$$

$$v_1(x, t) = g(x) + (b_1 f(x) \partial_x^\beta g(x) - b_2 \partial_x^3 g(x)) \frac{t^\rho}{\Gamma(\rho + 1)} \quad (3.20)$$

The 2nd residual power series solution for $m = 2$ can be obtained as follows:

$$u_2(x, t) = f(x) + f_1(x) \frac{t^\rho}{\Gamma(\rho + 1)} + f_2(x) \frac{t^{2\rho}}{\Gamma(2\rho + 1)}, \quad (3.21)$$

$$v_2(x, t) = g(x) + g_1(x) \frac{t^\rho}{\Gamma(\rho + 1)} + g_2(x) \frac{t^{2\rho}}{\Gamma(2\rho + 1)}. \quad (3.22)$$

Following is the 2nd residual function which can be expressed as follows:

$$Res_{u,2}(x, t) = \partial_t^\rho u_2 - a_1 u_2 \partial_x^\beta u_2 - a_2 v_2 \partial_x^\beta v_2 - a_3 \partial_x^3 u_2, \quad (3.23)$$

$$Res_{v,2}(x, t) = \partial_t^\rho v_2 - b_1 u_2 \partial_x^\beta v_2 - b_2 \partial_x^3 v_2. \quad (3.24)$$

Inserting (3.21), (3.22) into (3.23), (3.24) using initial condition, we obtain

$$\begin{aligned}
 Res_{u,2}(x, t) &= \partial_t^\rho \left(f(x) + f_1(x) \frac{t^\rho}{\Gamma(\rho + 1)} + f_2(x) \frac{t^{2\rho}}{\Gamma(2\rho + 1)} \right) \\
 &\quad - a_1 \left(f(x) + f_1(x) \frac{t^\rho}{\Gamma(\rho + 1)} + f_2(x) \frac{t^{2\rho}}{\Gamma(2\rho + 1)} \right) \\
 &\quad \times \partial_x^\beta \left(f(x) + f_1(x) \frac{t^\rho}{\Gamma(\rho + 1)} + f_2(x) \frac{t^{2\rho}}{\Gamma(2\rho + 1)} \right) \\
 &\quad - a_2 \left(g(x) + g_1(x) \frac{t^\rho}{\Gamma(\rho + 1)} + g_2(x) \frac{t^{2\rho}}{\Gamma(2\rho + 1)} \right) \\
 &\quad \times \partial_x^\beta \left(g(x) + g_1(x) \frac{t^\rho}{\Gamma(\rho + 1)} + g_2(x) \frac{t^{2\rho}}{\Gamma(2\rho + 1)} \right) \\
 &\quad - a_3 \partial_x^3 \left(f(x) + f_1(x) \frac{t^\rho}{\Gamma(\rho + 1)} + f_2(x) \frac{t^{2\rho}}{\Gamma(2\rho + 1)} \right)
 \end{aligned} \tag{3.25}$$

$$\begin{aligned}
 Res_{v,2}(x, t) &= \partial_t^\rho \left(g(x) + g_1(x) \frac{t^\rho}{\Gamma(\rho + 1)} + g_2(x) \frac{t^{2\rho}}{\Gamma(2\rho + 1)} \right) \\
 &\quad - b_1 \left(f(x) + f_1(x) \frac{t^\rho}{\Gamma(\rho + 1)} + f_2(x) \frac{t^{2\rho}}{\Gamma(2\rho + 1)} \right) \\
 &\quad \times \partial_x^\beta \left(g(x) + g_1(x) \frac{t^\rho}{\Gamma(\rho + 1)} + g_2(x) \frac{t^{2\rho}}{\Gamma(2\rho + 1)} \right) \\
 &\quad - b_2 \partial_x^3 \left(g(x) + g_1(x) \frac{t^\rho}{\Gamma(\rho + 1)} + g_2(x) \frac{t^{2\rho}}{\Gamma(2\rho + 1)} \right)
 \end{aligned} \tag{3.26}$$

From (3.25), (3.26) and (3.12), we have

$$\begin{aligned}
 f_2(x) &= a_1 (f(x) \partial_x^\beta f_1(x) + f_1(x) \partial_x^\beta f(x)) + a_2 (g_1(x) \partial_x^\beta g(x) \\
 &\quad + g(x) \partial_x^\beta g_1(x)) + a_3 \partial_x^3 f_1(x),
 \end{aligned} \tag{3.27}$$

$$g_2(x) = b_1 (f(x) \partial_x^\beta g_1(x) + f_1(x) \partial_x^\beta g(x)) + b_2 g_1(x). \tag{3.28}$$

Hence, the 2nd approximate solution of system (3.1) is written below

$$\begin{aligned}
 u_2(x, t) &= f(x) + (a_1 f(x) \partial_x^\beta f(x) - a_2 g(x) \partial_x^\beta g(x) - a_3 \partial_x^3 f(x)) \frac{t^\rho}{\Gamma(\rho + 1)} \\
 &\quad + (a_1 (f(x) \partial_x^\beta f_1(x) + f_1(x) \partial_x^\beta f(x)) + a_2 (g_1(x) \partial_x^\beta g(x) \\
 &\quad + g(x) \partial_x^\beta g_1(x)) + a_3 \partial_x^3 f_1(x)) \frac{t^{2\rho}}{\Gamma(2\rho + 1)},
 \end{aligned} \tag{3.29}$$

$$\begin{aligned}
 v_2(x, t) &= g(x) + (b_1 f(x) \partial_x^\beta g(x) - b_2 \partial_x^3 g(x)) \frac{t^\rho}{\Gamma(\rho + 1)} + (b_1 (f(x) \partial_x^\beta g_1(x) \\
 &\quad + f_1(x) \partial_x^\beta g(x)) + b_2 g_1(x)) \frac{t^{2\rho}}{\Gamma(2\rho + 1)}.
 \end{aligned} \tag{3.30}$$

Proceeding like this, we can obtain the remaining approximate solutions of order 3,4 and the rest one of system (3.1). Therefore, we have

$$u(x, t) = \sum_{m=0}^n f_m(x) \frac{t^{m\rho}}{\Gamma(m\rho + 1)}, \tag{3.31}$$

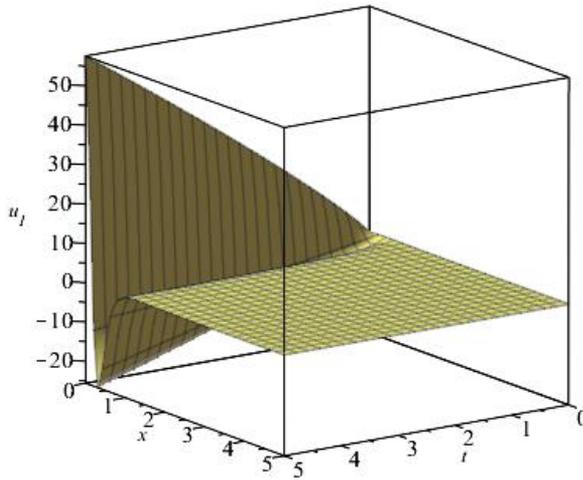
$$v(x, t) = \sum_{m=0}^n g_m(x) \frac{t^{m\rho}}{\Gamma(m\rho + 1)}, \quad x \in I, t > 0. \tag{3.32}$$

4 Numerical Discussion

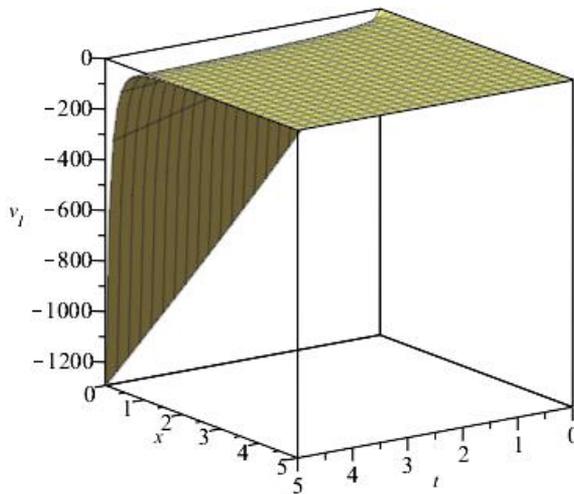
The efficacy and dependability of the proposed technique for fractional bi-Hamiltonian Boussinesq systems are evaluated graphically in the current section. Let us choose

$$\begin{aligned} f(x) &= \frac{\sin(x)}{5x^2 + 1}, \\ g(x) &= \frac{-\cos(x)}{2x^2 + 3x + 1}. \end{aligned} \quad (4.1)$$

The behavior of solution is discussed as:

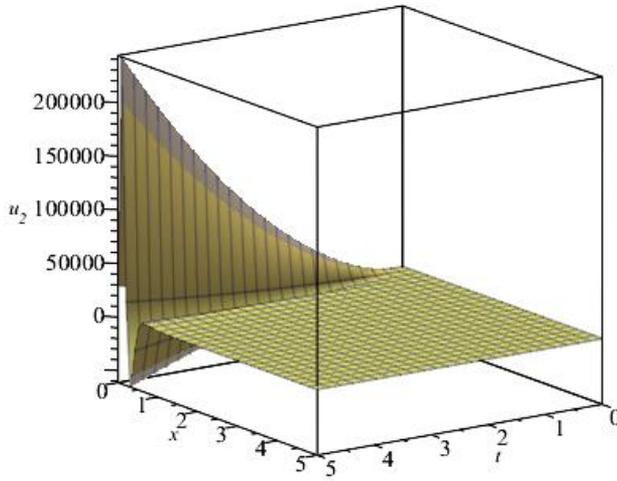


(a)

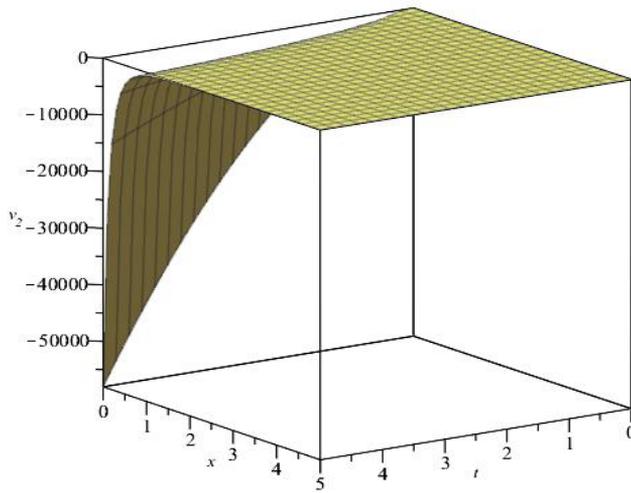


(b)

Figure 1: The modality of solution of (3.19) and (3.20) has been shown by (a) and (b), respectively, for $\alpha = 0.2$, $a_1 = \frac{8}{3}$, $a_2 = 1$, $a_3 = 0.5$, $b_1 = \frac{1}{3}$, $b_2 = 5$ and $f(x)$, $g(x)$ in (4.1).



(a)



(b)

Figure 2: The modality of solution of (3.29) and (3.30) has been shown by (a) and (b), respectively, for $\alpha = 0.2$, $a_1 = \frac{8}{3}$, $a_2 = 1$, $a_3 = 0.5$, $b_1 = \frac{1}{3}$, $b_2 = 5$ and $f(x)$, $g(x)$ in (4.1).

Therefore, As a result, it is clear from the graphs that the obtained solution converges to an analytical solution by more the terms of the approximate residual solution.

5 Conclusion

The approximate residual power series solution of the Space-time fractional coupled Korteweg-de-Vries (KdV) equation was successfully obtained in this study using RPSM. Based on the results, we may conclude that RPSM is a very effective and realistic method for studying fractional equations. As a result, we can conclude that the RPSM is a very useful and innovative approach for examining the precise and approximate solutions of physical phenomena that are appearing in many scientific fields. The behavior of approximate residual solutions is seen in the charts.

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Author information

Baljinder Kour, Department of Mathematics, Akal University, Talwandi Sabo, India.
E-mail: baljinder@virat172@gmail.com