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Dynamic Stability and Growth Analysis in Plants: A DDE Model

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Abstract In this paper, a mathematical model is proposed for plant growth dynamics consisting of root compartment and shoot compartment. The associated variables are nutrients concentration and structural dry weight of root and shoot compartment respectively. Under the premise of logistic growth, root compartment supply nutrients to the shoot compartment which is an essential element for the plant growth. There are some external and internal factors which obstruct the supply of nutrients from root to shoot compartment. These factors bring delay in nutrients consumption which causes the damage to structural dry weight. This effect is studied after adding the delay parameter in nutrients consumption term. The stability of the system is studied about the interior equilibrium. Beyond the critical value of the delay parameter, Hopf-bifurcation is seen. The sensitivity of solution of the model is examined by changing the value of parameters. MATLAB simulation is used to support analytical findings.

1 Introduction

In plant ecology, nutrients availability and solubility play significant role in growing plants. There are some other essential elements which help in the plant growth: oxygen, carbon and hydrogen comes from air and also water and nutrients taken up from soil. The interaction between plant-soil describes the process in which roots help to transport the nutrients from soil which give rise to the growth of plant. Hiltner was the first who to initiate the modelling between plant-soil interaction [1]. In some situations, due to lack of mobility in nutrients plants do not get sufficient amount of nutrients for their normal growth which cause decreases in plant productivity. Hence the soil properties and availability of water are the main factors responsible for plant growth [2]. Thornley was the first to propose mathematical models including these factors individually and in combination to the several issues in plant physiology to estimate the outcome of variables: humidity, temperature, transportation, respiration, guard cells, rate of photosynthesis etc. [3-4]. The models have been categorized and explained the growth of specific plants in crop [5]. For modeling purposes, the plant is split up into root compartment and shoot compartments in which, concentration of nutrients and dry weight are considered as the conditional variables [6-7]. It's assumed that due to the existence of toxic metal in root compartment, resistance increases in transport nutrients from root to shoot compartment [8-9]. Copper and zinc are the important macronutrients which are necessitated for the normal growth of the plants. Normally, these are found in very low concentration in soil. Whereas, metals like chromium, cadmium, lead, mercury, nickel etc. are toxic to plants [10-11]. Such heavy metals get in the way to the up taking and absorbing of vital nutrients minerals from the soil and results unbalancing nutrients levels in plant [12]. Nutrients influence stressed on discrete plant growth, that ensuing effect on non-linear population growth dynamics which arise the impact on production of standing crop yield. The theoretical model discussed in which the assumptions were that there was a dynamic growth process whose rate was dependent on the size of the plant, reduced the growth rates, possibly resulting in mortality [13]. There are 30 percent difference shown between the uptaking of nutrients theoretically and computed uptakes [14]. Models of biological phenomena whose dynamics is explained better by delay differential equation and numerical approaches are the tools considered for their solutions [15]. Rouches theorem plays an essential role during the analysis of exponential polynomials and their roots distribution [16-17]. The stability and oscillations studied in the set of non-linear differential equations with delay [18]. Also, the Boundedness and Persistence is calculated of mixed nonlinearity delay differential equations [19]. Under certain necessary and sufficient criteria, the system of delay differential equation models over its positive equilibrium point, has been asymptotically stable for all positive solutions [20]. The phenomenon of Hopf-bifurcation shown up at the equilibrium point when the delay increased from critical point [21]. The sensitivity analysis for nonlinear differential equation systems with time delays utilizing direct when the constituting model parameters were altered with respect to time, not remain constant only [22]. Theoretical findings for sensitivity are presented with relation to the delays. The study of the parametric sensitivity is used to examine the periodic responses to delay differential equations [23]. Dipesh and Kumar examined the allelopathic effect using the delay differential equations [24-26].

2 Mathematical Model

For the analysis of the plant growth, the plant has been divided into the shoot compartment and root compartment. Structural dry weight and nutrients concentration are corresponding state variables. Let the nutrients concentration in shoot and root as N_2 and N_1, respectively. Let structural dry weights in root and shoot as W_1 and W_2, respectively. It is supposed that the structural dry weight is damaged due to hinderance in up taking of nutrients caused by some exogenic activities. This hypothesis is in corporated in the model by delay parameter in nutrients utilization. The above notations generate following mathematical:

$$\frac{dN_1}{dt} = rN_1\left(1 - \frac{N_1}{K}\right) - \mu W_1 N_1(t-\tau) - d_1 N_1 \tag{1}$$

$$\frac{dN_2}{dt} = rN_2\left(1 - \frac{N_2}{K}\right) - \mu W_2 N_2 - d_2 N_2 \tag{2}$$

$$\frac{dW_2}{dt} = (\alpha N_2 - Y_2)W_2 - \Delta_2 W_2^2 \tag{3}$$

$$\frac{dW_1}{dt} = (\alpha N_1 - Y_1)W_1 - \Delta_1 W_1^2 \tag{4}$$

The definition of system parameters are as follows:
$r\left(1 - \frac{N_1}{K}\right), r\left(1 - \frac{N_2}{K}\right)$, are considered as logistic growth where K be the carrying capacity. r is natural rate of growth of nutrients, μ is utilization coefficient or consumption coefficient, α is considered as nutrient-use efficiency. Natural decay of W_1 and W_2 are represented as γ_1 and γ_2, respectively. Natural decay of N_1 and N_2 are represented as d_1 and d_2, respectively. Δ_2 is rate of self-limiting growth of W_2 and Δ_1 is rate of self-limiting growth of W_1.
With initial conditions:
$N_1(0) > 0, N_2(0) > 0, W_1(0) > 0$ and $W_2(0) > 0$, for all $t > 0$ and $N_1(t-\tau) =$ Constant, for all $t \in [0, \tau]$.

3 Boundedness

The lemma provides the boundedness of solutions of the model as (1)-(4):
Lemma 1. All solution of the model lying between the region

$$D_1 = \left[(N_1, N_2, W_1, W_2) \in Q_+^4 : 0 \leq N_1 + N_2 + \frac{\mu}{\alpha}W_1 + \frac{\mu}{\alpha}W_2 \leq \frac{r}{\varphi}\right], \text{ as } t \to \infty,$$

for all positively initial values $\{N_1(0), N_2(0), W_1(0), W_2(0), N_1(t-\tau) =$ Constant, for all $t \in [0, \tau]\} \in D_1 \subset Q_+^4$ where $\varphi = (d_1, d_2, \gamma_1, \gamma_2)$.

Proof. Assume the function $F(t)$, such that:

$$F(t) = N_1(t) + N_2(t) + \frac{\mu}{\alpha}W_1(t)$$
$$\frac{dF(t)}{dt} = \frac{d}{dt}\left[N_1(t) + N_2(t) + \frac{\mu}{\alpha}W_1(t) + \frac{\mu}{\alpha}W_2(t)\right]$$

Using Equations (1)-(4) and $\varphi = (d_1, d_2, \gamma_1, \gamma_2)$ and assuming that $N_1(t) \approx N_1(t-\tau)$ as $t \to \infty$, we get

$$\frac{dF(t)}{dt} \leq r_n - \varphi F(t).$$

By using comparison theorem, we get as $t \to \infty$: we get

$$F(t) \leq \frac{r_n}{\varphi}$$
$$N_1(t) + N_2(t) + \frac{\mu}{\alpha}W_1(t) + \frac{\mu}{\alpha}W_2(t) \leq \frac{r_n}{\varphi}.$$
$$\text{So, } 0 \leq N_1(t) + N_2(t) + \frac{\mu}{\alpha}W_1(t) + \frac{\mu}{\alpha}W_2(t) \leq \frac{r_n}{\varphi}.$$

□

4 Positivity of the model

Since the model defines the dynamics of plant growth, it is important to show that all variables are positive for all time t. Positivity defines that the dynamic system of plant growth is sustain. From equation (1)-(4), we prove that all represented variables shows positive solutions, such that $N_1(0) > 0, N_2(0) > 0, W_1(0) > 0, W_2(0) > 0$ for all $t > 0$ and $N_1(t) - \tau =$ Constant, $\forall t \in [0, \tau]$, then the solution $N_1(t), N_2(t), W_1(t), W_2(t)$ of the model remains positive $\forall$ time $t > 0$.

$$\text{From (2), } \frac{dN_2}{dt} = r\left(1 - \frac{N_2}{k}\right)N_2 - \mu W_2 N_2 - d_2 N_2$$
$$\frac{dN_2}{dt} \leq -\left(\mu W_2 + d_2\right)N_2$$
$$N_2 \geq c^{e^{-(\mu W_2 + d_2)t}}$$

Hence, $N_2 \geq 0$ as $t \to \infty$.
From (3), we get,

$$\frac{dW_2}{dt} = \left(\alpha N_2 - \gamma_2\right)W_2 - \Delta_2 W_2^2$$
$$\frac{dW_2}{dt} \geq -\left(\gamma_2 - \Delta_2 W_2\right)W_2$$
$$W_2 \geq e^{-(\gamma_2 + \Delta_2 W_2)t}$$

Hence, $W_2 \geq 0$ as $t \to \infty$.
From (4), we get,

$$\frac{dW_1}{dt} = \left(\alpha N_1 - \gamma_1\right)W_1 - \Delta_2 W_1^2$$
$$\frac{dW_1}{dt} \geq -\left(\gamma_1 - \Delta_2 W_1\right)W_1$$
$$W_1 \geq e^{-(\gamma_1 + \Delta_2 W_1)t}$$

Hence, $W_1 \geq 0$ as $t \to \infty$.
From (1), we get,

$$\frac{dN_1}{dt} = rN_1\left(1 - \frac{N_1}{k}\right) - \mu W_1 N_1(t-\tau) - d_1 N_1$$

$$\frac{dN_1}{dt} \geq -\left(\frac{r}{k} + d_1\right) N_1 - \mu W_1 N_1(t-\tau)$$

$$\frac{dN_1}{dt} + \left(\frac{r}{k} + d_1\right) N_1 \geq -\mu W_1 N_1(t-\tau)$$

$$\frac{dN_1}{dt} + \delta N_1 \geq -\mu W_1 N_1(t-\tau)$$

$$\frac{d\left(e^{\delta t} N_1\right)}{dt} \geq -\mu W_1 N_1(t-\tau) e^{\delta t}$$

$$N_1 \geq -\mu \int N_1(t-\tau) W_1 e^{-\delta(t-\tau)} dx$$

The solution of the above inequality N_1 will converge to 0 for $t \to \infty$ iff $N_1(t-\tau) < \left(\frac{r}{k} + d_1\right)(t-\tau)$ for all $t > \tau$.

5 Interior Equilibrium of the model

Here, we determine an interior equilibrium of the model, S_1. There is one feasible solution to the system of equations, $S_1(N_1^*, N_2^*, W_1^*, W_2^*)$, where

$$\frac{1}{\Delta_1}\left(\alpha N_1^* - \gamma_1\right) = W_1^* > 0, \text{ provided } \alpha N_1^* > \gamma_1$$

$$\frac{1}{\Delta_2}\left(\alpha N_2^* - \gamma_2\right) = W_2^* > 0, \text{ provided } \alpha N_2^* > \gamma_2$$

$$N_1^* = \frac{\frac{\mu\gamma_1}{\Delta_1}\epsilon}{\frac{r}{k} + \frac{\mu\alpha}{\Delta_1}\epsilon + d_1}$$

$$N_2^* = \frac{-x_2 \pm \sqrt{x_2^2 - 4x_{1_*} x_3}}{2x_1}$$

$$\text{where } x_1 = \frac{\mu\alpha}{\Delta_2}, x_2 = d - \frac{\mu\gamma_2}{\Delta_2}, x_3 = -r\left(1 - \frac{N_1^*}{k}\right)$$

6 Study of Stability Analysis of Interior Equilibrium and Hopf - Bifurcation

In this, we examine dynamic behaviour of the interior equilibrium point S_1 of the model. The following is the exponential characteristic equation for equilibrium S_1:

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda^1 + A_4 + de^{-\lambda\tau} = 0 \tag{5}$$

Here,

$A_1 = R_1 + R_5 + R_6 + R_9$,

$A_2 = \left(R_6R_9 + R_1R_6 + R_1R_9 + R_5R_6 + R_5R_9 + R_1R_5 + R_8R_7 + R_2R_4\right)$,

$A_3 = \left(R_6R_9R_1 + R_6R_9R_5 + R_8R_7R_1 + R_8R_7R_5 + R_6R_1R_5 + R_6R_2R_4 + R_9R_1R_5 + R_9R_2R_4\right)$,

$A_4 = \left(R_1R_5R_6R_9 + R_2R_4R_6R_9 + R_2R_4R_8R_7 + R_1R_5R_8R_7\right)$

where $R_1 = \mu W_2 + d_2$, $R_2 = \mu N_2$, $R_3 = \frac{r}{k}$,

$$R_4 = -aW_2, R_5 = 2\Delta_2 W_2 - \left(\alpha N_2 - \gamma_2\right),$$

$$R_6 = 2\Delta_1 W_1 - \left(\alpha N_1 - \gamma_1\right), R_7 = -\alpha W_1,$$

$$R_8 = \mu N_1 e^{-\lambda\tau}, R_9 = \left(\frac{r}{k} + d_1\right)$$

Clearly $\lambda = (i\omega)$ is a root of Equation

$$(i\omega)^4 + A_1(i\omega)^3 + A_2(i\omega)^2 + A_3\omega + A_4 + de^{-(i\omega)\tau} = 0$$
$$\omega^4 - iA_1\omega^3 - A_2\omega^2 + iA_3\omega + A_4 + d(\cos\omega\tau - i\sin\omega\tau) = 0$$

Separating real and imaginary

$$\omega^4 - A_2\omega^2 + A_4 = -d\cos\omega\tau \tag{6}$$
$$A_1\omega^3 - A_3\omega = -d\sin\omega\tau \tag{7}$$

Squaring and Adding

$$\omega^8 + (A_1^2 - 2A_2)\omega^6 + (A_2^2 + 2A_1 - 2A_1A_3)\omega^4 + (A_3^2 - 2A_2A_1)\omega^2 + (A_4^2 - d_2) = 0 \tag{8}$$

Let $\omega^2 = x$ and $A_1^2 - 2A_2 = a, (A_2^2 + 2A_4 - 2A_1A_3) = b, (A_3^2 - 2A_2A_4) = c, (A_4^2 - d_2) = r$,
Equations become:

$$x^4 + ax^3 + bx^2 + cx + r = 0 \tag{9}$$

Lemma 2. If $r < 0$, there is at least one real root is positive in equation (9).

Proof. Let $px = x^4 + ax^3 + bx^2 + cx + r$
Here $p(0) = r < 0, \lim_{y\to\infty} p(x) = \infty$.
So, $\exists x_0 \in (0,\infty)$ such that $p(x_0) = 0$ □

Also $p'(x) = 4x^3 + 3ax^2 + 2bx + c$ Let $p'(x) = 0$

$$4x^3 + 3ax^2 + 2bx = 0 \tag{10}$$
$$y^3 + qy + s = 0, \tag{11}$$

where $y = x + \frac{3a}{4}, q = \frac{b}{2} - \frac{3a^2}{16}, s = \frac{c}{4} - \frac{ab}{8} + \frac{a^3}{32}$
Assuming $y = (m+n)$ be the solution of equation (11), we get:

$$m^3 - \frac{p^3}{27m^3} + s = 0.$$

Assuming $m^3 = z$, we get $z^2 + sz - \frac{p^3}{27} = 0$.
Three roots are comes out from equation (11):

$$y_1 = \left(-\frac{s}{2} + \sqrt{D}\right)^{1/3} + \left(-\frac{s}{2} - \sqrt{D}\right)^{1/3}$$
$$y_2 = \left(-\frac{s}{2} + \sqrt{D}\sigma\right)^{1/3} + \left(-\frac{s}{2} + \sqrt{D}\sigma^2\right)^{1/3}$$
$$y_3 = \left(-\frac{s}{2} + \sqrt{D}\sigma^2\right)^{1/3} + \left(-\frac{s}{2} + \sqrt{D}\sigma\right)^{1/3}$$
$$x_i = y_i - \frac{3a}{4}, \qquad i = 1,2,3.$$

Where $D = \left(\frac{s}{2}\right)^2 + \left(\frac{q}{3}\right)^3$ and $\sigma = \frac{-1+\sqrt{3}i}{2}$
Lemma 3. Let $r \geq 0$

I. If $D \geq 0$, then positive roots are given by equation (9) iff $x_1 > 0, p(x_1) < 0$.

II. If $D < 0$, then positive roots are given by equation (9) iff there exists at least one $x^* \in (x_1, x_2, x_3)$ such that $x^* > 0$ and $p(x^*) \leq 0$.

III. If $D < 0$, then equation (11) has, three zeros only i.e. y_1, y_2, y_3 accordingly equation (10) has also gives three roots x_1, x_2, x_3 such that at least one real root exists between them.

Proof. (I) If $D \geq 0$, then equation (11) has a distinct real root y_1, which implies that equation (10) has a unique distinct real root x_1.
As consider $p(x)$ is a differentiable function and $\lim_{x\to\infty} p(x) = \infty$, results x_1 be the unique critical point of $p(x)$, which comes out as the minimum point of $p(x)$.
Assumed equation (9) have positive roots. Generally, we represent that the 4 positive roots are denoted by $x_i^*, i = 1, 2, 3, 4$. Therefore equation (8) also gives 4 positive roots that is $\omega_j = \sqrt{x_i^*}, i = 1, 2, 3, 4$.
From (7) $\sin \omega\tau = \dfrac{A_3\omega - A_1\omega^3}{d}$.
That gives $\tau = \frac{1}{\omega}\left[\sin^{-1}\left(\frac{A_3\omega - A_1\omega^3}{d}\right) + 2(j-1)\pi\right]; j = 1, 2, 3, 4, \ldots$
Let $\tau_k^{(j)} = \frac{1}{\omega_k}\left[\sin^{-1}\left(\frac{A_3\omega - A_1\omega^3}{d}\right) + 2(j-1)\pi\right]; k = 1, 2, 3, 4; j = 1, 2, 3, 4, \ldots$
Then $\mp i\omega_k$ is a pair of complex roots, i.e., purely imaginary of equation (5)
Where $\tau = \tau_k^{(j)}, k = 1, 2, 3, 4; j = 1, 2, 3, 4, \ldots$, we have $\lim_{j\to\infty} \tau_k^{(j)} = \infty, k = 1, 2, 3, 4$.
Therefore, we can state $\tau_0 = \tau_{k_0}^{(j_0)} = \min 1 \leq k \leq 4, j \geq 1 \left[\tau_k^{(j)}\right], \omega_0 = \omega_{k_0}, y_0 = y_{k_0}^*$ □

Lemma 4. Suppose that $A_1 > 0, A_2 > 0, A_3 > 0, (A_4 + d) > 0, A_1A_2 - A_3 > 0, A_1A_2A_3 - A_3^2 - A_1^2A_4 + d > 0$.

I. If any of the following condition satisfy: (i) $r < 0$ (ii) $r \geq 0, D \geq 0, x_1 > 0, p(x_1) \leq 0$ (iii) $r \geq 0, D < 0$ and $\exists a x^* \in (x_1, x_2, x_3)$ such that $x^* > 0$ and $p(x^*) \leq 0$, there will be negative real part in all roots of equation (5), when $\tau \in [0, \tau_0)$.

II. From (I), if any condition (i) - (iii) are not concluded, then there will be roots with negative real parts of equation (5) $\forall, \tau \geq 0$.

Proof. Equation (5) defines with $\tau = 0$;

$$\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 + d = 0 \tag{12}$$

All roots have negative real parts of equation (12) iff $A_1 > 0, A_2 > 0, A_3 > 0, A_4 + d > 0, A_1A_2 - A_3 > 0, A_1A_2A_3 - A_3^2 - A_1^2A_4 + d > 0$. (using Routh-Hurwitz's criteria)
We discuss that if any conditions from (i)-(iii) are not satisfy, then none of the roots of equation (5) having zero as real part $\forall, \tau \geq 0$, from lemma 1 & 2.

If any one of the states from lemma 3, (i), (ii), (iii) holds, with $\tau \neq \tau_k^{(j)}, k = 1, 2, 3, 4; j \geq 1$, then not any roots of the equation (5) having zero as real part. The least value of τ is τ_0 by which equation (3.1) has purely complex roots.

Assume equation (5) has roots, $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ satisfying:

$$\alpha(\tau_0) = 0, \omega(\tau_0) = \omega_0.$$

□

Lemma 5. Suppose $p'(x_0) \neq 0$. Then equation (5) has $\mp i\omega_0$ as set of simple and purely complex roots, if $\tau = \tau_0$. Also, if the lemma 3, conditions are satisfied, then $\dfrac{d}{d\tau}(Re\lambda(\tau_0)) > 0$.

Proof. The $i\omega_0$ should satisfy, whenever $i\omega_0$ is not a simple root:

$$\frac{d}{d\lambda}\left[\lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 + de^{-\lambda\tau}\right]_{\lambda = i\omega_0} = 0$$
$$-4i\omega_0^3 - 3A_1\omega_0^2 + 2iA_2\omega_0 + A_3 - \tau d(\cos\omega_0\tau - i\sin\omega_0\tau) = 0$$

By separating real and imaginary parts of the above equation:

$$A_3 - 3A_1\omega_0^2 = \tau d\cos\omega_0\tau$$
$$4\omega_0^3 - 2A_2\omega_0 = \tau d\sin\omega_0\tau$$

By dividing we obtain

$$\tan\omega_0\tau = \frac{4\omega_0^3 - 2A_2\omega_0}{A_3 - 3A_1\omega_0^2} \tag{13}$$

As also, ω_0 must satisfy equation (3.2) and (3.3), from where we get:

$$\tan\omega_0\tau = \frac{A_1\omega_0^3 - A_3\omega_0}{\omega_0^4 - A_2\omega_0^2 + A_4} \tag{14}$$

Comparing (13) and (14) we get

$$4\omega_0^6 + 3(A_1^2 - 2A_2)\omega_0^4 + 2(2A_4 + A_2^2 - 2A_1A_3)\omega_0^2 + (A_3^2 - 2A_2A_4) = 0$$

As we know $\omega_0^2 = x_0$,

$$4x_0^3 + 3(A_1^2 - 2A_2)x_0^2 + 2(2A_4 + A_2^2 - 2A_1A_3)x_0 + (A_3^2 - 2A_2A_4) = 0$$
$$4x_0^3 + 3lx_0^2 + 2mx_0 + n = 0$$

Where $l = (A_1^2 - 2A_2), m = (2A_4 + A_2^2 - 2A_1A_3), n = (A_3^2 - 2A_2A_4)$
Which gives $p'(x) = 4x^3 + 3lx^2 + 2mx + n$
Which is a contradiction as $p'(x_0) \neq 0$.
First part of the result is proved.
Differentiating equation (5) with respect to τ, and obtain

$$\frac{d\lambda(\tau)}{d\tau} = \frac{d\lambda e^{-\lambda\tau}}{4\lambda^3 + 3A_1\lambda^2 + 2A_2\lambda + A_3 - d\tau e^{-\lambda\tau}}$$

Putting $\lambda = i\omega$,

$$\frac{d\lambda(\tau)}{d\tau} = \frac{di\omega(\cos\omega - i\sin\omega\tau)}{(A_3 - 3A_1\omega^2 - d\tau\cos\omega\tau) + i(2A_2 - 4\omega^3 + d\tau\sin\omega\tau)}.$$

When $\tau = \tau_0, \omega = \omega_0, x = x_0$, we get

$$\frac{dRe\lambda(\tau_0)}{d\tau} = \frac{\omega_0^2}{\gamma}p'(x_0) \neq 0,$$

where $\gamma = (A_3 - 3A_1\omega^2 - d\tau\cos\omega\tau)^2 + (2A_2 - 4\omega^3 + d\tau\sin\omega\tau)^2$.
If $\frac{dRe\lambda(\tau_0)}{d\tau} < 0$, equation (5) gives positive real part root for $\tau < \tau_0$ with the closeness of τ_0 which contradicts, lemma 3. The proof is completed. □

7 Numerical simulation

In this model MATLAB simulation is used to numerically consolidate the analytical findings. The system behaves as follows.
The values of these parameters are assumed after going through previous literature.

$$r = 3.71, k = 3.02, \mu = 4.01, d_1 = 0.00009, d_2 = 0.0009,$$
$$\alpha = 1.5, \gamma_1 = 1.3, \gamma_2 = 1.3, \Delta_1 = 1.21, \Delta_2 = 1.21.$$

8 Sensitivity Analysis

In this study, the model has fixed parameters. The "Direct Method" is used to figure out the global sensitivity coefficient. All of the parameters $(r, \alpha, \mu, \Delta_1, \Delta_2, \gamma_1, \gamma_2, d_1, d_2)$ contained in the system (1)-(4) are taken to be constants, then finding the partial derivatives of the solution for each

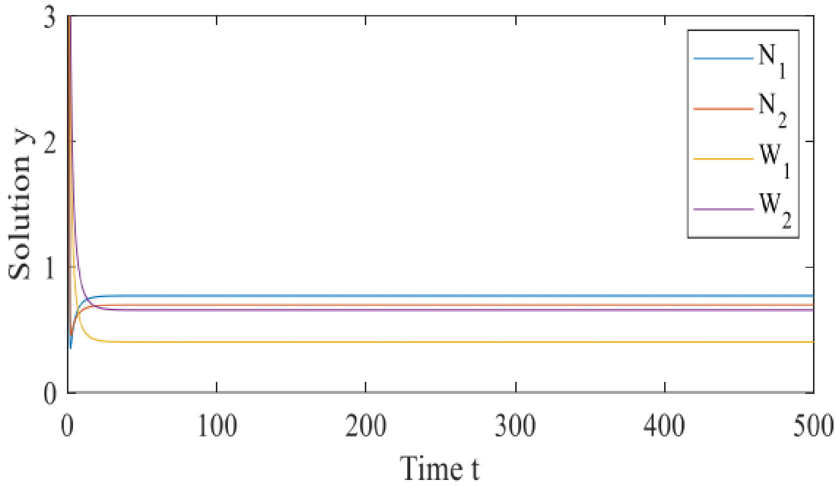

Figure 1. When there is absence of delay, $\tau = 0$, the system interior equilibrium point S_1 is stable.

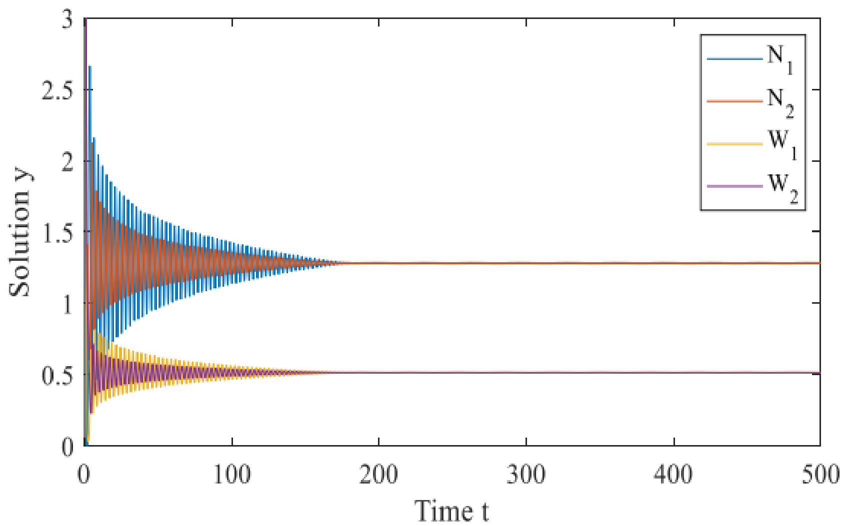

Figure 2. When there is delay that is $\tau < 2.15$, the system interior equilibrium point S_1 is asymptotically stable.

parameter may be sufficient for the sensitivity analysis in this case. The following set of sensitivity equations can be created by taking the partially derivative of the solution (N_1, N_2, W_2, W_1) in respect to the variables:

$$\frac{dS_1}{dt} = rS_1 - \frac{2rN_1S_1}{k} - \mu S_4 N_1(t-\tau) - \mu S_1(t-\tau)W_1 - d_1S_1 \tag{15}$$

$$\frac{dS_2}{dt} = rS_2 - \frac{2rN_2S_2}{k} - \mu S_3 N_2 - \mu S_2 W_2 - d_2S_2 \tag{16}$$

$$\frac{dS_3}{dt} = \alpha S_2 W_2 + \alpha N_2 S_3 - 2\Delta_2 W_2 S_3 - \gamma_2 S_3 \tag{17}$$

$$\frac{dS_4}{dt} = \alpha S_1 W_1 + \alpha N_1 S_4 - 2\Delta_1 W_1 S_4 - \gamma_1 S_4 \tag{18}$$

Where $S_1 = \dfrac{\partial N_1}{\partial \alpha}, S_2 = \dfrac{\partial N_2}{\partial \alpha}, S_3 = \dfrac{\partial W_2}{\partial \alpha}, S_4 = \dfrac{\partial W_1}{\partial \alpha}$.

The concentration of nutrients becomes unstable when $\alpha = 1.5$ and Hopf-bifurcation occurs. But when the nutrients efficiency coefficient declines from $\alpha = 1.5$ to $\alpha = 1.47$, the graph becomes

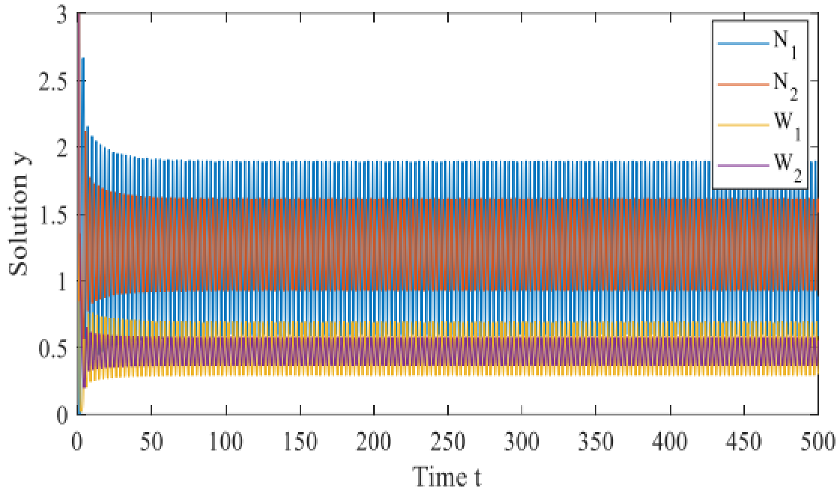

Figure 3. Shows that the interior equilibrium of the system loses stability and occurs the hopf bifurcation with delay $\tau > 1.25$.

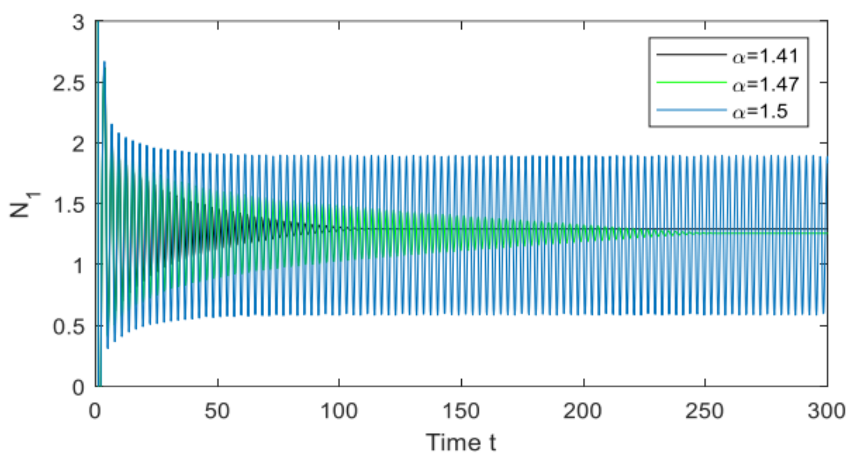

Figure 4. A time series graph shows the variation in the concentration of nutrients in roots for various values of the nutrients usage efficiency α.

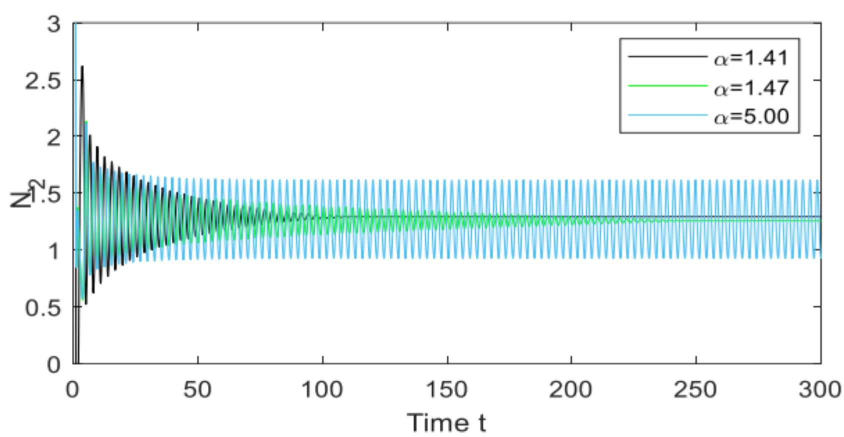

Figure 5. A time series graph shows the variation in the concentration of nutrients in shoot compartment for various values of the nutrients use efficiency α.

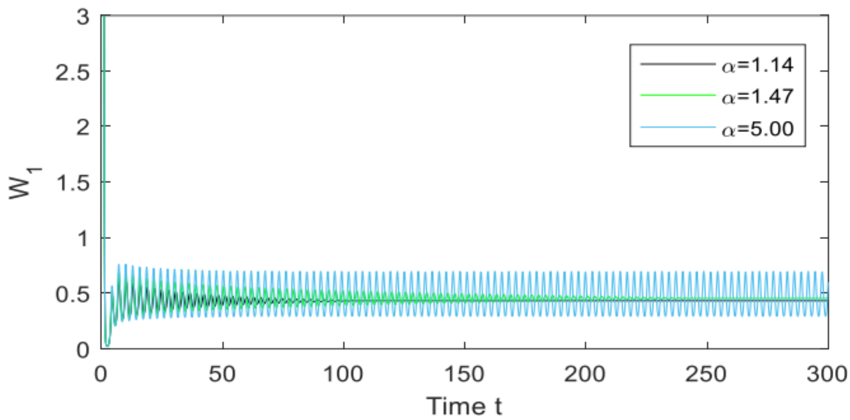

Figure 6. A time series graph shows the variation in the structural dry weight in roots for various values of the nutrients use efficiency α.

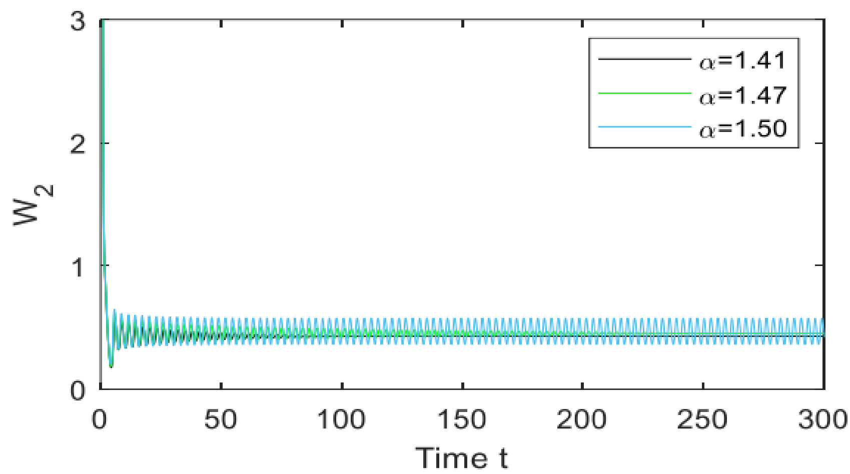

Figure 7. A time series graph shows the variation in the structural dry weight in shoot compartment for various values of the nutrients use efficiency α.

asymptotically stable, and it exhibits stability at $\alpha = 1.41$ as shown in figure 4. Similarly, as α drops from $\alpha = 1.5$ to $\alpha = 1.41$, as shown in figure 5, figure 6, figure 7 the concentration of nutrients in shoot compartment and dry weight of root and shoot decreases respectively.

9 Conclusion

The stability of the model has been examined about the interior equilibrium S_1. When there is no delay ($\tau = 0$) in the model, as shown in Figure 1, the interior equilibrium S_1 of the model is absolutely stable. This in accordance with lemma 4. The model has shown two distinct types of behaviour even after adding the delay. The interior equilibrium S_1 is asymptotically stable, when the value of delay is less than $1.25(\tau < 1.25)$ shown as Figure 2. The critical value of the parameter of delay is $\tau = 1.25$. When the value of delay parameter surpasses this critical value ($\tau > 1.25$), the equilibrium S_1 of the model shows oscillation that is Hopf bifurcation as shown in the Figure 3. This behaviour of system is in agreement with lemma 2 & lemma 3. The sensitivity is calculated of state variable in the model (1)-(4) by changing the parameter in delay deferential equation (15)-(18). Figures (4) to (7) depicts the phenomenon of sensitivity graphically that the state variable N_1, N_2, W_2 and W_1 changes the rate of oscillations with respect to the various values of α.

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