Palestine Journal of Mathematics

Vol 14 (Special Issue IV), (2025) , 87–93

Essential μ-hollow and essential μ-supplemented modules

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MSC 2010 Classifications: Primary 16D10 ; Secondary 16D80.

Keywords and phrases: μ-small submodules, essential submodules, μ-hollow modules, μ-supplemented modules.

The authors would like to express their thanks to the referee for her/his meticulous and insightful suggestions and comments on this paper.

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Abstract In this article, we introduce and present e-μ-small submodules, we stated that a nonzero submodule V of a module U is e-μ-small in U if $U = V + W$ with $\frac{U}{W}$ is cosingular, for some essential submodule W of U , implies that U = W, We list the fundamental characteristics of e-μ-small submodules. As an application, we introduce a module U is e-μ-hollow module if all submodules of U are $e-\mu$-small in U. As a generalization of $e-\mu$-hollow modules, we present e-μ-supplemented module when each of it's submodules have e-μ-supplement submodule. Some properties of these modules are considered in this paper.

1 Introduction

R is used as an associative ring throughout this paper using, U is right R - module. Following [1], defined a submodule $\mathrm{Z}^*(\mathrm{U}) = \{\mathrm{u} \in \mathrm{U} : u\mathrm{R} \ll \mathrm{E}(\mathrm{U})\}, \mathrm{E}(\mathrm{U})$ is the injective hull of U, U is called cosingular (noncosingular) when $Z^*(U) = U$ $(Z^*(U) = 0)$. A nonzero submodule V of a module U is essential in U if V intersects nonzeroly with any nonzero submodule of U, when each nonzero submodule of U is essential in U, U is called uniform module, see [2]. In [3], [4] and [5], the authors defined and study a new class of submodules of a module U, V is said to be μ-small in U if whenever $U = V + W$, with $\frac{U}{W}$ is cosingular, $U = W$. A submodule V of U is essential small in U (for short e-small), if whenever $U = V + W$, where W is essential in U, then $U = W$, (denoted by V $\ll_{\mathrm{e}} U$), see [6].
In this paper, we define and study submodules that are analogous to that essential small submodules which are presented by [6] and [7] , by replacing each small submodule by μ-small submodule, such submodules are called essential μ-small submodules (for short e- μ-small) and we denote V $\ll_{e\mu}$ U. Clearly that each e-small submodule is e- μ-small submodule and every μ-small submodule is e- μ-small. As a generalization of μ-hollow modules, we introduce essential μ-hollow (e- μ-hollow), a module U is e- μ-hollow if each submodule of U is $e - \mu$-small in U. Also, we use $e - \mu$-small submodule to introduce a new class of generalization of supplemented modules, a submodule V of U has e-μ-supplement submodule W in U if U = V + W and V$\cap$W $\ll_{e\mu}$ W. When each submodule of U has e- μ-supplement , U is called e- μ-supplemented module.
In Section 2, we introduce and investigate essential μ-small submodules with their properties and examples.
Section 3 is devoted to define and study e-μ-hollow modules as a generalization of each e-hollow modules and μ-hollow modules with characterization and examples.

In section 4, we present e-μ-supplemented modules, clearly that each e- μ-hollow module is e-μ-supplemented. Some basic features of these modules are stated in this section.

2 Essential μ-Small submodules

In this section, we think a new generalization of each of μ-small submodules and e-small submodules. Some of fundamented properties of this notion are presented within this section.

Definition 2.1. A submodule V of an R-module U is called essential μ-small (for short e-μ-small) submodule if $V + L \neq U$, for every proper essential submodule L of U with $\frac{U}{L}$ is cosingular, denoted by V $\ll_{e\mu}$ U. Equivalently, V $\ll_{e\mu}$ U if whenever V+L = U, L is essential submodule of U with $\frac{U}{L}$ is cosingular, then $U = L$.

Examples and Remarks 2.2.

(i) Clearly that each μ-small submodule is e- μ-small.

(ii) In general, the reverse implication of (i) is false, for example, in $\mathbb{Z}_6$ as $\mathbb{Z}$-module ($\bar{3}$) is e-μ-small in $\mathbb{Z}_6$ but not μ-small.

(iii) Clearly that each e-small submodule is e- μ-small.

(iv) When U is uniform R -module, e $-$ μ-small and μ-small submodules of U are equivalent.

The closeness of the concept of e-μ-small submodules to μ-small submodules is explained by the following proposition.

Proposition 2.3. *Put U be R-module, V and W be submodules of U, then.*

(i) If $V \leqslant W \leqslant U$ *and* $W \ll_{e\mu} U$, *then* $V \ll_{e\mu} U$ *and* $\frac{W}{V} \ll_{e\mu} \frac{U}{V}$.

(ii) $\mathrm{V} + \mathrm{W} \ll_{\mathrm{e}\mu} \mathrm{U}$ *if and only if* $\mathrm{V} \ll_{\mathrm{e}\mu} \mathrm{U}$ *and* $\mathrm{W} \ll_{\mathrm{e}\mu} U$.

(iii) If $V \leqslant W \leqslant U$ *and* $V \ll_{e\mu} W$, *then* $V \ll_{e\mu} U$.

(iv) Let $\mathrm{f} : \mathrm{U} \to \mathrm{U}'$ *be R-homomorphism and let* $\mathrm{V} \ll_{\mathrm{e}\mu} \mathrm{U}$, *then* $\mathrm{f}(\mathrm{V}) \ll_{\mathrm{e}\mu} \mathrm{U}'$.

(v) Let $\mathrm{U} = \mathrm{U}_1 \oplus \mathrm{U}_2$ *be a module and let* $\mathrm{V}_1 \leqslant \mathrm{U}_1$ *and* $\mathrm{V}_2 \leqslant \mathrm{U}_2$, *then* $\mathrm{V}_1 \oplus V_2 \ll_{\mathrm{e}\mu} U_1 \oplus U_2$ *if and only if* $V_1 \ll_{\mathrm{e}\mu} U_1$ *and* $V_2 \ll_{\mathrm{e}\mu} U_2$.

Proof. (i) Assume that $\mathrm{V} \leqslant \mathrm{W}$ and $\mathrm{W} \ll_{\mathrm{e}\mu} \mathrm{U}$. Take $\mathrm{U} = \mathrm{V} + \mathrm{L}$, for some essential submadule L of U with $\frac{U}{L}$ is cosingular. So, $U = W + L$, hence $L = U$, because $W \ll_{e\mu} U$.

Now, to prove that $\frac{W}{V} \ll_{e\mu} \frac{U}{V}$, let $\frac{X}{V} \leqslant_e \frac{U}{V}$ and $\frac{U}{X}$ is cosingular with $\frac{U}{V} = \frac{W}{V} + \frac{X}{V}$. One can easily show that $X \leqslant_e U$, as $W \ll_{e\mu} U$, then $U = X$ and hence $\frac{U}{V} = \frac{X}{V}$.

(ii) Suppose that $V + W \ll_{e\mu} U$. To show that $V \ll_{e\mu} U$, let $U = V + K, K \leqslant_e U$ and $\frac{U}{K}$ is cosingular, $\mathrm{U} = \mathrm{V} + \mathrm{W} + \mathrm{K}$. But $\mathrm{V} + \mathrm{W} \ll_{\mathrm{e}\mu} \mathrm{U}$, hence $\mathrm{U} = \mathrm{K}$. Similarly, $\mathrm{W} \ll_{\mathrm{e}\mu} \mathrm{U}$. Conversely, assume that $V \ll_{e\mu} U, W \ll_{e\mu} U$ and $U = V + W + L$, for some essential submodule L of U with $\frac{U}{L}$ is cosingular. Since $V \ll_{e\mu} U$, then $U = W + L$, similarly, $\mathrm{W} \ll_{\mathrm{e}\mu} \mathrm{U}$ implies that $\mathrm{U} = \mathrm{L}$.

(iii) Suppose that $V \leqslant W \leqslant U$ and $V \ll_{e\mu} W$, let $U = V + K$, where $K \leqslant_e U$ and $\frac{U}{K}$ is cosingular. Note that $\mathrm{W} = \mathrm{W} \cap (\mathrm{V} + \mathrm{K}) = \mathrm{V} + (\mathrm{W} \cap \mathrm{K})$, by modularity. It is clear that $\mathrm{W} \cap \mathrm{K} \leqslant_{\mathrm{e}} \mathrm{W}$ and $\frac{\mathrm{W}}{\mathrm{W} \cap \mathrm{K}} \cong \frac{\mathrm{W}+\mathrm{K}}{\mathrm{K}} = \frac{\mathrm{U}}{\mathrm{K}}$ is cosingular. But $\mathrm{V} \ll_{\mathrm{e}\mu} \mathrm{W}$, therefore $\mathrm{W} \cap \mathrm{K} = \mathrm{W}$ which implies that $\mathrm{V} \leqslant \mathrm{W} \leqslant \mathrm{K}$, hence $\mathrm{U} = \mathrm{K}$.

(iv) Let f: $\mathrm{U} \to \mathrm{U}'$ be R-homomorphisim and let $\mathrm{V} \ll_{\mathrm{e}\mu} \mathrm{U}$. By first isomorphism theorem $\frac{U}{\mathrm{Kerf}} \cong f(U)$ and $\frac{V}{\mathrm{Kerf}} \cong f(V)$. Since $V \ll_{e\mu} U$, then $f(V) \ll_{e\mu} U'$, from (i).

(v) Suppose that $V_1 \oplus V_2 \ll_{e\mu} U_1 \oplus U_2$, let P: $U_1 \oplus U_2 \to U_1$ be the projection map. Then $\mathrm{P}(\mathrm{V}_1 \oplus \mathrm{V}_2) \ll_{\mathrm{e}\mu} \mathrm{P}(\mathrm{U}_1 \oplus \mathrm{U}_2)$, hence $\mathrm{V}_1 \ll_{\mathrm{e}\mu} \mathrm{U}_1$. Similarly, $\mathrm{V}_2 \ll \mathrm{U}_2$. Conversely, let $V_1 \ll_{e\mu} U_1$ and $V_2 \ll_{e\mu} U_2$. Let $i_1 : U_1 \to U_1 \oplus U_2$ and $i_2 : U_2 \to U_1 \oplus U_2$ represent the inclusion maps, then $\mathrm{V}_1 \oplus 0 \ll_{\mathrm{e}\mu} \mathrm{U}_1 \oplus \mathrm{U}_2$ and $0 \oplus \mathrm{V}_2 \ll_{\mathrm{e}\mu} \mathrm{U}_1 \oplus \mathrm{U}_2$. Thus $\mathrm{V}_1 \oplus \mathrm{V}_2$ $\ll_{\mathrm{e}\mu} \mathrm{U}_1 \oplus \mathrm{U}_2$.

□

Example 2.4. The inverse image of e- μ-small submodule not necessary again e- μ-small. Let $\pi : \mathbb{Z} \to \frac{\mathbb{Z}}{2\mathbb{Z}}$ represent the natural epimorphism, it is well known that $0 \ll_{\mathrm{e}\mu} 2\mathbb{Z}$, however $\pi^{-1}(0) = 2\mathbb{Z}$ is not e- μ-small in $\mathbb{Z}$.

Proposition 2.5. *Let $V \leqslant W \leqslant U$ and W be a direct summand of U. If $V \ll_{e\mu} U$, then* $\mathrm{V} \ll_{\mathrm{e}\mu} \mathrm{W}$.

Proof. Let $\mathrm{U} = \mathrm{W} \oplus \mathrm{W}'$, for some submodule W' of U , let $\mathrm{W} = \mathrm{V} + \mathrm{L}$, L is essential in W with $\frac{\mathrm{W}}{\mathrm{L}}$ is consigular, hence $\mathrm{L} \oplus \mathrm{W}'$ is essential in U and $\frac{\mathrm{U}}{\mathrm{L}\oplus\mathrm{W}'} = \frac{\mathrm{W}\oplus\mathrm{W}'}{\mathrm{L}\oplus\mathrm{W}'} \cong \frac{\mathrm{W}}{\mathrm{L}}$ is cosingular. Note that $U = W + W' = \mathrm{V} + (\mathrm{L} \oplus \mathrm{W}')$, as $\mathrm{V} \ll_{e\mu} \mathrm{U}$ implies $\mathrm{U} = \mathrm{L} \oplus \mathrm{W}'$. So, $\mathrm{L} = \mathrm{W}$. □

Proposition 2.6. *For a module U , let* V, W *and Y be submodules of U with* $\mathrm{V} \leqslant \mathrm{W} \leqslant \mathrm{Y} \leqslant U$. *If* $\mathrm{W} \ll_{\mathrm{e}\mu} \mathrm{Y}$, *then* $\mathrm{V} \ll_{\mathrm{e}\mu} \mathrm{U}$.

Proof. Assume that $\mathrm{W} \ll_{e\mu} \mathrm{Y}, \mathrm{U} = \mathrm{V} + \mathrm{L}$, L is essential in U and $\frac{\mathrm{U}}{\mathrm{L}}$ is cosingualr, then $\mathrm{U} = \mathrm{W}+\mathrm{L}$. Note that, $\mathrm{Y} = \mathrm{Y}\cap\mathrm{U} = \mathrm{Y}\cap(\mathrm{W}+\mathrm{L}) = \mathrm{W}+(\mathrm{Y}\cap\mathrm{L})$, by modularity and $\frac{\mathrm{Y}}{\mathrm{Y}\cap\mathrm{L}} \cong \frac{\mathrm{Y}+\mathrm{L}}{\mathrm{L}} = \frac{\mathrm{U}}{\mathrm{L}}$ is cosingular. But $\mathrm{W} \ll_{\mathrm{e}\mu} \mathrm{Y}$, therefore $\mathrm{Y} = \mathrm{Y} \cap \mathrm{L}$, hence $\mathrm{V} \leqslant \mathrm{Y} \leqslant \mathrm{L}$, and $\mathrm{U} = \mathrm{L}$. Thus, $\mathrm{V} \ll_{\mathrm{e}\mu} \mathrm{U}$. □

Example 2.7. In general, the opposite of proposition 2.6 is untrue. Consider $\mathbb{Z}_{12}$ as $\mathbb{Z}$ module, note that $0 \leqslant< \overline{4} >\leqslant< \overline{2} >\leqslant \mathbb{Z}_{12}, 0 \ll_{\mathrm{e}\mu} \mathbb{Z}_{12}$ while $< \overline{4} >$ is not e- μ-small in $\langle\overline{2}\rangle$ because $\frac{\langle\overline{2}\rangle}{\langle\overline{6}\rangle}$ is cosingular, $\langle\overline{6}\rangle$ is essential in $\langle\overline{2}\rangle$ and $\langle\overline{6}\rangle \neq \langle\overline{2}\rangle$.

3 Essential μ-Hollow modules

In this part, we define e-μ-hollow modules which is based on e-μ-small condition. Some properties, examples and characterization of these modules are stated in this section.

Definition 3.1. A module U is contacted e- μ-hollow module, if all proper submodules of U are e- μ-small in U.

Examples and Remarks 3.2.

(i) Each μ-hollow module is essential μ-hollow .

(ii) Any R-module which lacks proper essential submodule is $\mathrm{e} - \mu$-hollow module.

(iii) In general, the converse of (i) is not true. For example, $\mathbb{Z}_6$ as $\mathbb{Z}$-module, the only essential submodule of $\mathbb{Z}_6$ is $\mathbb{Z}_6$, hence it is e- μ-hollow but not μ-hollow.

(iv) When U is uniform module, then U is μ-hollow if and only if U is $e - \mu$-hollow.

(v) Consider $\mathbb{Z}_{12}$ as $\mathbb{Z}$-module, let $\mathrm{V} = \{\overline{0},\overline{3},\overline{6},\overline{9}\}, \mathrm{L} = \{\overline{0},\overline{2},\overline{4},\overline{6},\overline{8},\overline{10}\}$ clearly, that $V + L = \mathbb{Z}_{12}$, L is essential in $\mathbb{Z}_{12}$, $\frac{\mathbb{Z}_{12}}{\mathrm{L}}$ is cosingular and $\mathrm{L} \neq \mathbb{Z}_{12}$, hence V is not e- $\mu-$small. Thus, $\mathbb{Z}_{12}$ is not e-μ-hollow.

Proposition 3.3. *"A module U is e-μ-hollow if and only if each of its proper essential submodules V of U with $\frac{U}{V}$ is cosingular is small in U."*

Proof. Let V be a proper essential submodule of U with $\frac{U}{V}$ is cosingular and assume that V is not small in U, there is a proper submodule B of U such that $U = V + B$. Since U is e-μ-hollow, $\frac{U}{V}$ is cosingular and $\mathrm{V} \leqslant_e \mathrm{U}$, then $\mathrm{U} = \mathrm{B}$ which is a contradiction, hence $\mathrm{V} \ll \mathrm{U}$. Conversely, to show that U is $e - \mu$ hollow, let V be a proper submodule of U and assume that V is not e-μ-small in U, there exists a proper essential submodule B of U such that $\frac{U}{B}$ is cosingular and $\mathrm{U} = \mathrm{V} + \mathrm{B}$, By our hypothesis, $\mathrm{B} \ll \mathrm{U}$, hence $\mathrm{B} = \mathrm{U}$ which is a contradiction. □

Proposition 3.4. *An epimorphic image of e- μ-hollow is e- μ-hollow.*

Proof. Let $\mathrm{f} : \mathrm{U} \to \mathrm{U}'$ be an epimorphism and assume that U is $\mathrm{e} - \mu$-hollow. Let V' be a proper submodule of U'. Note that $f^{-1}(V')$ is a proper submodule of U, if $f^{-1}(V') = U$, then $V' = U'$, which is a contraduction. But U is $e - \mu$-hollow, therefore $f^{-1}(V') \underset{e\mu}{\ll} U$, implies $V' \ll_{\mathrm{e}\mu} U'$. □

Corollary 3.5. *Let U be* $\mathrm{e} - \mu$*-hollow module and let V be a submodule of U, then* $\frac{\mathrm{U}}{\mathrm{V}}$ *is* $\mathrm{e} - \mu$*-hollow.*

In general, the opposite of corollary 3.5 is false, $\frac{\mathbb{Z}}{4\mathbb{Z}} \cong \mathbb{Z}_4$ as $\mathbb{Z}$-module is e- μ-hollow, while $\mathbb{Z}$ is not $\mathrm{e}-\mu$-hollow $\mathbb{Z}$-module.

Corollary 3.6. *Each direct summand of e- μ-hollow module is* $\mathrm{e}-\mu$*-hollow.*

The following proposition shows that under certain conditions the converse of corollary 3.5 is true.

Proposition 3.7. *Let* $\frac{\mathrm{U}}{\mathrm{V}}$ *be e-μ-hollow module, let V be a proper small and closed in U, if V is small in U, then U is e - μ-hollow.*

Proof. Let L be a proper submodule of U and let K be an essential submodule of U such that $U = L + K$ with $\frac{U}{K}$ is cosingular, then $\frac{U}{V} = \frac{L+K}{V} = \frac{L+V}{V} + \frac{K+V}{V}$. Note that $L + V$ is proper submodule of U, if $L + V = U$, then $L = U$ because of $V \ll U$, this is a contradiction. Since K is essential submodule of U, then $\frac{\mathrm{K+V}}{\mathrm{V}} \leqslant_e \frac{U}{V}$. Clearly that $\frac{U}{\mathrm{K+V}}$ is cosingular. Since $\frac{U}{V}$ is e-μ-hollow, therefore $\frac{L+V}{V} \underset{e\mu}{\ll} \frac{U}{V}$, then $\frac{U}{V} = \frac{K+V}{V}$, and $U = K + V$ implies that $U = K$, because $V \ll U$, then $L \underset{e\mu}{\ll}$. Thus, U is e- μ-hollow. □

The following example shows that the direct sum of e- μ-hollow modules may be not e- μ hollow.

Example 3.8. Put $U = \mathbb{Z}_8 \oplus \mathbb{Z}_2$ as $\mathbb{Z}$-module, each of $\mathbb{Z}_8$ and $\mathbb{Z}_2$ are e- μ-hollow modules while U is not e-μ-hollow, where at least there is a submodule $V =< (\bar{1}, \bar{1}) >$ of U with $U = V + L$, where $L =< (\bar{2}, \bar{0}) >$ is nonzero essential in U, $\frac{U}{L}$ is cosingular and $U \neq L$.

Recall that a submodule V of U is said to be fully invariant in U if $\varphi(\mathrm{V}) \leqslant \mathrm{V}$, for all nonzero $\varphi \in$ End (U). If each submodule of U is fully invariant in U, U is duo module, [8], [9] and [10]. A module U is distributive if for all submodules V, W and Y of U $V \cap (\mathrm{W+Y}) = (\mathrm{V}{\cap}\mathrm{W}) + (\mathrm{V}{\cap}\mathrm{Y})$, [11].

We end this section by the following proposition which gives some conditions under which the direct sums of e- μ-hollow again e- μ-hollow.

Proposition 3.9. *For R-module $U = U_1 \oplus U_2$, if each of U_1 and U_2 are e- μ-hollow modules, U is* $\mathrm{e}-\mu$*-hollow in each of the following cases.*

(i) U is duo module.

(ii) U is distributive.

(iii) ann $(\mathrm{U}_1) + \mathrm{ann}\,(\mathrm{U}_2) = \mathrm{R}$

4 Essential μ-Supplemented modules

As a generalization of μ-supplemented modules, we investigate essential μ-supplemented, we discuss the relationship between e-μ-hollow and e-μ-supplemented with examples, we show that the sum of $\mathrm{e}-\mu$-supplemented modules again e-μ-supplemented.

Definition 4.1. Let V and W be submodules of a module U, W is e- μ-supplement of V in U if $U = V + W$ and $\mathrm{V} \cap \mathrm{W} \ll_{\mathrm{e}\mu} \mathrm{V}$, U is called $e-\mu$-supplemented if each submodule of U has $e-\mu$-supplement in U.

Examples and Remarks 4.2.

(i) Every e- μ-hollow module is e- μ-supplemeted , but not conversely, $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ as $\mathbb{Z}$ module is e- μ-supplemented which is not e- μ-hollow.

(ii) $\mathbb{Z}$ as $\mathbb{Z}$-module is not e-μ-supplemented, since it is uniform which is not μ-supplemented.

(iii) In general e-μ-supplement is not a symmetric relation, $\mathbb{Z}_4$ as $\mathbb{Z}$-module is e- μ-supplement of $\{\bar{0}, \bar{2}\}$, while $\{\bar{0}, \bar{2}\}$ is not e- μ-supplement of $\mathbb{Z}_4$.

(iv) It is not necessary that e- μ-supplement submodule exists, $2\mathbb{Z}$ has no e- μ-supplement in $\mathbb{Z}$ as $\mathbb{Z}$-module.

Proposition 4.3. *For a module U, if V is $\mathrm{e}-\mu$-hollow submodule of U, then V is $\mathrm{e}-\mu$-supplement of every proper submodule W of U with* $\mathrm{U} = \mathrm{V} + \mathrm{W}$.

Proof. Put W as a proper submodule of U with $U = V + W$. Observe that $V \cap W \neq V$, if $\mathrm{V} \cap \mathrm{W} = \mathrm{V}, \mathrm{U} = \mathrm{W}$, which is a contradiction. Because of V is $\mathrm{e} - \mu$-hollow submodule of U , then $\mathrm{V} \cap \mathrm{W} \ll_{\mathrm{e}\mu} \mathrm{V}$. So, V is e- μ-supplement of W . □

The e-μ-supplement submodule is characterized by the following theorem.

Theorem 4.4. *For submodules V and W of a module U, the following properties are equivalent.*

(i) W is e- μ-supplement of V .

(ii) $U = V+W$ and for each proper essential submodule X of U, $\frac{W}{X}$ is cosingular, $U \neq V+X$.

Proof. $(i) \Rightarrow (ii)$ Let W be an e-μ-supplement of V in U, assume that X is proper essential submodule of U with $U = V + X$ and $\frac{W}{X}$ is cosingular. Note that $W = W \cap (V + X) = X+ (\mathrm{V} \cap \mathrm{W})$, by modularity. Since W is e-μ-supplement of V in U, $\frac{\mathrm{W}}{\mathrm{X}}$ is cosingular and X is essential in W, then $\mathrm{W} = \mathrm{X}$, that is a contradiction, hence $\mathrm{U} \neq \mathrm{V} + \mathrm{X}$.
$(ii) \Rightarrow (i)$ Let $U = V + W$, it is enough to show that $V \cap W \ll_{e\mu}$ W, let $\mathrm{W} = (\mathrm{V} \cap \mathrm{W})+L$, for some essential submodule L of W with $\frac{\mathrm{W}}{\mathrm{L}}$ is cosingular. If L is proper, then $\mathrm{U} \neq \mathrm{V} + \mathrm{L}$, that is a contradiction with $\mathrm{U} = \mathrm{V} + \mathrm{W} = \mathrm{V} + (\mathrm{V} \cap \mathrm{W}) + \mathrm{L} = \mathrm{V} + \mathrm{L}$, hence $\mathrm{W} = \mathrm{L}$. Thus, W is $\mathrm{e} - \mu$-supplement of V in U. □

The following proposition gives some basic properties of e- μ-supplement submodule.

Proposition 4.5. *For a module U, V and W are submodules of U . If W is $\mathrm{e} - \mu$-supplement of V in U, then.*

(i) If $\mathrm{U} = \mathrm{X} + \mathrm{W}$, for some submodule X of V , then W is $\mathrm{e} - \mu$-supplement of X in U.

(ii) For every submodule L of V, $\frac{W+L}{L}$ is e- μ-supplement of $\frac{V}{L}$ in $\frac{U}{L}$.

Proof. (i) If $\mathrm{U} = \mathrm{X} + \mathrm{W}$, for some submodule X of V, W is e- μ-supplement of V in U . Since $\mathrm{X} \cap \mathrm{W} \leqslant \mathrm{V} \cap \mathrm{W} \ll_{\mathrm{e}\mu} \mathrm{W}$, then $\mathrm{X} \cap \mathrm{W} \ll_{\mathrm{e}\mu} \mathrm{W}$, by proposition 2.3. Hence W is $\mathrm{e} - \mu-$ supplement of X in U.

(ii) Let L be a submodule of V, then $\frac{U}{L} = \frac{V+W}{L} = \frac{V}{L} + \frac{W+L}{L}$. Observe that $\frac{V}{L} \cap \frac{W+L}{L} = \frac{\mathrm{V}\cap(\mathrm{W}+\mathrm{L})}{\mathrm{L}} = \frac{\mathrm{L}+(\mathrm{V}\cap\mathrm{W})}{\mathrm{L}}$, let $\psi : \mathrm{W} \to \frac{\mathrm{W}+\mathrm{L}}{\mathrm{L}}$, since $\mathrm{W} \cap \mathrm{V} \ll_{\mathrm{e}\mu} \mathrm{W}$, then $\psi(\mathrm{W} \cap \mathrm{V}) = \frac{(\mathrm{V}\cap\mathrm{W})+\mathrm{L}}{\mathrm{L}} \ll_{\mathrm{e}\mu} \psi(\mathrm{W}) = \frac{\mathrm{W}+\mathrm{L}}{\mathrm{L}}$, by proposition 2.3. Therefore, $\frac{\mathrm{W}+\mathrm{L}}{\mathrm{L}}$ is e- μ-supplement of $\frac{\mathrm{V}}{\mathrm{L}}$ in $\frac{\mathrm{U}}{\mathrm{L}}$.

□

Proposition 4.6. *Let V and W be submodules of a module U, if $\mathrm{U} = \mathrm{U}_1 \oplus \mathrm{U}_2$ such that V is e- μ-supplement of V' in U_1 and W is e- μ-supplement of W' in U_2, then $\mathrm{V} + \mathrm{W}$ is e- μ-supplement of $\mathrm{V}' \oplus \mathrm{W}'$ in U.*

Proof. Take $\mathrm{U}_1 = \mathrm{V} + \mathrm{V}', \mathrm{V} \cap \mathrm{V}' \ll_{\mathrm{e}\mu} \mathrm{V}, \mathrm{U}_2 = \mathrm{W} + \mathrm{W}'$ and $\mathrm{W} \cap \mathrm{W}' \ll_{\mathrm{e}\mu} \mathrm{W}$, hence $\mathrm{U} = \mathrm{U}_1 \oplus \mathrm{U}_2 = (\mathrm{V} + \mathrm{V}')\oplus(\mathrm{W} + \mathrm{W}') = (\mathrm{V}\oplus\mathrm{W})+(\mathrm{V}' \oplus \mathrm{W}')$. Since $(\mathrm{V} \cap \mathrm{V}')\oplus(\mathrm{W}\cap \mathrm{W}') \ll_{\mathrm{e}\mu} \mathrm{V}+\mathrm{W}$, by proposition 2.3 It is easy to see that $(\mathrm{V} \oplus \mathrm{W}) \cap (\mathrm{V}' \oplus \mathrm{W}') = (\mathrm{V}\cap \mathrm{V}') \oplus (W \cap W') \ll_{e\mu} V \oplus W$. It follows that $V + W$ is e- μ-supplement of $V' \oplus W'$ in U. □

Lemma 4.7. *Let U_1 be e- μ-supplemented and $\mathrm{U}_1 + \mathrm{U}_2$ has e- μ-supplement in U. Then U_2 has e- μ-supplement in U.*

Proof. Suppose that V is e- μ-supplement of $U_1 + U_2$ in U, hence $U = U_1 + U_2 + V$ and $(U_1 + U_2) \cap V \ll_{\mathrm{e}\mu} \mathrm{V}$. Since U_1 is e-μ-supplemented, then $(\mathrm{U}_2 + \mathrm{V}) \cap \mathrm{U}_1$ has e- μ-supplement in U_1 say W, hence $U_1 = W + [(U_2 + V) \cap U_1]$ and $W \cap [(U_2 + V) \cap U_1] \ll_{e\mu} W$, we have $U = U_1 + U_2 + V = W + [(U_2 + V) \cap U_1] + U_2 + V = W + U_2 + V = U_2+ (\mathrm{W} + \mathrm{V})$. One can easily shows $\mathrm{U}_2 \cap (\mathrm{W} + \mathrm{V}) \leqslant [(\mathrm{U}_2 + \mathrm{W}) \cap \mathrm{V}] + [(\mathrm{U}_2 + \mathrm{V}) \cap \mathrm{W}] \leqslant [(U_2 + U_1) \cap V] + [(U_2 + V) \cap W] \ll_{e\mu} V + W$, by proposition 2.3, hence $U_2 \cap (\mathrm{W}+ \mathrm{V}) \ll_{\mathrm{e}\mu} (\mathrm{W} + \mathrm{V})$. Thus, U_2 has e-μ-supplement in U. □

The following proposition shows that sum of e- μ-supplemented modules again e$-\mu$-supplemented.

Proposition 4.8. *If $U = U_1 + U_2$, and each of U_1 and U_2 are e- μ-supplemented modules, then U is e- μ-supplemented.*

Proof. Let V be a submodule of U. Since $U = U_1 + U_2 + V$ trivially has e- μ-supplement in U, then $U_2 + V$ has e-μ-supplement in U, by Lemma 4.7. Since $U_2 + V$ has $e - \mu$-supplement and U_2 is e- μ-supplemented , then Lemma 4.7 implies V has e- μ-supplement in U. So, U is e-μ-supplemented. □

It is easy to show that any finite sum of e- μ-supplemented is again e- μ-supplemented , by induction.

Proposition 4.9. *An e-μ-supplemented modules are closed under epimorphisms.*

Proof. Put f : U $\to$ U$'$ be an epimorphism and U be an e-μ-supplemented, let Y be a submodule of U', hence $f^{-1}(Y) \leqslant U$. Since U is e- μ-supplemented, then $f^{-1}(Y)$ has e- μ -supplement W in U. It is simple to prove that $U' = f(W) + Y$ and $Y \cap f(W) \ll_{e\mu} f(W)$. So, U$'$ is e-μ-supplemented.

□

Corollary 4.10. *Any factor of e- μ-supplemented modules is e- μ-supplemented.*

In general, the reverse implication of the previous corollary is not true. For example, $\frac{\mathbb{Z}}{6\mathbb{Z}} \cong \mathbb{Z}_6$ is e- μ-supplemented while $\mathbb{Z}$ is not e- μ-supplemented $\mathbb{Z}$-module.

This section concludes with the diagram that follows.

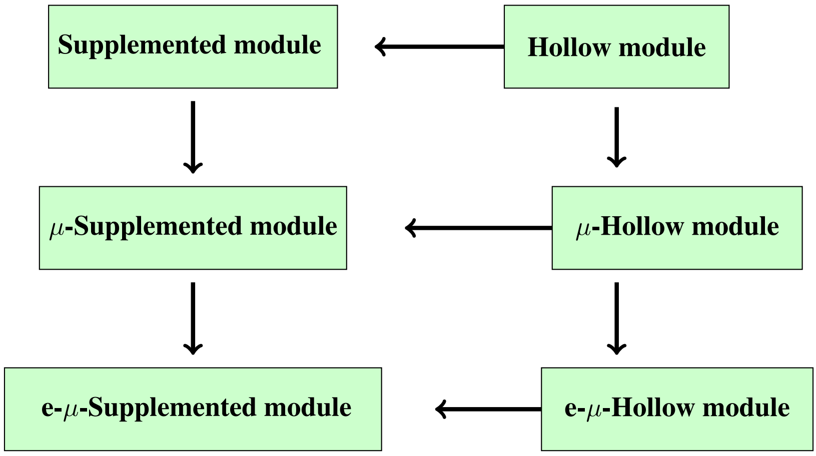

Figure 1. The relation among the concepts that we defined in this article extending property

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