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Topological Indices and Linear Codes of Generalized Franklin Graphs

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Abstract Inspired by the structure of the Franklin graph, this study presents a general construction of the Franklin graph $\mathbb{F}_p$ for infinitely many odd integers $p \geq 3$. We determine the chromatic number, independence number, domination number, Roman domination number, girth, and clique number for the family of graphs $\mathbb{F}_p$. Also, we prove that these graphs are Hamiltonian and triangle-free. Further, we calculate various topological indices for the graph $\mathbb{F}_p$. Finally, from an application point of view, we study linear codes of the generalized Franklin graphs, which are generated by incidence and distance matrices.

1 Introduction

The application of Graph Theory in Computer Science and other subjects, such as Mathematical Chemistry, shows the relevance of graphs to other domains of study. In molecular graphs, topological indices are the numerical descriptors of some basic properties of the molecules. Weiner index is the first known topological index introduced by Weiner [31] in 1947. Later, many topological indices related to the different graphs were introduced in [10, 13, 19, 20], and many interesting results were discussed related to these indices in [21, 23]. These indices have applications in mathematical chemistry and are one of the important ingredients in describing the characteristics of chemical compounds [15].
Besides application, many conjectures and new graphs have also been introduced to solve the conjectures or sometimes by generalizing the existing graphs. Towards this, in 1852, Guthrie [16] conjectured that any map on the sphere is four-colorable, and in 1976, Appel and Haken proved this conjecture [1]. In 1890, Heawood [17] conjectured a bound for the minimum number of colors required for graph coloring on a surface of genus g. This conjecture was proved in 1968 by Ringel and Youngs [24] except for the Klien bottle. However, Franklin [9] partially disproved the Heawood conjecture using the Franklin graph in 1934. In graph generalization, we recall the generalized Petersen graph introduced by Coxeter [5] in 1950, and in 1969, Watkins [30] discussed some properties of this family of graphs.
Inspired by this generalized Peterson graph [25, 30], we present the generalization of the Franklin graph. This paper initially explores the construction and properties of the generalized Franklin graphs. The Franklin graph is 3-regular, 2-chromatic, Hamiltonian consists of 12 vertices and 18 edges and named after P. Franklin in 1934 [9]. The decomposition of vertices of the Franklin graph is as $12 = 2 \times 3 + 2 \times 3$, or $12 = 2 \times p + 2 \times p$ where $p = 3$. A curious question arises as to whether the generalized Franklin graph possesses all the properties of the Franklin graph for all numbers $p \geq 3$. We answer this question affirmatively for all odd integers $p \geq 3$. A generalized Franklin graph possesses all such properties, similar to the Franklin graph, like 3-regular, 2-chromatic, Hamiltonian, etc.

Further, we calculate the first and second Zagreb indices, Forgotten index, Redefined third Zagreb index, inverse second Zagreb index, symmetric deg division index, inverse sum indeg index, Harmonic index, connectivity index, and atom-bond connectivity index for the generalized Franklin graphs. Apart from the applications of graphs in computer science, the study of codes from the row span of the adjacency matrices and incidence matrices of the graphs has emerged in [6, 11, 12, 14, 26] towards application in information technology. These works motivate us to find the associated linear codes to the generalized Franklin graphs.
The structure of this paper is as follows. Section 2 contains basic definitions and results. Section 3 provides the construction of a generalized Franklin graph while Section 4 establishes the basic properties of the generalized Franklin graphs. In section 5, we calculate degree-based topological indices, whereas, in Section 6, we obtain the parameter of the linear codes associated with the generalized Franklin graphs.

2 Preliminaries

Let G be an undirected, simple, and connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $|V(G)| = n$ and $|E(G)| = m$ be the order and size of the graph G, respectively. First, we recall here a few basic definitions and terminologies related to graphs. The degree of a vertex v in a graph is the number of vertices adjacent to v. The graph is said to be p-regular if all the vertices of the graph are of degree p. A path of length m in the graph G is a finite sequence $u_1e_1u_2e_2u_3 \dots u_me_mu_{m+1}$ of distinct vertices and edges. Also, a cycle of length m in the graph G is formed by the path of length m with the same initial and terminal vertices denoted by $C_m(u_1u_2u_3 \dots u_mu_1)$. The distance between two vertices is the length of the shortest path. The graph G is called triangle-free if it does not contain a cycle of length 3. A Hamiltonian cycle in a graph is a cycle that contains all the vertices. Proper graph vertex coloring is an assignment of colors to the vertices such that the endpoints of each edge receive different colors. The chromatic number is the smallest number of colors needed for proper graph vertex coloring. Throughout this paper, we follow the definitions and terminologies from Bondy and Murty [3].
Recall that a set $S \subseteq V(G)$ is said to be a dominating set if for any $v \in V(G)$ either $v \in S$ or v is adjacent to some vertex in S. A minimum cardinality of a dominating set is called a domination number, denoted by $\gamma(G)$. A total dominating set S is a set of vertices of G with no isolated vertex such that every vertex is adjacent to a vertex in S. If no proper subset of S is a total dominating set of G, then S is a minimal total dominating set of G. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. Further, A Roman dominating function of a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the Roman domination number of G, denoted by $\gamma_(G)$. Recently, Kumar and Prakash [18] studied the Roman domination of some zero-divisor graphs of the commutative rings and some of their properties.
On the other hand, an incidence matrix of the graph G is a Boolean matrix with rows and columns labeled by vertices and edges, respectively. If the edge contains a vertex, the corresponding entry is 1; otherwise, it is 0. The distance matrix of a graph is a matrix that represents the pairwise distances between vertices in the graph. Here, note that the vertices label the rows and columns of this matrix. Let $\mathcal{F}_q$ be a finite field with cardinality q. A q-ary code of length n is a nonempty subset of $\mathcal{F}_q^n$. This q-ary code is called a linear code if it is a subspace of $\mathcal{F}_q^n$ over $\mathcal{F}_q$. Also, for $\alpha \in \mathcal{F}_q^n$, the Hamming weight of α, denoted by $w_H(\alpha)$, is the number of non-zero entries in α. For $\alpha, \beta \in \mathcal{F}_q^n$, the Hamming distance between α and β, denoted by $d_H(\alpha, \beta)$, is the Hamming weight of $\alpha - \beta$. In this work, we find q-ary linear code over $\mathcal{F}_q^n$ and calculate the parameters $[n, k, d]_q$, where k represents dimension and d represents minimum Hamming distance of this code. Recall that the generator matrix of a linear code is a matrix in which rows constitute a basis for the code. The dimension of the linear code is the rank of the generator matrix. We refer to [28] for more details on basic coding theory.

3 Graph $\mathbb{F}_p$: Construction and Examples

Let $n \geq 12$ such that $n = 2 \times p + 2 \times p$ and $p \geq 3$ be an odd integer. The following steps present the construction of the graph $\mathbb{F}_p$.

- The two class of vertices are $x_1, x_2, \ldots, x_{2p}$ and $y_1, y_2, \ldots, y_{2p}$.
  For this construction, we consider modulo $2p$ for the value of $i > 2p$.

- The edge set consists of $x_i x_{i+1}$ for $1 \leq i \leq 2p$, together with the following edges

- $x_i y_i$ for $1 \leq i \leq 2p$,

- $y_i y_{i+1}$ for $i \in \{1, 3, 5, \ldots, 2p-1\}$, and

- $y_i y_{i+p}$ for $1 \leq i \leq p$.

We present the graphs pictorially for the values $p = 3, 5$, and 7.

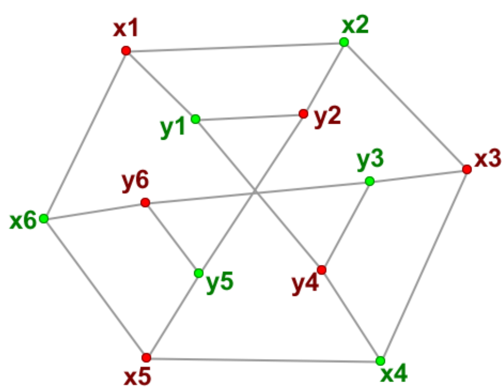

Figure 1. The Franklin Graph ($\mathbb{F}_3$)

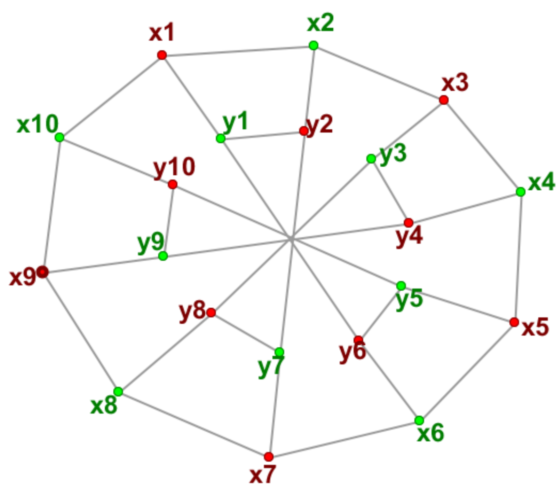

Figure 2. The Graph ($\mathbb{F}_5$)

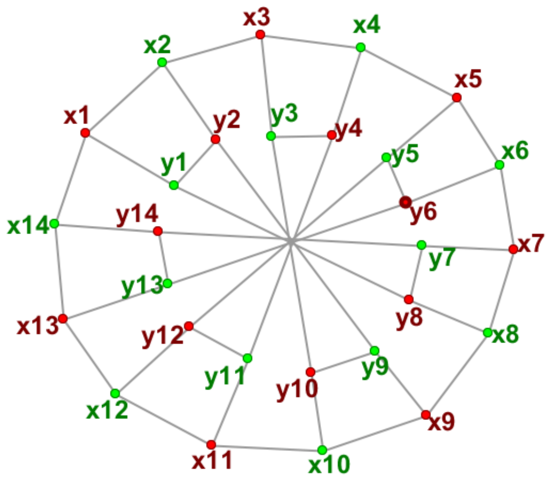

Figure 3. The Graph ($\mathbb{F}_7$)

4 Properties of the graph $\mathbb{F}_p$

Lemma 4.1. *The chromatic number of the graphs $\mathbb{F}_p$ is 2.*

Proof. For all even integers $2 \leq i \leq 2p$, assign color G to vertices x_i and color R to vertices y_i. Further, for all odd integers $1 \leq i \leq 2p-1$, assign color R to vertices x_i and color G to vertices y_i. By construction, this vertex coloring of the graph $\mathbb{F}_p$ is the proper coloring with the minimum number of colors.

□

Lemma 4.2. *The graphs $\mathbb{F}_p$ are Hamiltonian.*

Proof. We prove this lemma by finding a Hamiltonian cycle in the graph $\mathbb{F}_p$. In this connection, we consider the cycle $C_{2p+2p}(x_1y_1y_2x_2x_3y_3y_4x_4\dots x_{2p-1}y_{2p-1}y_{2p}x_{2p}x_1)$ of length $2p+2p$ consists of $2p+2p$ vertices. □

Lemma 4.3. *The graphs $\mathbb{F}_p$ are triangle-free.*

Proof. By symmetry of the graph $\mathbb{F}_p$, we need to check the triangle with the vertices x_1 and y_1. The set of adjacent vertices of x_1 is $\{x_2, y_1, x_{2p}\}$ and this set is independent. Hence, there is no triangle containing vertex x_1. Now, consider the vertex y_1. The set of adjacent vertices of y_1 is $\{x_1, y_2, y_{1+p}\}$ and this set is independent. Therefore, there is no triangle containing vertex y_1. Thus, the graph $\mathbb{F}_p$ is triangle-free. □

Remark 4.4. As the graph $\mathbb{F}_p$ is triangle-free, and there is a cycle $C_4(x_1y_1y_2x_2x_1)$ of length 4, implies that the girth of the graph $\mathbb{F}_p$ is 4. Since $\mathbb{F}_p$ is triangle-free, the clique number is 2.

In 2011, Willis [32] discussed an upper bound for the independence number as follows:

Theorem 4.5 ([32]). *For any graph G, $\alpha(G) \leq n - \frac{E}{\Delta}$.*

Now, using this result, we prove the following lemma.

Lemma 4.6. *The independence number of the graph $\mathbb{F}_p$ is $2p$ where $p \geq 3$ is an odd integer.*

Proof. We know that the graph $\mathbb{F}_p$ is 3−regular with $6p$ edges. By Theorem 4.5, $\alpha(\mathbb{F}_p) \leq 4p - \frac{6p}{3} = 2p$, i.e., independence number cannot exceed $2p$. Now, we consider a set $S = \{x_1, y_2, x_3, y_4, \dots, x_{2p-1}, y_{2p}\}$ of length $2p$. Since there is no edge between the two vertices, the set S is independent. Thus, the independence number of $\mathbb{F}_p$ is $2p$. □

Theorem 4.7. *Let $\mathbb{F}_p$ be the generalized Franklin graph of order $4p$ with odd integer $p \geq 3$. Then the domination number of $\mathbb{F}_p$ is $p + \lceil \frac{p}{4} \rceil$.*

Proof. It is obvious that for an $r-$regular graph G, $\gamma(G) \geq \frac{1}{r+1}n$ where n is the order of G. In the case of the generalized Franklin graph, $\gamma(\mathbb{F}_p) \geq \frac{4p}{3+1} = p$. Now, consider a set $S = \{x_1, y_3, x_5, y_7, \ldots, x_{2p-1}\}$. Then, every vertex of type x_k, $1 \leq k \leq 2p$, either in S or adjacent to a vertex in S. Also, there are $\lceil \frac{p}{4} \rceil$ remaining vertices that are not adjacent in S. Suppose the set V consists of $\lceil \frac{p}{4} \rceil$ vertices. Now, there are two cases: If $p = 2k+1$, $k \geq 1$ is an odd, then the set $V = \{y_p \mid p = 2k,\ k \text{ is an odd}\}$ is of length $\lceil \frac{p}{4} \rceil$. If $p = 2k+1$, k is an even, then the set $V = \{y_{p+1}, y_{p+4s+1} \mid 1 \leq s \leq \lfloor \frac{p}{4} \rfloor, s \in \mathbb{N}\}$. In both cases, the vertex set $S \cup V$ dominates the graph $\mathbb{F}_p$ and order of $S \cup V$ is $p + \lceil \frac{p}{4} \rceil$. To prove this set is minimal, we suppose by contrary that the dominating set consists of $p + \lceil \frac{p}{4} \rceil - 1$ elements. If we remove any one vertex, say x_i or y_j from the set $S \cup V$, then any of the adjacent vertices of x_i or y_j does not belong to $S \cup V$. Therefore, we get a contradiction. Hence, the domination number of $\mathbb{F}_p$ is $p + \lceil \frac{p}{4} \rceil$. □

Theorem 4.8. *For every odd integer $p \geq 3$, the graph $\mathbb{F}_p$ is perfect.*

Proof. A graph is perfect if the chromatic number is equal to the clique number of the graph. From Lemma 4.1, the chromatic number of the graph $\mathbb{F}_p$ is 2, and by Remark 4.4, the clique number is 2. Therefore, the graph $\mathbb{F}_p$ is perfect. □

In the next result, we have characterized the lower and upper bound of the Roman domination number of the Franklin graph. In 2004, Cockayne et al. [4] proved some important results related to $\gamma_R(G)$ as follows:

Lemma 4.9 ([4]). *For any graph G of order n and maximum degree $\Delta(G)$,*

$$\frac{2n}{\Delta(G)+1} \leq \gamma_R(G).$$

Lemma 4.10 ([4]). *For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

Theorem 4.11. *Let $\mathbb{F}_p$ be the generalized Franklin graph of order $4p$, $p \geq 3$ be an odd integer. Then $2p \leq \gamma_R(G) \leq 2p + 2\lceil \frac{p}{4} \rceil$.*

Proof. Since the Franklin graph has the maximum degree $\Delta(\mathbb{F}_p) = 3$, following Lemma 4.9, we get $\gamma_R(\mathbb{F}_p) \geq 2\frac{4p}{4} = 2p$. Also, by Lemma 4.10, we have $\gamma_R(\mathbb{F}_p) \leq 2\gamma(\mathbb{F}_p)$ and by Theorem 4.7, $\gamma(\mathbb{F}_p) = p + \lceil \frac{p}{4} \rceil$. Therefore, $\gamma_R(\mathbb{F}_p) \leq 2p + 2\lceil \frac{p}{4} \rceil$. □

5 Topological indices of the graph $\mathbb{F}_p$

Here, we present the definitions of some topological indices, which will be used in the subsequent section of this paper.

Definition 5.1. The first and second Zagreb indices (Gutman and Trinajestić [13])

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) = \sum_{v \in V(G)} d_v^2;$$

$$M_2(G) = \sum_{uv \in E(G)} d_u d_v.$$

Definition 5.2. The Forgotten index (Furtula et al.[10])

$$F(G) = \sum_{uv \in E(G)} (d_u^2 + d_v^2) = \sum_{v \in V(G)} d_v^3.$$

Definition 5.3. The Redefined third Zagreb index (Mansour et al. [20])

$$rZ(G) = \sum_{uv \in E(G)} (d_u + d_v) d_u d_v.$$

Definition 5.4. The inverse second Zagreb index (A. Miličević et al. [19])

$$ISZI(G) = \sum_{uv\in E(G)} \frac{1}{d_u d_v}.$$

Definition 5.5. The symmetric deg division index (D. Vukičević [27])

$$SDDI(G) = \sum_{uv\in E(G)} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u}\right).$$

Definition 5.6. The inverse sum indeg index (D. Vukičević and M. Gašperov [29])

$$ISI(G) = \sum_{uv\in E(G)} \left(\frac{d_u d_v}{d_u + d_v}\right).$$

Definition 5.7. The Harmonic index (Fajtlowicz et al. [8])

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d_u + d_v}.$$

Definition 5.8. The connectivity index or the Randić index (M. Randić [22])

$$R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$

Definition 5.9. The atom-bond connectivity index (Estrada et al. [7])

$$ABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}.$$

Now, we calculate the values of the above-mentioned topological indices for the generalized Franklin graph.
As we can see, in the graph $\mathbb{F}_p$, the degree of each vertex is equal to 3 for all odd integer $p \geq 3$. Therefore, $E(\mathbb{F}_p) = (4p)\frac{3}{2} = 6p$.

Lemma 5.10. *Let $\mathbb{F}_p$ be the generalized Franklin graph of order $n = 4p$, and $p \geq 3$ be an odd integer. Then*

(i) $M_1(\mathbb{F}_p) = 36p$. *(ii)* $M_2(\mathbb{F}_p) = 54p$. *(iii)* $F(\mathbb{F}_p) = 108p$. *(iv)* $rZ(\mathbb{F}_p) = 324p$.

(v) $ISZI(\mathbb{F}_p) = \frac{2p}{3}$. *(vi)* $SDDI(\mathbb{F}_p) = 12p$. *(vii)* $ISI(\mathbb{F}_p) = 9p$. *(viii)* $H(\mathbb{F}_p) = 2p$.

(ix) $R(\mathbb{F}_p) = 2p$. *(x)* $ABC(\mathbb{F}_p) = 4p$.

Proof. The graph shows that $\mathbb{F}_p$ is 3-regular. Hence, the degree of each vertex is 3. Hence,

(i) $M_1(\mathbb{F}_p) = \sum_{uv\in E(\mathbb{F}_p)} (d_u + d_v) = \sum_{v\in V(\mathbb{F}_p)} d(v)^2 = 4p(3^2) = 36p.$

(ii) $M_2(\mathbb{F}_p) = \sum_{uv\in E(\mathbb{F}_p)} d_u d_v = 6p(3\times 3) = 54p.$

(iii) $F(\mathbb{F}_p) = \sum_{uv\in E(\mathbb{F}_p)} (d_u^2 + d_v^2) = \sum_{v\in V(G)} d(v)^3 = (4p)(3^3) = 108p.$

(iv) $rZ(\mathbb{F}_p) = \sum_{uv\in E(\mathbb{F}_p)} (d_u + d_v)d_u d_v = 6p(3+3)(3)(3) = 324p.$

(v) $ISZI(\mathbb{F}_p) = \sum_{uv\in E(\mathbb{F}_p)} \frac{1}{d_u d_v} = \frac{6p}{3(3)} = \frac{2p}{3}.$

(vi) $SDDI(\mathbb{F}_p) = \sum_{uv\in E(\mathbb{F}_p)} \left(\frac{d_u}{d_v} + \frac{d_v}{d_u}\right) = 6p(\frac{3}{3} + \frac{3}{3}) = 12p.$

(vii) $ISI(\mathbb{F}_p) = \sum\limits_{uv \in E(\mathbb{F}_p)} \left(\frac{d_u d_v}{d_u + d_v} \right) = 6p \left(\frac{3(3)}{3+3} \right) = 9p.$

(viii) $H(\mathbb{F}_p) = \sum\limits_{uv \in E(\mathbb{F}_p)} \frac{2}{d_u + d_v} = 6p \frac{2}{3+3} = 2p.$

(ix) $R(\mathbb{F}_p) = \sum\limits_{uv \in E(\mathbb{F}_p)} \frac{1}{\sqrt{d_u d_v}} = \frac{6p}{\sqrt{3 \times 3}} = 2p.$

(x) $ABC(\mathbb{F}_p) = \sum\limits_{uv \in E(\mathbb{F}_p)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} = 6p \sqrt{\frac{3+3-2}{3 \times 3}} = 4p.$

□

6 Linear Codes from graphs $\mathbb{F}_3$, $\mathbb{F}_5$, and $\mathbb{F}_7$ with Hamming distances

This section presents the computation of the parameters of the associated linear codes to the incidence matrices and distance matrices, respectively, for the graphs $\mathbb{F}_3$, $\mathbb{F}_5$, and $\mathbb{F}_7$.

6.1 Linear Codes obtained from incidence matrix

The incidence matrix $I(\mathbb{F}_3)$ of the graph $\mathbb{F}_3$ is given below. The rows of the matrix $I(\mathbb{F}_3)$ are labeled by vertices of the graph $\mathbb{F}_3$ and columns of the matrix are labeled by edges of the graph $\mathbb{F}_3$.

$$I(\mathbb{F}_3) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Lemma 6.1 ([2]). *If G is a connected graph on n vertices, then the rank of the incidence matrix of G is* $|V| - 1$.

Theorem 6.2. *Let $C_q(I(\mathbb{F}_3)) \subseteq \mathcal{F}_q^{18}$ $(q - prime)$ be a code generated by incidence matrix $I(\mathbb{F}_3)$ of graph $\mathbb{F}_3$. Then the code $C_q(I(\mathbb{F}_3))$ has parameters* $[18, 11, 3]_q$.

Proof. The graph $\mathbb{F}_3$ has 12 vertices and 18 edges and the code $C_q(I(\mathbb{F}_3))$ is generated by incidence matrix $I(\mathbb{F}_3)$. Therefore, length of the code $C_q(I(\mathbb{F}_3))$ is 18. Following Lemma 6.1, the dimension of the code $C_q(I(\mathbb{F}_3))$ is 11. Also, by calculating in Magma software, we get $d_H(C_q(I(\mathbb{F}_3))) = 3$. □

The incidence matrix $I(\mathbb{F}_5)$ of the graph $\mathbb{F}_5$ is given below.

$$I(\mathbb{F}_5) = \begin{bmatrix}
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&1
\end{bmatrix}$$

Theorem 6.3. *Let $C_q(I(\mathbb{F}_5)) \subseteq \mathcal{F}_q^{30}$ $(q - prime)$ be a linear code generated by incidence matrix $I(\mathbb{F}_5)$ of graph $\mathbb{F}_5$. Then the code $C_q(I(\mathbb{F}_5))$ has parameters $[30, 19, 3]_q$.*

Proof. The graph $\mathbb{F}_5$ has 20 vertices and 30 edges and the code $C_q(I(\mathbb{F}_5))$ is generated by incidence matrix $I(\mathbb{F}_5)$. Therefore, length of the code $C_q(I(\mathbb{F}_5))$ is 30. Also, from the Lemma 6.1, dimension of the code $C_q(I(\mathbb{F}_5))$ is 19. By Magma computation, we get $d_H(C_q(I(\mathbb{F}_5))) = 3$. □

The incidence matrix $I(\mathbb{F}_7)$ of the graph $\mathbb{F}_7$ is given below.

$$I(\mathbb{F}_7) = \begin{bmatrix}
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&1
\end{bmatrix}$$

Theorem 6.4. *Let $C_q(I(\mathbb{F}_7)) \subseteq \mathcal{F}_q^{42}$ $(q - prime)$ be a linear code generated by incidence matrix $I(\mathbb{F}_7)$ of graph $\mathbb{F}_7$. Then, the code $C_q(I(\mathbb{F}_7))$ has parameters $[42, 27, 3]_q$.*

Proof. The graph $\mathbb{F}_7$ has 28 vertices and 42 edges and the code $C_q(I(\mathbb{F}_7))$ is generated by incidence matrix $I(\mathbb{F}_7)$. Therefore, length of the code $C_q(I(\mathbb{F}_7))$ is 42. Following Lemma 6.1, the dimension of the code $C_q(I(\mathbb{F}_7))$ is 27. Further, with the help of Magma software, we get $d_H(C_q(I(\mathbb{F}_7))) = 3$. □

Remark 6.5. It is easy to see that the parameters of linear codes generated by incidence matrices of the generalized Franklin graphs $\mathbb{F}_p$ are $[6p, 4p - 1, 3]_q$ for odd integers $p \geq 3$.

6.2 Linear Codes obtained from distance matrix

The distance matrix $D(\mathbb{F}_3)$ of the graph $\mathbb{F}_3$ is given below. The rows and columns of the matrix $I(\mathbb{F}_3)$ are labeled by vertices of the graph $\mathbb{F}_3$.

$$D(\mathbb{F}_3) = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}$$

Theorem 6.6. *Let $C_2(D(\mathbb{F}_3)) \subseteq \mathcal{F}_2^{12}$ be a linear code generated by distance matrix $D(\mathbb{F}_3)$ of graph $\mathbb{F}_3$. Then the code $C_2(D(\mathbb{F}_3))$ has parameters* $[12,2,6]_2$.

Proof. The graph $\mathbb{F}_3$ has 12 vertices and the code $C_2(D(\mathbb{F}_3))$ is generated by distance matrix $D(\mathbb{F}_3)$. Therefore, length of the code $C_2(D(\mathbb{F}_3))$ is 12. Further, with the help of Magma software, the dimension of the code 2 and $d_H(C_2(D(\mathbb{F}_3))) = 6$. □

The distance matrix $D(\mathbb{F}_5)$ of the graph $\mathbb{F}_5$ is given below.

$$D(\mathbb{F}_5) = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}$$

Theorem 6.7. *Let $C_2(D(\mathbb{F}_5)) \subseteq \mathcal{F}_2^{20}$ be a linear code generated by distance matrix $D(\mathbb{F}_5)$ of graph $\mathbb{F}_5$. Then, the code $C_2(D(\mathbb{F}_5))$ has parameters* $[20,2,10]_2$.

Proof. The graph $\mathbb{F}_5$ has 20 vertices and the code $C_2(D(\mathbb{F}_5))$ is generated by distance matrix $D(\mathbb{F}_5)$. Therefore, length of the code $C_2(D(\mathbb{F}_5))$ is 20. Further, with the help of Magma software, we get the dimension of the code 2 and $d_H(C_2(D(\mathbb{F}_5))) = 10$. □

The distance matrix $D(\mathbb{F}_7)$ of the graph $\mathbb{F}_5$ is given below.

$$D(\mathbb{F}_7) = \begin{bmatrix}
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\
1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\
0&1&0&1&0&1&0&1&0&1&0&1&0&1&1&0&1&0&1&0&1&0&1&0&1&0&1&0
\end{bmatrix}$$

Theorem 6.8. *Let $C_2(D(\mathbb{F}_7)) \subseteq \mathcal{F}_2^{28}$ be a linear code generated by distance matrix $D(\mathbb{F}_7)$ of graph $\mathbb{F}_7$. Then, the code $C_2(D(\mathbb{F}_7))$ has parameters $[28, 2, 14]_2$.*

Proof. The graph $\mathbb{F}_7$ has 28 vertices and the code $C_2(D(\mathbb{F}_7))$ is generated by distance matrix $D(\mathbb{F}_7)$. Therefore, length of the code $C_2(D(\mathbb{F}_7))$ is 28. Further, with the help of Magma software, we get the dimension of the code 2 and $d_H(C_2(D(\mathbb{F}_7))) = 14$. □

S.N.	Linear codes	Field	Parameter
1.	$C_2(D(\mathbb{F}_3))$	GF(2)	[12,2,6]
2.	$C_2(D(\mathbb{F}_5))$	GF(2)	[20,2,10]
3.	$C_2(D(\mathbb{F}_7))$	GF(2)	[28,2,14]

Remark 6.9. It is easy to see that the parameters of the linear codes generated by distance matrices of the generalized Franklin graphs $\mathbb{F}_p$ are $[4p, 2, 2p]_2$ for odd integers $p \geq 3$.

7 Conclusion

In this paper, we have introduced the graphs $\mathbb{F}_p$ for infinitely many odd integers $p \geq 3$ and studied their properties. It is proved that graph $\mathbb{F}_p$ is Hamiltonian, triangle-free with clique number 2, and girth 4. However, the chromatic number of the graph $\mathbb{F}_p$ is 2. This class of graphs is a generalization of the Franklin graph for infinitely many odd integers $p \geq 3$. This class of graphs possessed the property of the Franklin graph. We have determined the independence number, domination number, and Roman domination number of the graph $\mathbb{F}_p$. Further, the values of various degree-based topological indices of graphs $\mathbb{F}_p$ have been obtained. Finally, we have obtained parameters of the linear codes generated by the incidence and distance matrices of the generalized Franklin graphs.

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