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Spectral Method for Kolmogorov Equations: A Numerical Approach to Forward and Backward Diffusion Problems

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Abstract. This paper develops a numerical method for solving the linear and nonlinear Kolmogorov backward equation (KBE) and its adjoint, the Kolmogorov forward equation (KFE), which describe continuous-time Markov diffusion processes.

We apply a spectral discretization of the homogeneous initial–boundary value problem, using quadrature formulas to approximate both the right-hand side and the linear form in the variational formulation. The method relies on orthogonal polynomials and matrix-analytic techniques.

An orthogonal transformation reduces the multidimensional problem to a one-dimensional system of ordinary differential equations, greatly simplifying computation.

The resulting scheme is accurate, efficient, and well suited for numerical simulation of KBE and KFE in diffusion models.

1 Introduction and Motivation

In one spatial dimension, we consider an Itô process that is governed by the standard Wiener process (Brownian Motion) $(W_t)_t$ and can be mathematically described using a stochastic differential equation (SDE) (see, e.g., [17]):

$$dX_t = \mu\left(X_t, t\right) dt + \sigma\left(X_t, t\right) dW_t \tag{1.1}$$

where $(X_t)_t$ is the stochastic process, and $\mu\left(X_t, t\right)$ is the drift coefficient, $\beta\left(X_t, t\right) = \sigma^2\left(X_t, t\right)/2$ is the diffusion coefficient.

The backward Kolmogorov equation is a partial differential equation (PDE) that governs how the probability distribution function (PDF) of a stochastic process evolves, particularly for Markov processes. Unlike the Fokker-Planck equation (forward Kolmogorov equation), which progresses forward in time, this equation operates backward in time [23].

The backward Kolmogorov equation is derived from the Chapman-Kolmogorov equation and describes how the probability distribution evolves backward in time. It has the following general form:

$$\frac{\partial P(x,t)}{\partial t} + \mu(x,t)\frac{\partial P(x,t)}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 P(x,t)}{\partial x^2} = 0 \tag{1.2}$$

The boundary condition at $t = 0$ would typically be a known function, such as $P(x, 0) = f(x)$.

The relationship between Itô stochastic differential equations (SDEs) and the Fokker–Planck equation is essential in modeling natural [13], engineering, and financial systems [19]. Differential equations, both ordinary and partial, linear and nonlinear, are fundamental tools across the sciences. They are widely used in physics, chemistry, biology, and engineering to describe complex phenomena.

Differential equation models play a key role in engineering, where they help design efficient and reliable systems [5]. Many of these equations are difficult to solve exactly, and analytical solutions are often not feasible.

The development of computers changed this situation dramatically [22]. Scientists shifted from searching for exact solutions to computing numerical approximations. These approximations are practical, efficient, and suitable for studying complex systems.
Several semi-analytical techniques such as the Adomian Decomposition Method (ADM) [28], the Homotopy Perturbation Method (HPM) [30], the Variational Iteration Method (VIM) and its interpolated variants [1], as well as the Daftardar–Gejji and Jafari method (DJM) [9], often struggle to produce high-accuracy solutions for Kolmogorov and Fokker–Planck equations. These approaches rely on recursive correction terms or perturbation structures, which tend to converge slowly when the diffusion operator is stiff or when the solution exhibits sharp gradients, boundary layers, or multi-modal profiles.

For nonlinear drift or diffusion coefficients—as appears in the general Fokker–Planck equation—the correction terms generated by these methods become increasingly intricate, leading to higher computational cost and, in some cases, reduced numerical stability. Furthermore, such iterative techniques do not take advantage of the self-adjoint or symmetrizable structure of Kolmogorov operators , and their convergence rate is typically algebraic rather than exponential.

In contrast, spectral methods [16, 10, 20] expand the solution in orthogonal polynomial bases that naturally capture the smoothness of the exact solution. This representation leads to matrices that are diagonal or nearly diagonal after an orthogonal transformation, reducing the PDE to a system of uncoupled or weakly coupled ODEs. Such structure enables exponential accuracy with relatively few grid points, making spectral discretizations particularly suitable for high-precision numerical solutions of the Kolmogorov backward and forward equations.

The present paper has been organized as follows. Section 2 is devoted to the description of the Fokker–Planck equation. In Section 3 the basic idea of the Spectral Method and its convergence are illustrated. In Section 4 we present the description of the new numerical method. In Section 5 some test examples are solved by our new method. Finally in Section 6 the conclusion is presented.

2 The Fokker-Planck Equation: Theory and Formulations

The Fokker–Planck equation, introduced by Fokker and Planck [23], is a central tool for modeling Brownian motion and general stochastic processes [25, 29, 3]. It plays a key role in many areas of science, including solid-state physics, quantum optics, chemical physics, theoretical biology, and electrical circuit theory.

2.1 Physical Origins and Basic Form

Originally formulated to describe Brownian motion (see, e.g., [14]), the Fokker–Planck equation governs the time evolution of the probability density $P(v,t)$.

$$\frac{\partial P}{\partial t} = \gamma \frac{\partial (vP)}{\partial v} + \gamma \frac{kT}{m} \frac{\partial^2 P}{\partial v^2}, \tag{2.1}$$

where m is the mass of the particle, v is the velocity for the Brownian motion, γ represents the friction coefficient, k is Boltzmann's constant, and T denotes the fluid temperature.
$P(v,t)$ should satisfy the condition of conservation of mass

$$\int P(v,t)\, dv = \int P(v,0)\, dv = 1.$$

where This fundamental formulation provides the framework for analyzing the stochastic dynamics of particles. To fully specify the evolution of $P(v,t)$, one must define appropriate initial conditions. These initial states determine how the probability distribution begins and directly influence the subsequent temporal evolution governed by the Fokker–Planck equation.

Typical choices for the initial condition $P(v,0)$ include a Dirac delta (localized start), a uniform distribution (random initialization), a Gaussian density (uncertain initial state), or multi-modal mixtures (multiple starting configurations) [11, 15, 17, 23, 18, 21].

The Fokker–Planck (FP) equation, also known as the forward Kolmogorov equation, describes the time evolution of a probability density $P(x,t)$ under drift and diffusion processes; in one dimension with space-dependent coefficients it reads

$$\frac{\partial P}{\partial t} = \left[-\frac{\partial}{\partial x}\mu(x) + \frac{\partial^2}{\partial x^2}\beta(x)\right] P, \qquad P(x,0) = f(x),\ x \in \mathbb{R}, \tag{2.2}$$

and admits a time-dependent generalization

$$\frac{\partial P}{\partial t} = \left[-\frac{\partial}{\partial x}\mu(x,t) + \frac{\partial^2}{\partial x^2}\beta(x,t)\right] P. \tag{2.3}$$

The adjoint formulation, known as the backward Kolmogorov equation, is given by

$$\frac{\partial P}{\partial t} = \left[-\mu(x,t)\frac{\partial}{\partial x} + \beta(x,t)\frac{\partial^2}{\partial x^2}\right] P, \tag{2.4}$$

while for nonlinear systems the drift and diffusion coefficients may depend explicitly on the probability density, leading to the nonlinear Fokker–Planck equation

$$\frac{\partial P}{\partial t} = \left[-\frac{\partial}{\partial x}\mu(x,t,P) + \frac{\partial^2}{\partial x^2}\beta(x,t,P)\right] P, \tag{2.5}$$

which exhibits rich dynamical behavior.

This section outlines the numerical method employed to solve the parabolic equation under study. The equation, defined on the space–time domain $\Omega = \mathcal{B} \times [0,+\infty)$, contains advection and diffusion with spatially varying coefficients. A spectral collocation approach is applied in space to achieve high-order accuracy, resulting in a semi-discrete system that is readily integrated in time. The main steps of the scheme are described below.

3 Details of the Proposed Numerical Method

We consider the problem,

$$\begin{cases} \frac{\partial}{\partial t}P(x,t) = -\mu(x)\frac{\partial}{\partial x}(P(x,t)) + \beta(x)\frac{\partial^2}{\partial x^2}(P(x,t)), & (x,t) \in \Omega \\ P(x,0) = f(x) \end{cases} \tag{3.1}$$

where $\Omega = \mathcal{B} \times [0,+\infty[$.

In this work, we construct an approximate solution to the boundary value problem (3.1) in the following form (see, e.g., [4, 27]):

$$P_N(x,t) = \sum_{n=0}^{N} a_n(t)\phi_n(x). \tag{3.2}$$

where $\phi_n(x), 0 \le n \le N$, are the Lagrangian interpolates at the points $x_i \in B = [-1,1]$, $0 \le i \le N$ defined by the formula

$$\phi_n(x) = \prod_{k=0,k\neq n}^{N} \frac{x - x_k}{x_n - x_k} \tag{3.3}$$

If $\mathcal{B} = [a,b], -\infty < a < b < +\infty$, we can return to [-1,1] via the bijection $k : [a,b] \to [-1,1]$ be a function such that: $k(x) = \frac{2(x-a)}{b-a} - 1$.

In the remainder of this work, we consider the interval $\mathcal{B} = [-1, 1]$.

These interpolates satisfy the property $\phi_n(x_j) = \delta_{nj}$, $1 \leq n, j \leq N - 1$, where δ_{nj} is the Kronecker delta, and the points $x_j \in [-1, 1]$, $0 \leq j \leq N$, are the collocation points on the Gauss-Lobatto Legendre grid [26, 12].

The choice of the form (3.2) for the approximate solution, combined with some techniques, leads to a linear system which can be written in matrix form as $\Lambda a' + Aa = 0$, where A is a square matrix, Γ is a diagonal invertible matrix.
We write $a = \Delta v$ where Δ is an orthogonal matrix such that $\Delta^{-1}(\Lambda^{-1}A)\Delta = C$.

We work on the domain $\mathcal{B}$ using the Legendre polynomials L_n, $n \geq 0$. Each polynomial L_n has degree n and the family $\{L_n\}_{n\geq 0}$ is orthogonal in the Hilbert space

$$L^2(\mathcal{B}) = \left\{ g : \mathcal{B} \to \mathbb{R} \,\middle|\, g \text{ measurable and } \int_{-1}^{1} g^2(x)\, dx < \infty \right\},$$

with the orthogonality relation

$$\int_{-1}^{1} L_n(x) L_m(x)\, dx = \frac{2}{2n+1}\, \delta_{nm},$$

where δ_{nm} denotes the Kronecker delta (see, e.g., [8, 6]).

$$h_n'(x) = -n(n+1)L_n(x), \qquad h_n(x) = (1-x^2)\, L_n'(x), \quad n \geq 0,$$

$$h_n(x) = \frac{n(n+1)}{2n+1}\big(L_{n-1}(x) - L_{n+1}(x)\big),$$

$$\|h_n\|^2_{L^2(\mathcal{B})} = \frac{4[n(n+1)]^2}{(4n^2-1)(2n+3)}.$$

4 Numerical Experiment

To introduce the variational formulation for the continuous problem (3.1), we define the subspace of the variational space with homogeneous Dirichlet boundary conditions [6, 7]:

$$\mathcal{A}_0(\mathcal{B}) = \left\{ Q \in L^2(\mathcal{B}) \,\middle|\, Q = f \text{ at } t = 0 \right\}.$$

We introduce the product in $L^2(\mathcal{B})$.
For functions P, Q in $L^2(\mathcal{B})$ as:

$$\langle P, Q \rangle = \int_{\mathcal{B}} P(x,t)\, Q(x,t)\, dx. \tag{4.1}$$

Then, the variational formulation of the continuous problem (3.1) is:
Find $u \in L^2(\mathcal{B})$ such that

$$a(P, Q) = \langle P, Q \rangle, \quad \forall Q \in \mathcal{A}_0(\mathcal{B}), \tag{4.2}$$

where $a(\cdot,\cdot)$ is a bilinear form defined on $L^2(\mathcal{B}) \times \mathcal{A}_0(\mathcal{B})$, and $\langle P, Q \rangle$ is the duality pairing associated with the right-hand side P.

4.1 Quadrature Formula

The approximate spaces are essentially generated by the finite-dimensional subspace of $L^2(\mathcal{B})$.
We denote by $\mathbb{P}_N(\mathcal{B})$ the space of polynomials of degree at most N on $\mathcal{B}$.

$$IP_N(\mathcal{B}) = \left\{ P_N \in \mathbb{P}_N(\mathcal{B}), \quad P_N = \sum_{i=0}^{N} f(x_j)\phi(x_j) \right\}$$

Proposition 4.1. *There exists a unique set of* $N-1$ *nodes* x_j, $1 \leq j \leq N-1$ *in* L *and, with the condition* $x_0 = -1$, $x_N = 1$, *there exists* $N+1$ *positive weights* ρ_j, $0 \leq j \leq N$, *such that the following exactness property holds:*
For all $g \in \mathbb{P}_{2N-1}(L)$,

$$\int_{-1}^{1} g(x)\, dx = \sum_{j=0}^{N} g(x_j)\rho_j \tag{4.3}$$

where x_j, $1 \leq j \leq N-1$, *are the roots of the Legendre polynomial* $L'_N(x)$, *and the weights are given by:*

$$\begin{cases} \rho_0 = \rho_N = \dfrac{2}{N(N+1)} \\ \qquad \rho_j = \dfrac{\rho_0}{L_N^2(x_j)} \qquad 1 \leq j \leq N-1 \end{cases} \tag{4.4}$$

Proof. See [2].
The proposition describes the Gauss–Lobatto quadrature rule, which exactly computes the integral of any polynomial up to degree $2N-1$ using $N+1$ points, including the endpoints -1 and 1. The interior nodes are chosen as the zeros of the derivative of the Legendre polynomial $L'_N(x)$, and they cluster near the endpoints for optimal accuracy. The positive weights ρ_j are selected so that the discrete weighted sum matches the exact integral for all polynomials of degree $\leq 2N-1$—the maximum possible exactness for a rule with $N+1$ nodes including the endpoints.

We define the discrete inner product for all polynomials $P_N, Q_N \in \mathbb{P}_N(L)$ as:

$$a_N(P_N, Q_N) = \sum_{l=0}^{N} P_N(x_l, t)\, Q_N(x_l, t)\, \rho_l \tag{4.5}$$

5 Discrete Variational Formulation

In this work, we construct the approximate solution to the problem in the form

$$P_N(x,t) = \sum_{n=0}^{N} a_n(t)\phi_n(x)$$

The variational formulation of the problem (3.1) can be written as:

$$\begin{cases} \text{Find } P_N \in IP_N(\mathcal{B}) \text{ such that} \\ (\partial_t P_N,\, Q_N) = a_N(P_N, Q_N), \qquad \forall\, Q_N \in IP_N(\mathcal{B}). \end{cases} \tag{5.1}$$

The formulation (5.1) holds for every test function $Q_N \in \mathbb{P}_N(\mathcal{B})$. In particular, it is valid for the basis functions $Q_m(x) = \phi_m(x)$, $m = 0, \ldots, N$, where $\{\phi_m(x)\}_{m=0}^{N}$ spans the polynomial space $\mathbb{P}_N(\mathcal{B})$.
Since the polynomial P_N has degree $2N$ with respect to the variable x, and the nodes $x_j \in [-1, 1]$, $0 \leq j \leq N$, represent the collocation points of the Gauss–Lobatto–Legendre grid, the formulation (5.1) can then be rewritten as:

$$\begin{cases} \text{Find } P_N \in IP_N(\mathcal{B}) \text{ such that for all } \phi_m \in IP_N(\mathcal{B}), \\ (\partial_t P_N,\, Q_N) = \displaystyle\sum_{n=0}^{N} \left(\sum_{k=0}^{N} \phi_n(x_k)\, \phi_m(x_k)\, \rho_k \right) a'_n(t), \\ a_N(P_N, Q_N) = \displaystyle\sum_{n=0}^{N} \left(\sum_{k=0}^{N} \big(-\mu(x_k, t)\, \phi'_n(x_k) + \beta(x_k, t)\, \phi''_n(x_k) \big) \phi_m(x_k)\, \rho_k \right) a_n(t). \end{cases} \tag{5.2}$$

Using the property $\phi_n(x_k) = \delta_{nk}$, (where δ_{nk} is the Kronecker delta), equation (5.2) simplifies to

$$\begin{cases} a'_m(t)\,\rho_m = \Big(-\mu(x_m,t)\,\phi'_m(x_m) + \beta(x_m,t)\,\phi''_m(x_m)\Big)\,\rho_m\,a_m(t), & m = 0,\dots,N, \\ P_N(x_m,0) = f(x_m), & t = 0. \end{cases} \tag{5.3}$$

5.1 Approximate Solution

We obtain a linear system that can be expressed in matrix form as

$$\Lambda\,a' + A\,a = 0, \tag{5.4}$$

where $A = (\alpha_{mn})_{0\le n,m\le N}$ is a symmetric positive definite matrix of order $N+1$ whose entries are given by

$$\alpha_{mn} = \big(-\mu(x_m,t)\,\phi'_n(x_m) + \beta(x_m,t)\,\phi''_n(x_m)\big)\,\rho_m, \qquad m,n = 0,\dots,N,$$

and $\Lambda = (\lambda_{mn})_{0\le n,m\le N}$ is an invertible diagonal matrix with strictly positive diagonal entries.

$$\lambda_{mn} = \rho_m\,\delta_{mn}, \qquad m,n = 0,\dots,N,$$

where δ_{mn} denotes the Kronecker symbol. The unknown vector a is defined by

$$a = \big(a_0(t), a_1(t), \dots, a_N(t)\big)^t.$$

Since $\Lambda = \mathrm{diag}(\rho_0,\dots,\rho_N)$, we have

$$\Lambda^{-1} = \mathrm{diag}\left(\frac{1}{\rho_0},\dots,\frac{1}{\rho_N}\right).$$

Therefore, the entries of the matrix $\Lambda^{-1}A$ are

$$(\Lambda^{-1}A)_{mn} = \frac{1}{\rho_m}\,\alpha_{mn} = \frac{1}{\rho_m}\Big(-\mu(x_m,t)\,\phi'_n(x_m) + \beta(x_m,t)\,\phi''_n(x_m)\Big)\rho_m, \qquad m,n = 0,\dots,N.$$

The factors ρ_m cancel, and we obtain

$$(\Lambda^{-1}A)_{mn} = -\mu(x_m,t)\,\phi'_n(x_m) + \beta(x_m,t)\,\phi''_n(x_m), \qquad m,n = 0,\dots,N.$$

Hence,

$$\Lambda^{-1}A = \big(-\mu(x_m,t)\,\phi'_n(x_m) + \beta(x_m,t)\,\phi''_n(x_m)\big)_{0\le m,n\le N}.$$

Multiplying (5.4) by the inverse matrix Λ^{-1}, which exists since Λ is diagonal with strictly positive entries, we obtain the equivalent system

$$a' + \Lambda^{-1}A\,a = 0. \tag{5.5}$$

The matrix $\Lambda^{-1}A$ has positive eigenvalues, and there exists an orthogonal invertible matrix Δ such that

$$\Delta^{-1}(\Lambda^{-1}A)\,\Delta = C,$$

where C is a diagonal matrix whose diagonal elements c_m, $m = 0,\dots,N$, are the eigenvalues of $\Lambda^{-1}A$.

If we define the vector Q by

$$a = \Delta\,Q,$$

then the system (5.5) becomes

$$\Delta\,Q' + (\Lambda^{-1}A)\,\Delta\,Q = 0. \tag{5.6}$$

Multiplying (5.6) by Δ^{-1} yields the decoupled system

$$Q' + C\,Q = 0, \tag{5.7}$$

which corresponds to $N+1$ independent linear equations:

$$Q'_m(t) + c_m\,Q_m(t) = 0, \quad m = 0,\dots,N. \tag{5.8}$$

The solution of (5.8) is explicitly

$$Q_m(t) = \lambda_m\,e^{-c_m t}, \quad m = 0,\dots,N. \tag{5.9}$$

Using the initial conditions, the coefficients λ_m can be written as

$$\lambda_m = \sum_{j=0}^{N} p^{-1}_{mj} f(x_j), \qquad Q_m(t) = \sum_{j=0}^{N} p^{-1}_{mj} f(x_j)\,e^{-c_m t}.$$

where p^{-1}_{mj} are the elements of the matrix Δ^{-1}.

Finally, the coefficients $a_n(t)$ are given by

$$a_n(t) = \sum_{m=0}^{N} p_{nm} \sum_{j=0}^{N} p^{-1}_{mj} f(x_j)\,e^{-c_m t}, \quad n = 0,\dots,N,$$

where p_{nm} are the elements of the matrix Δ.
The approximate solution of the original problem is then

$$P_N(x,t) = \sum_{n=0}^{N} a_n(t)\,\phi_n(x) = \sum_{n=0}^{N} \left(\sum_{m=0}^{N} p_{nm} \sum_{j=0}^{N} p^{-1}_{mj} f(x_j)\,e^{-c_m t} \right) \phi_n(x).$$

The computational complexity of the spectral collocation method consists of a one-time setup cost and a low-cost evaluation stage. Assembling the system matrices requires $\mathcal{O}(N^2)$ operations, while solving the resulting semi-discrete problem by diagonalizing the dense matrix incurs a cost of $\mathcal{O}(N^3)$. This overhead is paid only once and can be amortized over the full time-dependent simulation. After diagonalization, evaluating the spectral approximation at a given space–time point requires only $\mathcal{O}(N)$ operations, consistent with classical results for spectral discretizations and dense eigenvalue problems (see, e.g., [8]).

5.2 The Behavior of the A Posteriori Error

If the exact solution $P(\cdot,t)$ is analytic with respect to the spatial variable x for each $t \in [0,T]$, the coefficients of the differential operator and the boundary data are analytic, and the spatial domain (a,b) has a smooth boundary, then the spectral approximation $P_N(\cdot,t)$ converges exponentially fast.

Theorem 5.1 (Spectral Accuracy for Analytic Solutions). *Let $P(x,t)$ be the solution of a homogeneous linear initial–boundary value problem on the spatial domain (a,b) with smooth boundary. Assume that, for each $t \in [0,T]$, the solution $P(\cdot,t)$ is analytic in x, the coefficients of the differential operator are analytic, and the boundary data are analytic. Let $P_N(x,t)$ denote the spectral approximation of $P(x,t)$ using polynomials of degree N. Then there exist constants $C > 0$ and $\alpha > 0$, independent of N, such that*

$$\|P(\cdot,t) - P_N(\cdot,t)\|_{L^2(a,b)} \le C\,\mathrm{e}^{-\alpha N}, \qquad 0 \le t \le T.$$

This estimate shows that the spectral method converges exponentially fast for analytic solutions, a property commonly referred to as spectral accuracy *(see, e.g., [24, 27]).*

5.3 Figure Illustration

The figures of the section (5.3) present the approximate solution P_N and the true solution P respectively, these plots occur when $N = 10$ and the true solution is $P(x,t) = 2(x-t)$ of the equation

$$\begin{cases} \frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}(P(x,t)) + \frac{\partial^2}{\partial x^2}(P(x,t)), & (x,t) \in [0,1] \times [0,100], \\ P(x,0) = 2x \end{cases}$$

with the normalization

$$\int_0^1 P(x,0)\,dx = 1.$$

We use the change of variables

$$y = 2x - 1, \qquad x = \frac{y+1}{2}, \qquad Q(y,t) = P\left(\frac{y+1}{2}, t\right).$$

Thus the Partial Differential Equation on $[-1,1]$ becomes

$$\begin{cases} Q_t(y,t) = -2\,Q_y(y,t) + 4\,Q_{yy}(y,t), & (y,t) \in [-1,1] \times [0,100], \\ Q(y,0) = \dfrac{y+1}{2}, & y \in [-1,1]. \end{cases}$$

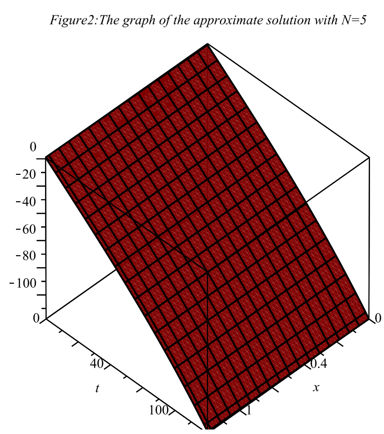

Figure 1. The Approximate Solution

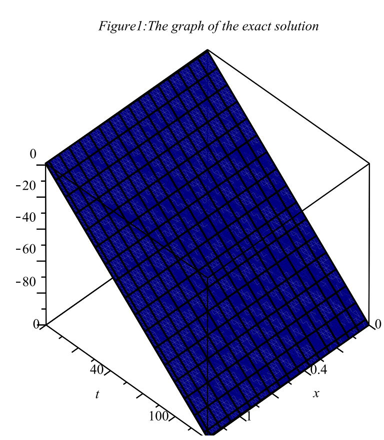

Figure 2. The exact solution

Table 1. The behavior of the error when N vary from 5 to 12

N	5	6	7	8
	6.906×10^{-3}	3.164×10^{-4}	1.175×10^{-6}	4.425×10^{-9}
N	9	10	11	12
	2.617×10^{-10}	1.522×10^{-11}	9.558×10^{-13}	5.506×10^{-14}

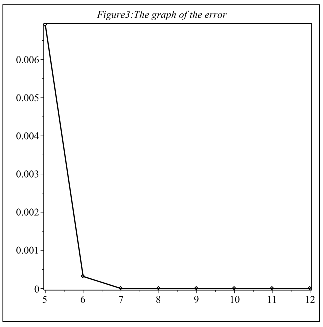

Figure 3. The behavior of the error when N vary from 5 to 12

Table 1 reports the $L^2(\Omega)$-error for polynomial degrees $N = 5, \ldots, 12$, over the domain $\Omega = [0, 1] \times [0, 100]$.

Figure (3) illustrates the behavior of the numerical error as the parameter N varies from 5 to 12. It is clearly observed that the error decreases monotonically with increasing N, which indicates an improvement in accuracy and confirms the convergence of the proposed numerical method.

6 Conclusion

This paper presented a spectral method for solving Kolmogorov backward and forward equations. The approach combines a variational spectral discretization using Legendre polynomials and Gauss–Lobatto quadrature for high-order accuracy and stability with an orthogonal transformation that diagonalizes the discretized system, reducing the PDEs to uncoupled ODEs solvable analytically. Numerical results demonstrate exponential convergence, achieving errors below 10^{-14} for $N = 12$, while the method remains flexible in handling nonlinear coefficients .
Future work will extend the method to fractional Kolmogorov equations and multidimensional domains, leveraging the spectral framework's ability to handle non-local operators and complex boundary conditions.

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Algorithmic Implementation

Algorithm 6.1 (H). Spectral Collocation Algorithm for the Approximate Solution

Polynomial degree $N = 5$; coefficients $\mu(x)$, $\beta(x)$; exact solution $P(x,t)$; source function $P(x,0)$ Approximate solution $P_a(x,t)$

Step 1: Initialization

Step 2: Auxiliary Legendre-based functions $k = 1\ N$

$$h_1(k,s) = P_k(2s-1)^2, \qquad h(k,s) = \frac{d}{ds}P_k(2s-1).$$

Step 3: Gauss–Lobatto collocation nodes Solve $h(N,s)=0$ to obtain the interior roots. Remove duplicate roots, reorder them, and define

$$x_0=-1,\quad x_N=1,\quad x_k=s_k,\ k=1,\ldots,N-1.$$

Step 4: Quadrature weights Set

$$\rho_0=\rho_N=\frac{2}{N(N+1)}.$$

$k=1\ N-1$

$$\rho_k=\frac{\rho_0}{h_1(N,x_k)}.$$

Step 5: Evaluate the source function $k=0\ N$

$$f_k=f(x_k).$$

Step 6: Lagrange basis polynomials $n=0\ N$ Construct

$$\phi_n(x)=\prod_{\substack{j=0\\ j\neq n}}^{N}\frac{x-x_j}{x_n-x_j}.$$

Compute first and second derivatives $\phi_n'(x)$ and $\phi_n''(x)$.

Step 7: Evaluation at collocation points $k_1,k_2=0\ N$

$$\phi_{k_1}(x_{k_2}),\quad \phi_{k_1}'(x_{k_2}),\quad \phi_{k_1}''(x_{k_2}).$$

Algorithm 6.2 (H). **Step 8: Assembly of matrices** Define the diagonal matrix

$$A_1=\mathrm{diag}(\rho_0,\rho_1,\ldots,\rho_N).$$

Construct the mass matrix

$$(A_{11})_{nm}=\sum_{j=0}^{N}\phi_n(x_j)\phi_m(x_j)\rho_j.$$

Define

$$g_n(x,t)=\frac{d}{dx}\big[-\mu(x)\phi_n(x)\big]+\frac{d^2}{dx^2}\big[\beta(x)\phi_n(x)\big].$$

Construct the stiffness-like matrix

$$(A_2)_{nm}=\sum_{j=0}^{N}2\,g_n(x_j,t)\phi_m(x_j)\rho_j.$$

Step 9: Generalized eigenvalue problem Compute

$$G_1=A_1^{-1},\qquad M=G_1A_2.$$

Compute eigenvalues and eigenvectors of M. Form matrix P from eigenvectors and compute P^{-1}.

Step 10: Diagonalization Compute

$$C=P^{-1}MP=\mathrm{diag}(c_0,\ldots,c_N).$$

Step 11: Time-dependent coefficients $k=0\ N$

$$\beta_k=\sum_{j=0}^{N}(P^{-1})_{kj}f(x_j),\qquad v_k(t)=\beta_k e^{c_k t}.$$

Step 12: Reconstruction of solution coefficients $k = 0\ N$

$$a_k(t) = \sum_{j=0}^{N} P_{kj} v_j(t).$$

Step 13: Approximate solution

$$u_a(x,t) = \sum_{n=0}^{N} a_n(t)\, \phi_n(x).$$

Step 14: Visualization Plot the exact solution $u(x,t)$ and the approximate solution $u_a(x,t)$.

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