Palestine Journal of Mathematics

Vol 15(1)(2026) , 10–20

JORDAN MAPPINGS OF SEMIRINGS

S. Ganesh and V. Selvan

Communicated by: Francesco Rania

MSC 2020 Classifications: Primary 16Y60, 16W10; Secondary 16N60, 16W25.

Keywords and phrases: Semirings, Jordan homomorphism, Jordan derivations, Jordan triple homomorphisms, Jordan triple derivations.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: V. Selvan.

Abstract. This article examines Jordan homomorphisms and Jordan derivations in semirings. We prove that if f is a Jordan homomorphism from an additively cancellative semiring R to an additively cancellative yoked prime semiring R', then f is either a homomorphism or an anti-homomorphism. We define Jordan triple derivation in semirings and prove that it is just a usual derivation in additively cancellative 2-torsion free yoked semiprime semirings. Furthermore, we demonstrate that every generalized Jordan derivation and generalized Jordan triple derivation is, in fact, a derivation in a certain class of semirings.

1 Introduction

Morphisms and Derivations are specialized functions in rings that have been extensively studied by several researchers. Herstein [10], in 1956, proved that if f is a Jordan homomorphism from a ring R to a prime ring R' of characteristic different from 2 and 3, then f is either a homomorphism or an anti-homomorphism. In 1979, Baxter and Martindale [2] showed that Herstein's result does hold in semiprime rings by constructing a mapping as the direct sum of a homomorphism and an anti-homomorphism. Subsequently, Brešer [4], in 1989, extended on this result in semiprime rings and obtained further results. A major challenge in extending these classical ring theory results to semirings is the lack of an additive inverse. Therefore, generalizing these results to semirings is nontrivial and is an important direction. In 2017, Shafiq and Aslam [17] defined Jordan homomorphism and Jordan triple homomorphism in the class of inverse semirings and extended some results of Herstein and Brešer. In 2019, Ibraheem and Majeed [13] introduced the concept of U-S Jordan homomorphism of inverse semirings, further extending Herstein's results. Motivated by this line of research, in Section 2, we define Jordan homomorphism and Jordan triple homomorphism in a general semiring setting. We then generalize the results of Herstein and Brešer to additively cancellative yoked semirings, a particular class of semirings. Note that this class of semirings has been less explored in the literature.

In the latter half of the 20th century, derivations and Jordan derivations were studied by various researchers, and numerous results were derived. Posner's theorems in prime rings [15] are a pioneering work in this direction. In 1957, Herstein [11] established that for a 2-torsion free prime ring, Jordan derivation is just a usual derivation. Brešer [3], in 1988, extended Herstein's result to semiprime rings. Some related studies on generalized Jordan higher homomorphisms can be seen in [6] and on generalized derivations can be found in [16]. The authors have generalized the results of Herstein and Brešer to additively cancellative yoked prime (and semiprime) semirings in [7, 8].

In 1989, Brešer [4] extended the research and proved that a Jordan triple derivation of a 2-torsion free semiprime ring is just a derivation. Wu Jing and Shijie Lu [18] generalized this result in the year 2003. They showed that in a 2-torsion free prime ring, every generalized Jordan derivation and generalized Jordan triple derivation is just a derivation. In 2007, Vukman [14] extended these results to the class of semiprime rings. Shafiq and Aslam [17], in 2017, explored Brešer's results in the context of inverse semirings. Recently, in 2021, Ahmed and Dudek [1] established conditions for a left Jordan derivation on an MA-semiring to be a left derivation. Inspired by these, in Section 3 of this research work, we prove that in additively cancellative 2-torsion free yoked prime (and resp. semiprime) semirings, Jordan triple derivation, generalized Jordan derivation and generalized Jordan triple derivation are all just a derivation.

We refer to Golan [9] for the basic definitions and fundamental concepts in semirings. The following definitions are helpful throughout the paper.

Definition 1.1. [9] Let R be a semiring and $a, b, c \in R$. R is called additively cancellative if and only if $a + b = a + c$ implies $b = c$.

Definition 1.2. [9] Let R be a semiring and $a, b \in R$. R is called a yoked semiring if there exists an element $r \in R$ such that either $a = b + r$ or $b = a + r$.

Definition 1.3. [5, 9] Let R be an additively cancellative semiring and then the set of differences, defined by $R^\Delta := \{a - b : a, b \in R\}$ becomes a ring under component wise addition, while the multiplication is given by $(a - b)(c - d) = (ac + bd) - (ad + bc)$. In R^Δ, we have $a - b = c - d$ if and only if there exists $r, r' \in R$ such that $a + r = c + r'$ and $b + r = d + r'$. We note that the embedding of R to the ring of differences R^Δ is due to the map $r \mapsto r - 0$, for each $r \in R$.

Definition 1.4. A semiring R is said to be 2-torsion free, if $2r = 0$ implies $r = 0$ for all $r \in R$.

Definition 1.5. [7] Let R be an additively cancellative semiring and f be an additive mapping from R into R. Let R^Δ be the corresponding ring of differences of the semiring R. Then, f^Δ is a mapping in R^Δ induced by f, defined as follows.

$$f^\Delta \ : \ R^\Delta \to R^\Delta$$
$$f^\Delta(a - b) = f(a) - f(b), \quad \forall\, a, b \in R.$$

Note that f^Δ is well-defined because if $a - b = c - d$ then there exists $r, r' \in R$ such that $a + r = c + r'$ and $b + r = d + r'$. Then applying f to these equations results in

$$f(a + r) = f(c + r') \qquad \text{and} \qquad f(b + r) = f(d + r')$$
$$f(a) + f(r) = f(c) + f(r') \qquad \text{and} \qquad f(b) + f(r) = f(d) + f(r')$$

and the last set of equations imply $f(a) - f(b) = f(c) - f(d)$ and therefore we obtain $f^\Delta(a - b) = f^\Delta(c - d)$.

Definition 1.6. [10] A mapping f from a semiring R into a semiring R' is said to be a Jordan homomorphism if for all $r, s \in R$, the following holds.

(i) $f(r + s) = f(r) + f(s)$, and

(ii) $f(rs + sr) = f(r)f(s) + f(s)f(r)$

In the next section it is shown that a Jordan homomorphism f of a semiring R induces a Jordan homomorphism f^Δ in the corresponding ring of differences R^Δ.

Definition 1.7. [8] A function d of R into R is called a Jordan derivation of a semiring R if it satisfies the following conditions.

(i) $d(r + s) = d(r) + d(s), \quad \forall\, r, s \in R$ and

(ii) $d(r^2) = d(r)r + rd(r), \quad \forall\, r, s \in R.$

Remark 1.8. The following points are noteworthy to be recalled from [8].

(i) Ganesh and Selvan [8] proved that the condition (ii) in the above definition is equivalent to $d(rs + sr) = d(r)s + sd(r) + rd(s) + d(s)r$ whenever the semiring R is additively cancellative 2-torsion free and yoked.

(ii) f^Δ defined in Definition 1.5 is indeed a Jordan derivation in R^Δ whenever f is a Jordan derivation of R [8].

Definition 1.9. A map d of R into R is called a Jordan triple derivation of a semiring R if it satisfies the following conditions.

(i) $d(r+s) = d(r) + d(s), \quad \forall\, r, s \in R$ and

(ii) $d(rsr) = d(r)sr + rd(s)r + rsd(r), \quad \forall\, r, s \in R.$

The following example illustrates the existence of Jordan triple derivations that are neither Jordan derivations nor just derivations. This makes it clear that the structure of Jordan triple derivations is more complex than that of basic ones, such as derivations and Jordan derivations. It is therefore intriguing to analyse the conditions under which complex structures become simple.

Example 1.10. Consider the basic semiring $B(3,1) = \{0,1,2\}$ wherein $1+2 = 2+1 = 1$ and $2+2 = 2$. Now let $S := \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in B(3,1) \right\}$. It is easy to see that S is a semiring with I_2 being the identity element in S. Let us define a map $d : S \to S$ by $d\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. For $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in S$, we have $d(ABA) = d(A)BA + Ad(B)A + ABd(A) = \begin{pmatrix} aca & 0 \\ 0 & 0 \end{pmatrix}$. Thus the map d is a Jordan triple derivation. However, $d(A^2) = \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix}$ and $d(A)A + Ad(A) = \begin{pmatrix} 2a^2 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, we see that d is not a Jordan derivation but a Jordan triple derivation. It is also easy to verify that d is not a derivation.

Following definitions are motivated by the work of Jing and Lu [18] and are analogous to the ones in rings.

Definition 1.11. Let R be a semiring and $F : R \to R$ be an additive map. If there exists a Jordan derivation $d : R \to R$ such that

$$F(r^2) = F(r)r + rd(r)$$

for all $r \in R$, then F is said to be a generalized Jordan derivation of R corresponding to d.

Definition 1.12. Let R be a semiring and $F : R \to R$ be an additive map. If there exists a Jordan triple derivation $d : R \to R$ such that

$$F(rsr) = F(r)sr + rd(s)r + rsd(r)$$

for all $r, s \in R$, then F is said to be a generalized Jordan triple derivation of R corresponding to d.

2 Jordan homomorphisms in semirings

In this section, we prove that a Jordan homomorphism f of R induces a corresponding Jordan homomorphism f^Δ in the ring of differences R^Δ corresponding to R. We present some of the basic properties of Jordan homomorphisms in semirings. The main result of this section is that the Jordan homomorphism from an additively cancellative semiring into an additively cancellative prime semiring is either a homomorphism or an anti-homomorphism.

Lemma 2.1. *If R be a 2-torsion free yoked semiring, then $2r = 2s$ implies $r = s$, for all $r, s \in R$.*

Proof. Since R is yoked, for any $r, s \in R$, there exists $r' \in R$ such that $r = s + r'$ or $s = r + r'$. Hence, we have either $r' = r - s \in R$ or $r' = s - r \in R$. Consider the case when $r - s \in R$. If $2r = 2s$ then by yoked property of R, we obtain $2(r - s) = 0$. Since R is 2-torsion free, we further deduce that $r - s = 0$. Now, by yoked property, we have $r - s \in R$ and therefore we obtain $r = s + (r - s) = s + 0 = s$. A similar argument for the case when $s - r \in R$ would yield $s = r$. Hence, in both cases we have proved the result. □

Lemma 2.2. *Let R be a 2-torsion free yoked semiring. If f is a Jordan homomorphism of R to R, then $f(r^2) = (f(r))^2$, for every $r \in R$.*

Proof. Since f is a Jordan homomorphism, we have $f(rs + sr) = f(r)f(s) + f(s)f(r)$, for all $r, s \in R$. In particular, when $r = s$, we obtain

$$\begin{aligned} f(r^2 + r^2) &= f(r)f(r) + f(r)f(r) \\ f(r^2) + f(r^2) &= (f(r))^2 + (f(r))^2 \\ 2f(r^2) &= 2(f(r)^2) \end{aligned}$$

Now, by Lemma 2.1, we get $f(r^2) = (f(r))^2$. □

Lemma 2.3. *Let R and S be an additively cancellative semirings. If f is a Jordan homomorphism of R into S then f^Δ defined by $f^\Delta(r - s) = f(r) - f(s), r - s \in R^\Delta$, where $r, s \in R$, is a Jordan homomorphism of R^Δ into S^Δ.*

Proof. We prove the conditions in Definition 1.6. Let $x, y \in R^\Delta$ such that $x = a-b$ and $y = c-d$, where $a, b, c, d \in R$. Then, we prove condition (i) as follows.

$$\begin{aligned} f^\Delta(x + y) &= f^\Delta((a + c) - (b + d)) \\ &= f(a + c) - f(b + d) \\ &= f(a) - f(b) + f(c) - f(d) \\ &= f(a) - f(b) + f(c) - f(d) \\ &= f^\Delta(x) + f^\Delta(y) \end{aligned}$$

Now, we prove condition (ii).

$$\begin{aligned} f^\Delta(xy + yx) &= f^\Delta\Big((ac + bd) - (ad + bc) + (ca + db) - (da + cb)\Big) \\ &= \Big(f(ac + bd) + f(ca + db)\Big) - \Big(f(ad + bc) + f(da + cb)\Big) \\ &= \Big(f(ac + ca) + f(db + bd)\Big) - \Big(f(ad + da) + f(bc + cb)\Big) \\ &= f(a)f(c) + f(c)f(a) + f(d)f(b) + f(b)f(d) - f(a)f(d) - f(d)f(a) \\ &\qquad - f(b)f(c) - f(c)f(b) \\ &= \Big(f(a) - f(b)\Big)f(c) - \Big(f(a) - f(b)\Big)f(d) + f(c)\Big(f(a) - f(b)\Big) \\ &\qquad - f(d)\Big(f(a) - f(b)\Big) \\ &= \Big(f(a) - f(b)\Big)\Big(f(c) - f(d)\Big) + \Big(f(c) - f(d)\Big)\Big(f(a) - f(b)\Big) \\ &= f^\Delta(x)f^\Delta(y) + f^\Delta(y)f^\Delta(x) \end{aligned}$$

□

We extend some of the famous the results of Herstein [10] as follows. In particular, the following result is referred as Jordan triple homomorphism in the literature.

Lemma 2.4. *If f is a Jordan homomorphism of a semiring R into an additively cancellative 2-torsion free yoked semiring R', then $f(rsr) = f(r)f(s)f(r)$, for all $r, s \in R$.*

Proof. We use the Jordan product $r \circ s = rs + sr$, for all $r, s \in R$. Consider $r \circ (s \circ r)$, then we have

$$\begin{aligned}
f\Big(r\circ(r\circ s)\Big) &= f\Big(r\circ(rs+sr)\Big)\\
&= f\Big(r(rs+sr)+(rs+sr)r\Big)\\
&= f(r)f(rs+sr)+f(rs+sr)f(r)\\
&= f(r)[f(r)f(s)+f(s)f(r)]+[f(r)f(s)+f(s)f(r)]f(r)\\
&= (f(r))^2f(s)+f(r)f(s)f(r)+f(r)f(s)f(r)+f(s)(f(r))^2 \qquad (2.1)
\end{aligned}$$

Also,

$$\begin{aligned}
f\Big(r\circ(r\circ s)\Big) &= f\Big(r\circ(rs+sr)\Big)\\
&= f\Big(r(rs+sr)+(rs+sr)r\Big)\\
&= f(r^2s+rsr+rsr+sr^2)\\
&= f(r^2)f(s)+f(rsr)+f(rsr)+f(s)f(r^2) \qquad (2.2)
\end{aligned}$$

Using equations (2.1) and (2.2) we have,

$$f(r^2)f(s)+2f(rsr)+f(s)f(r^2)=2f(r)f(s)f(r)+(f(r))^2f(s)+f(s)(f(r))^2 \qquad (2.3)$$

Since R' is 2-torsion free and yoked, we have $f(r^2)=(f(r))^2$ by Lemma 2.2 and hence we obtain the following from (2.3).

$$(f(r))^2f(s)+2f(rsr)+f(s)(f(r))^2=2f(r)f(s)f(r)+(f(r))^2f(s)+f(s)(f(r))^2 \qquad (2.4)$$

Now using the fact that R' is additively cancellative, (2.4) becomes

$$2f(rsr)=2f(r)f(s)f(r)$$

Again using Lemma 2.1 we obtain $f(rsr)=f(r)f(s)f(r)$. □

Lemma 2.5. *If f is a Jordan homomorphism of a semiring R into and additively cancellative semiring R', then we have $f(rst+trs)=f(r)f(s)f(t)+f(t)f(r)f(s)$, for all $r,s,t\in R$.*

Proof. We linearize $f(rsr)=f(r)f(s)f(r)$ by replacing r with $r+t$.

$$\begin{aligned}
f\Big((r+t)s(r+t\Big) &= f(r+t)f(s)f(r+t)\\
f(rsr)+f(rst)+f(tsr)+f(tst) &= f(r)f(s)f(r)+f(r)f(s)f(t)+f(t)f(s)f(r)\\
&\quad +f(t)f(s)f(t)\\
f(rsr)+f(rst+tsr)+f(tst) &= f(rsr)+f(r)f(s)f(t)+f(t)f(s)f(r)+f(tst)
\end{aligned}$$

Since R' is additively cancellative, we get

$$f(rst+trs)=f(r)f(s)f(t)+f(t)f(r)f(s)$$

□

Theorem 2.6. *Let f be a Jordan homomorphism of a semiring R into an additively cancellative 2-torsion free yoked semiring R'. If R' has no zero-divisors then f is either a homomorphism or an anti-homomorphism.*

Proof. We let $x=f(rs), y=f(r)f(s)$ and $z=f(s)f(r)$. Since R' is yoked, we have either $x-y\in R'$ or $y-x\in R'$ and $x-z\in R'$ or $z-x\in R'$. We need to use the combination of these scenarios in order to obtain the desired result. The various cases are listed below.

(i) $(x-y) \in R'$ and $(x-z) \in R'$

(ii) $(x-y) \in R'$ and $(z-x) \in R'$

(iii) $(y-x) \in R'$ and $(x-z) \in R'$

(iv) $(y-x) \in R'$ and $(z-x) \in R'$

Let us consider case (i) and note the following.

$$\begin{aligned}(x-y)(x-z) &= \Big(f(rs)-f(r)f(s)\Big)\Big(f(rs)-f(s)f(r)\Big)\\ &= (f(rs))^2 - f(rs)f(s)f(r) - f(r)f(s)f(rs) + f(r)f(s)f(s)f(r)\\ &= f(rs)^2 - f\Big((rs)(sr)+(rs)(rs)\Big) + f(r)f(s^2)f(r)\\ &\qquad\qquad\qquad\qquad\qquad\qquad [\text{using Lemma 2.5 and Lemma 2.2}]\\ &= f\Big((rs)^2 - rs^2r - (rs)^2 + rs^2r\Big)\\ &= f(0)\\ &= 0\end{aligned}$$

We therefore have

$$\Big(f(rs)-f(r)f(s)\Big)\Big(f(rs)-f(s)f(r)\Big) = 0, \qquad \text{for all } r,s \in R. \tag{2.5}$$

Since R' has no zero-divisors, then equation (2.5) must imply that either $f(rs) - f(r)f(s) = 0$ or $f(rs) - f(s)f(r) = 0$. That is, $f(rs) = f(r)f(s)$ (meaning f is a homomorphism) or $f(rs) = f(s)f(r)$ (meaning f is an anti-homomorphism).
Using the same procedure as in case (i), it is easy to prove the following.

$$\begin{aligned}\text{Case (ii)} &\implies (x-y)(z-x) = 0.\\ \text{Case (iii)} &\implies (y-x)(x-z) = 0.\\ \text{Case (iv)} &\implies (y-x)(z-x) = 0.\end{aligned}$$

Again, use the fact that R' has no zero-divisors and apply a similar argument after equation (2.5) to see that each case leads to the desired result. □

Herstein's famous result states that: a Jordan homomorphism f of a prime ring of characteristic different from 2 and 3 is either a homomorphism or an anti-homomorphism (Theorem H in [10]). We prove the analogous result in semirings below.

Theorem 2.7. *Let R and S be two semirings that are additively cancellative (not necessarily 2-torsion free) and also let S be a yoked prime semiring. If f is a Jordan homomorphism from R onto S, then f is either a homomorphism or f is an anti-homomorphism.*

Proof. Let R^Δ and S^Δ be the ring of differences corresponding to R and S respectively. By Lemma 2.3 we note that f^Δ is Jordan homomorphism corresponding to f. Since R and S are additively cancellative, both R^Δ and S^Δ is of characteristic zero (Remark 2.9 in [7]). Also, S^Δ is prime as S is prime (Lemma 2.6 in [7]). Thus, by Theorem H of [10] we have f^Δ as either a homomorphism or an anti-homomorphism. Hence f must be either a homomorphism or an anti-homomorphism since the restriction of f^Δ to R is f due to embedding of R into R^Δ. □

3 Jordan derivations in semirings

We begin with generalizations of some famous results of Herstein and Brešer. We prove that a Jordan triple derivation d of R induces a corresponding Jordan triple derivation d^Δ in the ring of differences R^Δ corresponding to R. We extend the results of Brešer [4] by proving that in a certain class of semirings, Jordan triple derivation is just a usual derivation. We also generalize the results of Jing and Lu [18] and Vukman [14] to semirings. It is proved that generalized Jordan derivations and generalized Jordan triple derivations are nothing but generalized derivations. We start with the following Lemma, which is very useful throughout the article.

Lemma 3.1. *Let R be an additively cancellative yoked semiring. If R is 2-torsion free then the ring of differences R^{Δ} corresponding to R is also 2-torsion free.*

Proof. For $x \in R^{\Delta}$, assume $2x = 0$. Let $x = a - b$, where $a, b \in R$. Then $2(a - b) = 0$ implies $2a = 2b$. Since R is yoked and 2-torsion free, by Lemma 2.1 we get $a = b$ which implies $a - b = 0$. That is $x = 0$, proving that R^{Δ} is 2-torsion free as well. □

The next couple of Lemmas are generalization of some famous results in [3, 12] which are of independent interest.

Lemma 3.2. *Let R be an additively cancellative yoked prime semiring and u, v be any two elements of R. If $urv + vru = 0$, for all $r \in R$, then $u = 0$ or $v = 0$.*

Proof. Let R^{Δ} be the ring of differences corresponding to the semiring R. Then we note that R^{Δ} is prime since R is prime (Lemma 2.6 in [7]). Let $u, v \in R$ and $x \in R^{\Delta}$ such that $x = a - b$, where $a, b \in R$. Then, $uxv + uxu = u(a - b)v + v(a - b)u = (uav + vau) - (ubv + vbu) = 0$, for all $x \in R^{\Delta}$. by Lemma 3.10 in [12], we must have either $u = 0$ or $v = 0$ in R^{Δ}. Thus $u = 0$ or $v = 0$ in R, since R is embedded in R^{Δ}. □

Lemma 3.3. *Let R be an additively cancellative yoked 2-torsion free semiprime semiring. If $u, v \in R$ such that $urv + vru = 0$, for all $r \in R$, then $urv = vru = 0$, for all $r \in R$.*

Proof. Let R^{Δ} be the ring of differences corresponding to the semiring R. At first, we note that, R^{Δ} is 2-torsion free due to Lemma 3.1 and R^{Δ} is semiprime since R is semiprime (Lemma 3.4 in [8]). Let $u, v \in R$ and $x \in R^{\Delta}$ such that $x = a - b$, where $a, b \in R$. Then, by similar argument in Lemma 3.2, we have $uxv + uxu = 0$. Hence by Lemma 4 in [3], we get $uxv = vxu = 0$, for all $x \in R^{\Delta}$. This implies that $urv = vru = 0$, for all $r \in R$. □

Lemma 3.4. *Let R be an additively cancellative semiring. If d is a Jordan triple derivation, then for $r, s, t \in R$, we have*

$$d(rst + tsr) = d(r)st + rd(s)t + rsd(t) + d(t)sr + td(s)r + tsd(r)$$

Proof. We linearize the equation in Definition 1.9 by replacing r with $r + t$ to get

$$\begin{aligned}
d\Big((r+t)s(r+t)\Big) &= \Big(d(r+t)\Big)s(r+t) + (r+t)d(s)(r+t) \\
&\qquad + (r+t)s\Big(d(r+t)\Big) \\
&= \Big(d(r)+d(t)\Big)\Big(sr+st\Big) + \Big(rd(s)+td(s)\Big)\Big(r+t\Big) \\
&\qquad + \Big(rs+ts\Big)\Big(d(r)+d(t)\Big) \\
&= d(r)sr + d(r)st + d(t)sr + d(t)st \\
&\qquad + rd(s)r + rd(s)t + td(s)r + td(s)t \\
&\qquad + rsd(r) + rsd(t) + tsd(r) + tsd(t) \\
&= \Big[d(r)sr + rd(s)r + rsd(r)\Big] + d(r)st \\
&\qquad + rd(s)t + rsd(t) + d(t)sr + td(s)r \\
&\qquad + tsd(r) + \Big[d(t)st + td(s)t + tsd(t)\Big] \\
&= d(rsr) + d(r)st + rd(s)t + rsd(t) \\
&\qquad + d(t)sr + td(s)r + tsd(r) + d(tst) \qquad (3.1)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d\Big((r+t)s(r+t)\Big) &= d\Big((rs+ts)(r+t)\Big) \\
&= d(rsr + rst + tsr + tst) \\
&= d(rsr) + d(rst + tsr) + d(tst) \qquad (3.2)
\end{aligned}$$

From (3.1) and (3.2), we note that

$$d(rsr) + d(rst + tsr) + d(tst) = d(rsr) + d(r)st + rd(s)t + rsd(t) + d(t)sr + td(s)r + tsd(r) + d(tst) \tag{3.3}$$

Since R is additively cancellative, we get the desired result as follows.

$$d(rst + tsr) = d(r)st + rd(s)t + rsd(t) + d(t)sr + td(s)r + tsd(r)$$

□

We extend the map given in Definition 1.5 by assuming a map d to be Jordan triple derivation of a semiring R and prove the following in the ring of differences.

Theorem 3.5. *Let R be an additively cancellative semiring and d be a Jordan triple derivation of R into R then d^Δ defined by $d^\Delta(r-s) = d(r) - d(s), r - s \in R^\Delta$ for $r, s \in R$, is a Jordan triple derivation of R^Δ into R^Δ.*

Proof. It is clear that d^Δ is additive in R^Δ whenever d is additive in R. For $x, y \in R^\Delta$, we prove the following.

$$d^\Delta(xyx) = d^\Delta(x)yx + xd^\Delta(y)x + xyd^\Delta(x)$$

Let $x = a - b$ and $y = c - d$ where $a, b, c, d \in R$. Then we have

$$\begin{aligned}
d^\Delta(xyx) &= d^\Delta\Big((a-b)(c-d)(a-b)\Big)\\
&= d^\Delta\Big\{[(ac+bd)-(ad+bc)](a-b)\Big\}\\
&= d^\Delta(aca - acb + bda - bdb - ada + adb - bca + bcb)\\
&= d(aca) + d(adb + bda) + d(bcb) - d(acb + bca) - d(bdb) - d(ada)\\
&= d(a)ca + ad(c)a + acd(a) + d(a)db + ad(d)b + add(b)\\
&\quad + d(b)da + bd(d)a + bdd(a) + d(b)cb + bd(c)b + bcd(b)\\
&\quad - d(a)cb - ad(c)b - acd(b) - d(b)ca - bd(c)a - bcd(a)\\
&\quad - d(b)db - bd(d)b - bdd(b) - d(a)da - ad(d)a - add(a)\\
&\qquad\qquad\qquad\qquad \text{[using Definition 1.9 and Lemma 3.4]}\\
&= \Big(d(a)-d(b)\Big)\Big(ca+db-da-cb\Big) + \Big(a-b\Big)\Big(d(c)a - d(c)b - d(d)a + d(d)b\Big)\\
&\quad + \Big(ac+bd-ad-bc\Big)\Big(d(a)-d(b)\Big)\\
&= \Big(d^\Delta(a-b)\Big)(c-d)(a-b) + (a-b)\Big(d^\Delta(c-d)\Big)(a-b)\\
&\quad + (a-b)(c-d)\Big(d^\Delta(a-b)\Big)\\
&= d^\Delta(x)yx + xd^\Delta(y)x + xyd^\Delta(x)
\end{aligned}$$

Hence d^Δ is a Jordan triple derivation of R^Δ corresponding to d of R. □

Theorem 3.6. *Let R be an additively cancellative yoked semiprime semiring. If R is 2-torsion free and d is a Jordan triple derivation in R, then d is a usual derivation of R.*

Proof. Let R^Δ be the ring of differences corresponding to R and then by Theorem 3.5 we have d^Δ, a Jordan triple derivation induced by d. Note that R^Δ is semiprime since R is semiprime (Lemma 3.4 in [8]). Also, by Lemma 3.1 R^Δ, is 2-torsion free since R is 2-torsion free and yoked. Now, applying Theorem 4.3 of [4], yields that d^Δ is an ordinary derivation of R^Δ. If we restrict d^Δ to R then the result follows due to embedding of R into R^Δ. □

We state the following Lemma which induces a generalized Jordan derivation F^{Δ} in R^{Δ} corresponding to a generalized Jordan derivation F in R. It is an easy routine to check F^{Δ} satisfies the conditions in the Definition 1.11 and hence we skip the proof.

Lemma 3.7. *Let R be an additively cancellative semiring and F be a generalized Jordan derivation of R into R corresponding to a Jordan derivation d. Then $F^{\Delta}(r-s) = F(r) - F(s), r-s \in R^{\Delta}$, where $r, s \in R$, is a generalized Jordan derivation of R^{Δ} into R^{Δ} corresponding to the Jordan derivation d^{Δ}.*

Lemma 3.8. *Let R be an additively cancellative 2-torsion free yoked semiring and F be a generalized Jordan derivation of R into R corresponding to a Jordan derivation d. For each $r, s, t \in R$, the following holds.*

(i) $F(rs + sr) = F(r)s + rd(s) + F(s)r + sd(r)$;

(ii) $F(rsr) = F(r)sr + rd(s)r + rsd(r)$;

(iii) $F(rst + tsr) = F(r)st + rd(s)t + rsd(t) + F(t)sr + rd(s)r + tsd(r)$.

Proof. (i) Linearize r by $r + s$ in Definition 1.11 to see that

$$\begin{aligned} F\Big((r+s)^2\Big) &= \Big(F(r+s)\Big)(r+s) + (r+s)\Big(d(r+s)\Big) \\ &= F(r)r + F(r)s + F(s)r + F(s)s + rd(r) + rd(s) + sd(r) + sd(s) \end{aligned} \tag{3.4}$$

On the other hand,

$$\begin{aligned} F\Big((r+s)^2\Big) &= F(r^2 + rs + sr + s^2) \\ &= F(r^2) + F(rs + sr) + F(s^2) \\ &= F(r)r + rd(r) + F(rs + sr) + F(s)s + sd(s) \end{aligned} \tag{3.5}$$

Using (3.4) and (3.5) and applying the fact that R is additively cancellative, we have

$$F(rs + sr) = F(r)s + rd(s) + F(s)r + sd(r)$$

(ii) Linearize the result (i) by replacing s with $rs + sr$, we get

$$\begin{aligned} F\Big(r(rs+sr) + (rs+sr)r\Big) &= \Big(F(r)\Big)\Big(rs+sr\Big) + r\Big(d(rs+sr)\Big) \\ &\quad + F(rs+sr)r + (rs+sr)d(r) \\ &= F(r)rs + F(r)sr + rd(r)s + rsd(r) + rd(s)r \\ &\quad + r^2d(s) + F(r)sr + rd(s)r + F(s)r^2 \\ &\quad + sd(r)r + rsd(r) + srd(r) \\ &= F(r)rs + 2F(r)sr + rd(r)s + 2rsd(r) + 2rd(s)r \\ &\quad + r^2d(s) + F(s)r^2 + sd(r)r + srd(r) \end{aligned} \tag{3.6}$$

On the other hand,

$$\begin{aligned} F\Big(r(rs+sr) + (rs+sr)r\Big) &= F(r^2s + rsr + rsr + sr^2) \\ &= F(r^2s + sr^2 + 2rsr) \\ &= F(r^2)s + r^2d(s) + F(s)r^2 + sd(r^2) + 2F(rsr) \\ &= F(r)rs + rd(r)s + r^2d(s) + F(s)r^2 \\ &\quad + sd(r)r + srd(r) + 2F(rsr) \end{aligned} \tag{3.7}$$

From (3.6) and (3.7) and since R is additively cancellative, we get

$$2F(rsr) = 2(F(r)sr + rd(s)r + rsd(r))$$

Note that R is 2-torsion free and yoked and hence by Lemma 2.1, we must have

$$F(rsr) = F(r)sr + rd(s)r + rsd(r)$$

(iii) We skip the proof since it is very similar to (i) and (ii). That is, linearize the result (ii) by replacing r with $r + s$, the result will then follow immediately. □

The last result (iii) above is a generalization of Lemma 3.4.

Theorem 3.9. *If R be an additively cancellative 2-torsion free yoked prime semiring, then every generalized Jordan derivation of R is just a generalized derivation of R.*

Proof. Let F be a generalized Jordan derivation of R corresponding to a Jordan derivation d in R. Let R^Δ be the corresponding ring of differences of R and d^Δ be a Jordan derivation corresponding to d. By Lemma 3.7, F^Δ is a generalized Jordan derivation corresponding to d^Δ. Note that R^Δ is prime since R is prime (Lemma 2.6 in [7]). Also, by Lemma 3.1, R^Δ is 2-torsion free since R is 2-torsion free and yoked. Then, by Theorem 2.5 in [18], we must have F^Δ to be a generalized derivation of R^Δ. Thus, the restriction of F^Δ to R, namely F, is a generalized derivation of R. □

Lemma 3.10. *Let R be an additively cancellative semiring and F be a generalized Jordan triple derivation of R into R corresponding to a Jordan triple derivation d. Then the map F^Δ defined by $F^\Delta(r - s) = F(r) - F(s), r - s \in R^\Delta$ for $r, s \in R$, is a generalized Jordan triple derivation of R^Δ into R^Δ corresponding to a Jordan triple derivation d^Δ.*

Proof. It is a simple routine to prove that F^Δ is a generalized Jordan triple derivation of R^Δ if we follow along the lines of proof in Theorem 3.5. □

Theorem 3.11. *If R is an additively cancellative 2-torsion free yoked prime semiring, then every generalized Jordan triple derivation of R is just a generalized derivation of R.*

Proof. Let F be a generalized Jordan triple derivation of R corresponding to a Jordan triple derivation d in R. Let R^Δ be the ring of differences of R, then by Theorem 3.5, d^Δ is a Jordan triple derivation of R^Δ corresponding to d. By Lemma 3.10, F^Δ is a generalized Jordan triple derivation corresponding to d^Δ. Since R is prime we should have R^Δ as prime (Lemma 2.6 in [7]). Also by Lemma 3.1 R^Δ is 2-torsion free since R is 2-torsion free and yoked. Then, by Theorem 3.5 in [18] we must have F^Δ to be a generalized derivation of R^Δ. Thus, the restriction of F^Δ to R, namely F, is a generalized derivation of R. □

We conclude the paper by extending the results of Vukum [14] to semiprime semirings, which is a generalization of the above results in Theorem 3.9 and Theorem 3.11. The proof of the following results are quite similar to the ones in the above Theorems and hence we dispose them with a note that we apply Lemma 3.4 in [8] to see that R^Δ is semiprime whenever R is semiprime.

Theorem 3.12. *If R is an additively cancellative 2-torsion free yoked semiprime semiring, then every generalized Jordan derivation of R is just a generalized derivation of R.*

Theorem 3.13. *If R is an additively cancellative 2-torsion free yoked semiprime semiring, then every generalized Jordan triple derivation of R is just a generalized derivation of R.*

4 Conclusion

In this research work, we generalized classical results from ring theory to semirings. In the second section, we showed that a Jordan homomorphism from an additively cancellative semiring to an additively cancellative yoked prime semiring, is either a homomorphism or an anti-homomorphism. Section 3 dealt with Jordan derivations, Jordan triple derivations and their generalizations. We used the ring of differences to travel from semirings to rings. In Lemmas 3.7-3.10 and Theorems 3.9-3.13, we applied ring theoretic results from the literature to prove that the induced Jordan derivations, Jordan triple derivations, generalized Jordan derivations and generalized Jordan triple derivations are all regular derivations in the ring of differences. The induced mapping served as a bridge to travel from semirings to the ring of differences. Due to the embedding nature of additively cancellative semirings in the ring of differences, the induced map can be restricted to semirings that produce the desired outcomes. The absence of additive inverse elements within the semiring renders generalizations non-trivial and our methodology offers a systematic framework for such generalizations to semirings.

References

[1] Y. Ahmed and W. A. Dudek, *Left Jordan derivations on certain semirings*, Hacettepe Journal of Mathematics & Statistics, **50**, 624–633, (2021).

[2] W. E. Baxter and W. S. Martindale, *Jordan homomorphisms of semiprime rings*, Journal of Algebra, **56**, 457-471, (1979).

[3] M. Brešar, *Jordan derivations on semiprime rings*, Proceedings of the American Mathematical Society, **104**, 1003-1006, (1988).

[4] M. Brešar, *Jordan Mappings of Semiprime Rings*, Journal of Algebra, **127**, 218-228, (1989).

[5] T. C. Craven, *Orderings on semirings*, Semigroup Forum, **43**, 45–52, (1991).

[6] A. K. Faraj, A. H. Majeed, C. Haetinger and Nadeem-ur-Rahman, *On Generalized Jordan Higher Homomorphisms*, Palestine Journal of Mathematics, **3**, 406–421, (2014).

[7] S. Ganesh and V. Selvan, *Posner's theorems in Semirings*, Indian Journal of Pure and Applied Mathematics, **55**, 597-602, (2024).

[8] S. Ganesh and V. Selvan, *Jordan Structures in Semirings*, Rendiconti del Circolo Matematico di Palermo Series 2, **72**, 3587-3596, (2023).

[9] J. S. Golan, *Semirings and Their Applications*, Kluwer Academic Publishers, Isreal, (1999).

[10] I. N. Herstein, *Jordan homomorphisms*, Transactions of the American Mathematical Society, **81**, 331–341, (1956).

[11] I. N. Herstein, *Jordan Derivations of Prime Rings*, Proceedings of American Mathematical Society, **8**, 1104–1110, (1957).

[12] I. N. Herstein, *Topics in Ring Theory*, University of Chicago Press, Chicago, (1969).

[13] R. K. H. Ibraheem and A. H. Majeed, *U-S Jodan homomorphism of inverse semirings*, Iraqi Journal of Science, **60**, 1783–1790, (2019).

[14] Joso Vukman, *A note on generalized derivations of semiprime rings*, Taiwanese Journal of Mathematics, **11**, 367–370, (2007).

[15] E. C. Posner, *Derivations in prime rings*, Proceedings of the American Mathematical Society, **8**, 1093-1100, (1957).

[16] Gurninder S. Sandhu and Deepak Kumar, *Generalized derivations acting as homomorphisms or anti-homomorphisms on Lie ideals*, Palestine Journal of Mathematics, **7**, 137–142, (2018).

[17] Sara Shafiq and Muhammad Aslam, *On Jordan mappings of inverse semirings*, Open Mathematics, **15**, 1123–1131, (2017).

[18] Wu Jing and Shiijie Lu, *Generalized Jordan derivations on prime rings and standard operator algebras*, Taiwanese Journal of Mathematics, **7**, 605–613, (2003).

Author information

S. Ganesh, Research Scholar, Department of Mathematics, Ramakrishna Mission Vivekananda College, Chennai, India.
E-mail: `madmaths007@gmail.com`

V. Selvan, Associate Professor, Department of Mathematics, Ramakrishna Mission Vivekananda College, Chennai, India.
E-mail: `venselvan@gmail.com`

Received: 2024-05-17
Accepted: 2026-01-16