Palestine Journal of Mathematics

Vol 15(1)(2026) , 120–128

A VARIANT OF NEWTON'S METHOD WITH ACCELERATED SIXTH ORDER OF CONVERGENCE

Inderjeet and Rashmi Bhardwaj

Communicated by: Geeta Arora

MSC 2020 Classifications: 65A05, 65D05, 65D30, 65D32.

Keywords and phrases: Newton's formula, Nonlinear equations, Iterative methods, Order of convergence, Function evaluations.

Authors are thankful to the "University Grants Commission (UGC)" and "Guru Gobind Singh Indraprastha University (GGSIPU)" for financial support and research facilities, and also both the authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: Prof. Rashmi Bhardwaj

Abstract. In this paper, we present a new modification of Newton's method for solving nonlinear equations. Derivation of Newton's method involves an indefinite integral of the derivative of the function, and the relevant area is approximated by a rectangle. In the proposed scheme we approximate this indefinite integral with the average of by the parabola i.e. a polynomial of second order (Simpson's rule) and the Mid– point rule of integration, thereby reducing the error in the approximation. It is shown that the order of the convergence of the new method is six, and computed results support this theory. For most of the function we tested, the order of convergence in Newton's method was two and for our method, it was always close to six.

1 Introduction

The nonlinear equations, often arise from the numerical modeling of problems in many branches of science and engineering [2, 4] . There are many papers deals with the nonlinear algebraic equations, such as, improving Newton Raphson method for nonlinear equations by modified Adomian decomposition method [5], by approximating the function's derivative with finite differences [3], iterative method improving Newton's method by the decomposition method [6], variants of Newton's method using fifth order quadrature formulas [7, 8, 9]. Khattri and Argyros discuss the four-parameter family of sixth order convergent iterative methods for solving nonlinear scalar equations. Methods of the family require evaluation of four functions per iteration. These methods are totally free of derivatives. Convergence analysis shows that the family is sixth order convergent, which is also verified through the numerical work. Though the methods are independent of derivatives, computational results demonstrate that family of methods are efficient and demonstrate equal or better performance as compared with other six order methods, and the classical Newton method [10, 11]. Wang et al. present a variant of Jarratt method with order of convergence six for solving non-linear equations. Per iteration the method requires two evaluations of the function and two of its first derivatives. The new multistep iteration scheme, based on the new method, is developed and numerical tests verifying the theory are also given [12]. Newton' method that approximate the roots of a nonlinear equation in one variable using the value of the function and its derivative, in an iterative fashion, is probably the best known and most widely used algorithm, and it converges to the root quadratically. In other words, after some iterations, the process doubles the number of correct decimal places or significant digits at each iteration. It is worth noting that the averaging of the midpoint and trapezoidal rules yields a method of third-order convergence. Similarly, the averaging of the trapezoidal and Simpson's rules also results in third-order convergence. In contrast, the proposed method achieves sixth-order convergence, thereby providing a substantially higher rate of accuracy compared to these existing averaged schemes. Inderjeet and Bhardwaj [13, 14, 15, 16] presented modified Newton Raphson approaches to tackle the nonlinear equations with improved accuracy and robustness of the technique, and the proposed approaches are better than the existing techniques like Secant technique, Fixed–Point iteration etc. In this study, we suggest an improvement to the iteration of Newton's method at the expense of two additional first derivative evaluation. Derivation of Newton's method involves an indefinite integral of the derivative of the function, and the relevant area is approximated by a rectangle. Here, we approximate this indefinite integral by the parabola i.e. a polynomial of second order (Simpson's rule) and the Mid– point rule of integration, and the result is a method of sixth- order convergence. It is shown that the suggested method converges to the root, and the order of the convergence is at least six in a neighborhood of the root, whenever the first and higher order derivatives of the function exist in a neighborhood of the root; i.e. our method approximately sixth times the number of significant digits after some iterations. Computed results overwhelmingly support this theory.

2 Motivation of the Method

While extensive research has been conducted on traditional iterative methods and also some refined iterative techniques developed by the researchers, there remains a significant gap in the literature regarding proposed modified method. This research gap motivates our current study, which aims to:

- Develop novel modifications to Newton method that address the respective limitations while preserving their strengths.
- Create newly approach that intelligently modify the existing Newton method, leveraging their complementary characteristics to achieve superior performance across a wide range of nonlinear equations.
- Provide a comprehensive comparative analysis of the modified approach against classical technique, using a diverse set of test problems that include both algebraic and transcendental equations.
- Explore the theoretical foundations of modified approach, including convergence analysis, to provide a solid mathematical basis for its application.

By addressing these points, our research aims to contribute significantly to the field of numerical analysis, offering a new tool for solving nonlinear equations that combines the reliability of Newton method with the potential for rapid convergence offered by modified Newton method.

3 Numerical Scheme

3.1 Newton's Method

Newton's algorithm to approximate the root α of the nonlinear function $f(x) = 0$ is to start with an initial approximation x_0 sufficiently close to α and to use the one-point iteration scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3.1}$$

Where x_n is the nth iterate. It is well known that Newton's method as given above is quadratically convergent. It is important to understand how Newton's method is constructed. At each iterative step, we construct a local linear model of our function $f(x)$ at the point x_n and solve for the root x_{n+1} of the local model. In Newton's method, this local linear model is the tangent drawn to the function $f(x)$ at the current point x_n. The local linear model at x_n is

$$M_n(x) = f(x_n) + f'(x_n)(x - x_n) \tag{3.2}$$

This local linear model can be interpreted [1] in another way. From Newton's theorem,

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\lambda)\, d\lambda \tag{3.3}$$

In Newton's method, the indefinite integral is approximated by the rectangle i.e.

$$\int_{x_n}^{x} f'(\lambda)\,d\lambda \approx f'(x_n)(x - x_n) \tag{3.4}$$

which will result in the model given in (3.2).

3.2 A variant of Newton's Method

From Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\lambda)\,d\lambda \tag{3.5}$$

In the proposed scheme, we approximate the indefinite integral involved in (3.5) with the average of by the parabola i.e. a polynomial of second order (Simpson's rule) and the Mid – point rule of integration i.e.

$$\int_{x_n}^{x} f'(\lambda)\,d\lambda = \frac{x - x_n}{12}\left[f'(x) + 10f'\left(\frac{x + x_n}{2}\right) + f'(x_n)\right] \tag{3.6}$$

Thus, the local model equivalent to (3.2) is

$$\hat{M}_n(x) = f(x_n) + \frac{x - x_n}{12}\left[f'(x) + 10f'\left(\frac{x + x_n}{2}\right) + f'(x_n)\right] \tag{3.7}$$

Note that not only the model and the derivative of the model agree with the function $f(x)$ and the derivative of the function $f'(x)$ respectively, but also the second and higer order derivative of the model and the second and higher order derivative of the function agree at the current iterate $x = x_n$. Even though the model for the Newton's method matches with the values of the slope $f'(x_n)$ of the function, it does not match with its curvature in terms of $f''(x_n)$. We take the next iterative point as the root of the local model (3.7)

$$(\hat{M}_n)(x_{n+1}) = 0 \quad \text{i.e.} \quad f(x_n) + \frac{x - x_n}{12}\left[f'(x) + 10f'\left(\frac{x + x_n}{2}\right) + f'(x_n)\right] = 0 \tag{3.8}$$

$$x_{n+1} = x_n - \frac{12f(x_n)}{f'(x_{n+1}) + 10f'\left(\frac{x_{n+1} + x_n}{2}\right) + f'(x_n)} \tag{3.9}$$

Equation (3.9) is implicit in nature and is rendered explicit in the present study using the Newton–Raphson method. Alternatively, this implicit scheme may also be transformed into an explicit form by employing other iterative techniques, such as the JM or the ACM. However, the use of JM and ACM iterative techniques can lead to variations in the order of convergence, which may either increase or decrease depending on the method employed,as each technique possesses distinct convergence characteristics. We could overcome this difficulty by making use of Newton's iterative step to compute the x_{n+1} iterate on the right - hand side. Thus, the resulting new scheme is

$$x_{n+1} = x_n - \frac{12f(x_n)}{f'(y_n) + 10f'\left(\frac{y_n + x_n}{2}\right) + f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{3.10}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$

4 Analysis of Convergence

Theorem 4.1. *Let $f : D \to \mathbb{R}$ for an open interval D. Assume that f has first, second and so on sixth order derivative in the interval D. If $f(x)$ has a simple root at $\alpha \in D$ and x_0 is sufficiently close to α then the new method defined by (3.10) satisfied the following error equation:*
$e_{n+1} = \left(\frac{3}{4}c_3^3c_2^3 - \frac{3}{64}c_3^4c_2 - 3c_3^2c_2^5\right)e_n^6 + o(e_n^7)$ *where* $e_n = x_n - \alpha$ *and*
$c_k = \frac{1}{k!} \cdot \frac{f^{(k)}(\alpha)}{f'(\alpha)}, \quad k = 1, 2, 3, \ldots$

Proof. The suggested variant of Newton's method (VNM) is

$x_{n+1} = x_n - \frac{12 f(x_n)}{f'(y_n)+10f'\left(\frac{y_n+x_n}{2}\right)+f'(x_n)}, \quad n = 0, 1, 2, \ldots$ where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$

Let α be simple root of $f(x)$ i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$ and $x_n = \alpha + e_n$

We use the following Taylor expansions:

$$f(x_n) = f(\alpha+e_n) = f(\alpha)+f^{(1)}(\alpha)e_n+f^{(2)}(\alpha)e_n^2+f^{(3)}(\alpha)e_n^3+f^{(4)}(\alpha)e_n^4+f^{(5)}(\alpha)e_n^5+f^{(6)}(\alpha)e_n^6+o(e_n^7)$$

$$f(x_n) = f^{(1)}(\alpha)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + o(e_n^7)\right] \tag{4.1}$$

$$f^{(1)}(x_n) = f^{(1)}(\alpha)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + o(e_n^7)\right] \tag{4.2}$$

Divide (4.1) by (4.2)

$\frac{f(x_n)}{f^{(1)}(x_n)} = e_n - c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 + (8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 + (-16c_2^5 + 56c_2^3c_3 - 24c_3^2c_2 - 30c_2^2c_4 + 12c_3c_4 + 12c_2c_5 - 5c_6)e_n^6 + o(e_n^7)$

$y_n = x_n - \frac{f(x_n)}{f^{(1)}(x_n)} = \alpha + c_2e_n^2 - (2c_2^2 - 2c_3)e_n^3 - (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 - (8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 - (-16c_2^5 + 56c_2^3c_3 - 24c_3^2c_2 - 30c_2^2c_4 + 12c_3c_4 + 12c_2c_5 - 5c_6)e_n^6 + o(e_n^7)$

$$f(y_n) = f'(\alpha)\Big[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 + (-6c_3^2 + 24c_3c_2^2 - 10c_2c_4 + 4c_5 - 12c_2^4)e_n^5 + (-17c_3c_4 + 34c_2^2c_4 - 13c_2c_5 + 5c_6 + 37c_2c_3^2 - 73c_2^3c_3 + 28c_2^5)e_n^6 + o(e_n^7)\Big]$$

$$\begin{aligned} f'(y_n) = f'(\alpha)\Big[&1 + 2c_2^2e_n^2 + (4c_2c_3 - 4c_2^3)e_n^3 + (6c_2c_4 - 11c_3c_2^2 + 8c_2^4)e_n^4 \\ &+ (28c_2^2c_3^2 - 20c_4c_2^2 + 8c_2c_5 - 16c_2^5)e_n^5 \\ &+ (-164c_2c_3c_4 - 68c_3c_2^4 + 12c_3^3 + 60c_4c_2^3 - 26c_5c_2^2 + 10c_2c_6 + 32c_2^6)e_n^6 + o(e_n^7)\Big] \end{aligned} \tag{4.3}$$

$$\begin{aligned} f'\left(\frac{y_n+x_n}{2}\right) = f'(\alpha)\Big[&1 + c_2e_n + \left(c_2^2 + \frac{3}{4}c_3\right)e_n^2 + \left(\frac{7}{2}c_2c_3 - 2c_2^3 + \frac{c_4}{2}\right)e_n^3 \\ &+ \left(\frac{9}{2}c_2c_4 - \frac{29}{4}c_2^2c_3 + \frac{5}{16}c_5 + 4c_2^4 + 3c_3^2\right)e_n^4 \\ &+ \left(\frac{11}{2}c_2c_5 - \frac{71}{4}c_2^2c_4 + \frac{45}{8}c_3c_4 + \frac{121}{8}c_2c_3^2 + 6c_6 + 10c_2^5 - 15c_2c_3c_4\right)e_n^5 \\ &+ \left(-\frac{205}{8}c_2^2c_3 + \frac{75}{8}c_2c_3^2 + \frac{15}{8}c_4c_2^2 - \frac{1}{6}c_2c_5 + \frac{3}{10}c_6 + 10c_2^5 + 6c_4^2\right)e_n^6 + o(e_n^7)\Big] \end{aligned} \tag{4.4}$$

Adding equations (4.2), (4.3), and (4.4), we get

$$\begin{aligned} f'(y_n) + 10f'\left(\frac{y_n+x_n}{2}\right) + f'(x_n) = 12f'(\alpha)\Big[&1 + c_2e_n + \left(3c_2^2 + \frac{15}{4}c_3\right)e_n^2 + \left(\frac{15}{2}c_2c_3 - 6c_2^3 + \frac{9}{2}c_4\right)e_n^3 \\ &+ \left(\frac{21}{2}c_2c_4 - \frac{73}{4}c_2^2c_3 + 12c_2^4 + \frac{85}{16}c_5 + 3c_3^2\right)e_n^4 \\ &+ \left(\frac{27}{2}c_2c_5 - \frac{71}{4}c_2^2c_4 + \frac{45}{8}c_3c_4 + \frac{121}{8}c_2c_3^2 + 12c_6 - 6c_2^5 - 15c_2c_3c_4 - \right. \\ &+ \left(- \frac{205}{8}c_2^2c_3 + \frac{75}{8}c_2c_3^2 + \frac{15}{8}c_4c_2^2 - \frac{1}{6}c_2c_5 + \frac{3}{10}c_6 + 10c_2^5 + 6c_4^2 - \right. \\ &\left. + 12c_3^3 + 60c_4c_2^3 - 26c_5c_2^2 + 10c_2c_6 + 32c_2^6 + 7c_7\right)e_n^6 + o(e_n^7)\Big] \end{aligned} \tag{4.5}$$

From equations (4.10), (4.1), (4.5) we get

$x_{n+1} = \alpha + \left(\frac{3}{4}c_3^3c_2^3 - \frac{3}{64}c_3^4c_2 - 3c_3^2c_2^5\right)e_n^6 + o(e_n^7)$

$$e_{n+1} = \left(\frac{3}{4}c_3^3c_2^3 - \frac{3}{64}c_3^4c_2 - 3c_3^2c_2^5\right)e_n^6 + o(e_n^7) \tag{4.6}$$

Equation (4.6) establishes the sixth-order convergence of the VNM. □

Practical Implication of the above theorem are

- The sixth-order convergence is derived using Taylor expansions up to the sixth derivative.
- Without these derivatives, the constants c_k and the error equation are not valid.
- Therefore, the smoothness requirement is not technical decoration it is fundamental.
- In practice, lack of smoothness, noisy data, or numerical differentiation can reduce or destroy the expected convergence rate.
- For this reason, the requirement $f \in C^6(D)$should be stated early and emphasized, and its practical limitations should be clearly acknowledged.

5 Numerical Experiments

The benchmark problems were selected to cover a variety of nonlinear equations with different characteristics, such as polynomial equations, transcendental equations. The following benchmark problems were considered:

Problem 1: $f(x) = x^3 + 4x^2 - 10 = 0$, with initial point -0.5
Problem 2: $f(x) = \sin^2(x) - x^2 + 1$, with initial point 3
Problem 3: $f(x) = x^2 - e^x - 3x + 2$, with initial point 3
Problem 4: $f(x) = \cos x - x$, with initial point -0.3
Problem 5: $f(x) = (x - 1)^3 - 1$, with inital point 3.5
Problem 6: $f(x) = x^{\frac{1}{3}}$, with initial point 0.1 (ill conditioned problem)

The performance of each method was evaluated based on the following metrics:

- Number of iterations required for convergence
- Number of functions evaluation
- Computational time (in seconds)

All simulations and computations in this study were performed using MATLAB R2021 on a system equipped with anIntel Core i5 processor. It is important to note that using a different version of MATLAB or a different hardware configuration may result in variations in computational time,although the qualitative outcomes are expectedto remain consistent. We compared the Newton method (NM), Jarrat's method (JM), Astrowski-Chun method (ACM), and proposed method (PM), introduced in this article.

The stopping criterion are used in the proposed algorithm is $|x_{n+1} - x_n| < \epsilon = 10^{-10}$

Table 1. Numerical Results for Benchmark Problems

Problem	Method	Number of Iterations	Number of Function Evaluations	Computational Time (s)
1	NM	109	218	0.006841
	JM	Fails to converge	-	-
	ACM	Fails to converge	-	-
	VNM	6	12	0.002948
2	NM	6	12	0.004637
	JM	6	12	0.004097
	ACM	3	6	0.002408
	VNM	3	6	0.002776
3	NM	6	12	0.003744
	JM	5	10	0.003714
	ACM	3	6	0.003362
	VNM	4	8	0.003626
4	NM	5	10	0.003843
	JM	4	8	0.003334
	ACM	5	10	0.003521
	VNM	3	6	0.002375
5	NM	7	14	0.004753
	JM	5	10	0.004451
	ACM	5	10	0.004376
	VNM	4	8	0.002993
6	NM	Fails to converge	-	-
	JM	Fails to converge	-	-
	ACM	Fails to converge	-	-
	VNM	8	16	0.008723

Table 2. Efficiency index for existing and proposed technique

Method	Convergence order	Number of Functions Evaluations	Efficiency index
NM	2	2	1.414
JM	6	2	2.449
ACM	6	2	2.449
PM	6	2	2.449

Table 3. Residual Norms for each Problems

Problem	Method	Residual Norms
1	NM	1.4×10^{-28}
	JM	-
	ACM	-
	VNM	-2.65×10^{-56}
2	NM	-1.04×10^{-50}
	JM	0
	ACM	0
	VNM	0
3	NM	2.93×10^{-55}
	JM	1.00×10^{-59}
	ACM	-3.70×10^{-52}
	VNM	-1.00×10^{-59}
4	NM	1.52×10^{-47}
	JM	0
	ACM	0
	VNM	0
5	NM	2.06×10^{-42}
	JM	0
	ACM	0
	VNM	0
6	NM	-
	JM	-
	ACM	-
	VNM	0

6 Advantages

By using the idea and with the help of the technique used in this paper one can analyze and developed the multistep iterative techniques of higher order for the simulation of nonlinear equations.

7 Limitations and Potential Improvements

While the modified Newton-Raphson technique demonstrates superior performance compared to other iterative methods, it is essential to acknowledge its limitations and potential areas for improvement. One limitation of modified Newton-Raphson technique is its reliance on the initial derivative approximation. If the initial derivative is not a good approximation of the true derivative, the method may exhibit slower convergence or even diverge. This sensitivity to the initial derivative can be mitigated by using more accurate approximations, such as higher-order finite differences or interpolation techniques. Another potential limitation is the method's performance in the presence of multiple roots. In such cases, modified Newton-Raphson method may converge slowly or fail to converge to the desired root. Specialized techniques, such as the deflation method or the modified Halley's method, can be employed to handle these situations more effectively.

Potential improvements to the modified Newton Raphson technique include the use of adaptive strategies for updating the derivative approximation. This adaptive approach can help improve the convergence rate and reduce the sensitivity to the initial derivative. Another possible enhancement is the combination of modified Newton-Raphson technique with other iterative techniques, such as Secant technique or Bisection technique. These hybrid techniques can leverage the strengths of different techniques to achieve faster convergence and improved robustness.

8 Conclusions

We have shown that VNM is at least sixth-order convergence provided the first, second up-to sixth derivatives of the function exist. Computed results (Table 1, 2 and 3) overwhelmingly support the sixth order convergence. The most important characteristics of the VNM is that unlike all other sixth-order or higher order methods, it is not required to compute second or higher derivatives of the function to carry out iterations. Apparently, the VNM needs two more function evaluation at each iteration, when compared to Newton's method. However, it is evident by the computed results (Table 1) that the total number of function evaluations required is less than that of Newton's method.

Conflict of Interest
Authors declare no conflict of interest.

References

[1] J.E. Dennis and R.B. Schnable, *Numerical Methods for Unconstrained Optimization and Nonlinear Equation*, Prentice Hall, (1983).

[2] R.L. Burden and D.J. Faires, *Numerical Analysis*, PWS, Kent Publishing, Boston (1993).

[3] T.G.I. Fernando and S. Weerakoon, *Improved Newton's Method for Finding Roots of a Nonlinear Equation*, Proceedings of the 53th Sri Lanka Association for the Advancement of Science (SLAAS) EI-22, (1997).

[4] J.M. Ortega and W.C. Rheinboldt, *Iterative solution of nonlinear equations in several variables*, SIAM (2000).

[5] S. Abbasbandy, *Improving Newton Raphson Method for Nonlinear Equations by Modified Adomian Decomposition Method*, Appl. Math. Comput., **145**, 887–893, (2003).

[6] C. Chun, *Iterative Method Improving Newton's Method by the Decomposition Method*, Comput. Math. Appl., **50**, 1559–1568, (2005).

[7] A. Cordero and J.R. Torregrosa,*Variant of Newton's Method using Fifth Order Quadrature Formulas*, Appl. Math. Comput., **190**, 686–698, (2007).

[8] R. Rani, S. Sahlot and G. Arora, *Numerical Simulation of Nonlinear SG Equation via DQM and Significant Applications in Josephson Junctions*, PJM, **14**, 1–11, (2025).

[9] P. Devshali and G. Arora, *Application of Differential Transform Method to Solve Fractional Relaxation Oscillation Equation*, IAENG International Journal of Applied Mathematics, **55**, 1685–1694 (2025).

[10] M. Kapoor, N. Singh and G. Arora, *Implementation of Reduced Differential Transform Method upon Fractional Coupled Burgers' Equations in various Dimensions*, Journal of Mathematical Sciences, 1–36, (2025).

[11] S.K. Khattri and I.K. Argyros, *Sixth order derivative free family of Iterative methods*, Appl. Math. Comput.,**217**, 5500–5507, (2011).

[12] X. Wang, J. Kou and Y. Li, *Modified Jarratt method with sixth-order convergence*, Appl. Math. Lett., **22**, 1798–1802, (2009).

[13] Inderjeet and R. Bhardwaj, *A new iterative Newton-Raphson technique for the numerical simulation of nonlinear equations*, JIST, **13**, 1080, (2025).

[14] Inderjeet and R. Bhardwaj, *Newton- Raphson Based Iterative Method for Simulating Nonlinear Equations*, IRJMS, **6**, 857–866, (2025).

[15] Inderjeet and R. Bhardwaj, *Numerical simulation of nonlinear equations by converting quadrature rule to iterative technique*, JIM, **28**, 1837–1845, (2025).

[16] Inderjeet and R. Bhardwaj, *Numerical Simulation of Non- Equations by modified Secant Method*, IAPQR, **49**, 100–117, (2024).

Author information

Inderjeet, Research Scholar, University School of Basic and Applied Sciences, Guru Gobind Singh Indraprastha University, Dwarka - 110078, Delhi, India., India.
E-mail: yadavinderjeet386@gmail.com

Rashmi Bhardwaj, Professor, Head, Nonlinear Dynamics Research Lab, University School of Basic and Applied Sciences, Guru Gobind Singh Indraprastha University, Dwarka - 110078, Delhi, India, India.
E-mail: rashmib@ipu.ac.in

Received: 2025-10-05
Accepted: 2026-02-18