

# On Codes over $\mathfrak{R} = \mathbb{Z}_2\mathbb{Z}_{2^s}$ and their Covering Radii

Abdollah Alhevaz, Fariba Mahmoudi and P. Chella Pandian

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**Abstract** This paper presents a study of repetition codes over  $\mathfrak{R} = \mathbb{Z}_2\mathbb{Z}_{2^s}$ , focusing on the Lee and Euclidean distances. We analyze the covering radius of certain repetition codes constructed using zero divisors and units in  $\mathfrak{R}$ . Additionally, we derive the covering radius for simplex codes and MacDonal codes of types  $\alpha$  and  $\beta$  over  $\mathfrak{R}$  with respect to the Lee and Euclidean distances, based on the aforementioned principles.

## 1 Introduction

The study of codes constructed over finite commutative rings has seen significant development over the past five decades, largely motivated by their capacity to model codes over finite fields via the Gray map. More recently, coding theory has turned its attention to more complex algebraic structures, making the investigation of codes over finite commutative non-chain rings, particularly additive codes, a prominent research area. This exploration draws foundational inspiration from algebraic combinatorics, where the characterization of subgroups within association schemes, as pioneered by Delsarte, provides essential structural insight into these codes (see [14, 15]).

Central to evaluating any code's efficiency is its primary performance metric: the parameters  $(n, k, d)$ . However, the covering radius stands out as a crucial geometric property that not only dictates the maximum error-correcting capability but also has direct implications for practical applications such as data compression and secure transmission. Consequently, the covering radius of various code classes has attracted intensive research over the last three decades (see [6, 7, 8, 9, 10, 11, 13]). While initial studies often focused on linear codes over fields or simpler rings, subsequent efforts extended this analysis to codes over mixed structures (see [1, 4, 5, 6, 19]).

A significant algebraic context arises from the structure of  $\mathbb{Z}_m$ , where  $\mathbb{Z}_m^n$  represents the set of all  $n$ -tuples over  $\mathbb{Z}_m$ . Building upon this, authors have defined additive codes as specific subgroups within translation association schemes, notably those related to the binary Hamming scheme. The underlying abelian groups for these codes have been structurally characterized, primarily taking the form  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ , where  $\alpha + 2\beta = n$ . Codes defined as subgroups of this structure are thus known as  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes (see [14, 15]). More recently, this concept has been broadened to the more general  $\mathfrak{R}$ -additive codes, defined based on abelian structures involving a product of power-of-two rings [2].

Given the significance of simplex and MacDonal codes, which have been defined over various finite commutative rings, and considering the crucial role of the covering radius in determining a code's error-correcting capability, this paper studies the covering radius for these codes over the ring  $\mathfrak{R} = \mathbb{Z}_2\mathbb{Z}_{2^s}$ , with respect to both the Lee and Euclidean distances. This investigation provides necessary characterizations for error-correction performance in this generalized

non-chain ring structure.

The remainder of this paper is organized as follows. Section 2 presents some preliminary and Section 3 is devoted to the study of the covering radius of codes. In Section 4, repetition codes and their covering radii are investigated. Finally, Sections 5 and 6 examine the covering radii of Simplex and MacDonal codes of types  $\alpha$  and  $\beta$ , respectively.

## 2 Preliminaries

In [3] and [4], the authors present preliminary results related to the topic of this section. A code  $C$  is defined as a non-empty subset of  $\mathbb{Z}_m^n$ , where  $n$  denotes the length of the code. A  $\mathbb{Z}_m$ -linear code is a subgroup of  $\mathbb{Z}_m^n$ . In particular, a binary linear code is a subgroup of  $\mathbb{Z}_2^n$ .

The *Hamming weight*  $wt_H(x)$  of a vector  $x$  is defined as the number of nonzero components. The *Lee weight* is defined by  $wt_L(x) : \mathbb{Z}_{2^s} \rightarrow \mathbb{Z}$ , where  $wt_L(i) = \min\{i, 2^s - i\}$ . For  $x \in \mathbb{Z}_{2^s}^n$ ,  $wt_L(x)$  is the sum of the Lee weights of its components. The *Euclidean weight* is defined by  $wt_E(x) : \mathbb{Z}_{2^s} \rightarrow \mathbb{Z}$ , where  $wt_E(i) = \min\{i^2, (2^s - i)^2\}$ . For  $x \in \mathbb{Z}_{2^s}^n$ ,  $wt_E(x)$  is the sum of the Euclidean weights of its components.

The Hamming, Lee, and Euclidean distances are defined as  $d_H(x, y) = wt_H(x - y)$ ,  $d_L(x, y) = wt_L(x - y)$ , and  $d_E(x, y) = wt_E(x - y)$ , respectively. The minimum Hamming, Lee, and Euclidean weights among all nonzero codewords of  $C$  are denoted by  $wt_H(C)$ ,  $wt_L(C)$ , and  $wt_E(C)$ , respectively. There are alternative definitions of the Gray map. The one used in this paper is defined in [18]. The Gray map  $\varphi : \mathbb{Z}_{2^s} \rightarrow \mathbb{Z}_2^{2^s-1}$  is defined by

$$\varphi(i) = \begin{cases} 0_{2^{s-1}-i} 1_i & 0 \leq i \leq 2^{s-1}, \\ 1_{2^{s-1}} + \varphi(i - 2^{s-1}) & i > 2^{s-1}. \end{cases} \quad (2.1)$$

This Gray map is extended to  $\mathbb{Z}_{2^s}^n$  as follows:

$$\varphi(v_1, v_2, \dots, v_n) = (\varphi(v_1), \varphi(v_2), \dots, \varphi(v_n)).$$

Note that the Gray map is an isometry that transforms the Lee distance over  $\mathbb{Z}_{2^s}^n$  to the Hamming distance over  $\mathbb{Z}_2^{2^s-1n}$ .

Let  $C$  be a  $\mathfrak{R}$ -additive code; it is isomorphic to an abelian structure of the form  $\mathbb{Z}_2^{k_1} \times \mathbb{Z}_4^{k_2} \times \dots \times \mathbb{Z}_{2^s}^{k_s}$ . Thus,  $|C| = 2^{k_1+2k_2+\dots+sk_s}$ , where  $|C|$  is the number of codewords in  $C$ . The  $\mathfrak{R}$ -additive codes can also be viewed as binary codes, referred to as  $\mathfrak{R}$ -linear codes, by considering the extension of the usual Gray map  $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta \rightarrow \mathbb{Z}_2^n$ , where  $n = \alpha + 2^{s-1}\beta$ , given by

$$\Phi(x, y) = (x, \varphi(y_1), \varphi(y_2), \dots, \varphi(y_\beta)),$$

for all  $x = (x_1, x_2, \dots, x_\alpha) \in \mathbb{Z}_2^\alpha$  and  $y = (y_1, y_2, \dots, y_\beta) \in \mathbb{Z}_{2^s}^\beta$ , where  $\varphi : \mathbb{Z}_{2^s} \rightarrow \mathbb{Z}_2^{2^s-1}$  is a generalization of the usual Gray map defined in (2.1).

The Gray map defined above is an isometry that transforms the Lee distance defined on  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta$  to the Hamming distance defined over  $\mathbb{Z}_2^n$ , with  $n = \alpha + 2^{s-1}\beta$ . Let  $v = (v_1, v_2) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta$  be a vector; then the weight of  $v$ , denoted by  $wt_L(v)$ , is defined as  $w_H(v_1) + w_L(v_2)$ . Also,  $wt_E(v)$ , is defined as  $w_H(v_1) + w_E(v_2)$ . The Gray map  $\Phi(v)$  is an isometry that transforms the distance defined in a  $\mathfrak{R}$ -additive code  $C$  over  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta$  to the Hamming distance defined in the corresponding  $\mathfrak{R}$ -linear code  $\Phi(C)$ . Note that the length of  $\Phi(C)$  is  $n = \alpha + 2^{s-1}\beta$ .

Let  $X$  be the set of  $\mathbb{Z}_2$  and  $Y$  be the set of  $\mathbb{Z}_{2^s}$ , respectively, with the coordinate positions, so  $|X| = \alpha$  and  $|Y| = \beta$ . Unless otherwise stated, the set  $X$  corresponds to the first  $\alpha$  coordinates, and  $Y$  corresponds to the last  $\beta$  coordinates. We denote  $C_X$  (resp.,  $C_Y$ ) as the punctured code of  $C$  obtained by deleting the coordinates outside  $X$  (resp.,  $Y$ ). Let  $C_b$  be the subcode of  $C$  that contains all order two codewords, and let  $k_0$  be the dimension of  $(C_b)_X$ , which is a binary linear

code. For the case  $\alpha = 0$ , we write  $k_0 = 0$ . Considering all these parameters, we will say that  $C$  (or equivalently  $\Phi(C)$ ) is of type  $(\alpha, \beta; k_1, k_2, \dots, k_s; k_0)$ .

Define an inner product of vectors  $u$  and  $v$  in  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta$  as follows:

$$\langle u, v \rangle = 2^{s-1} \sum_{i=1}^{\alpha} u_i v_i + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j.$$

Let  $C$  be a  $\mathfrak{R}$ -additive code, and the additive dual code of  $C$  is denoted by  $C^\perp$ , defined as follows:

$$C^\perp = \{v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta \mid \langle u, v \rangle = 0 \text{ for all } u \in C\}.$$

The corresponding binary code  $\phi(C^\perp)$  is denoted by  $C_\perp$  and is called the  $\mathfrak{R}$ -dual code of  $C$ . The generator matrices of  $C$  and  $C^\perp$  were given in [2].

Throughout  $\mathbf{0}, \mathbf{1}, \mathbf{0}_i, \mathbf{1}_i$  and  $2^{s-1}$  denote for the all-zero vector, the all-one vector, the all-zero vector of length  $i$ , the all-one vector of length  $i$ , and the vector whose components are all equal to  $2^{s-1}$ , respectively. The length of these vectors will be clear from the context.

### 3 The covering radii of codes over $\mathfrak{R}$

In this section, we introduce the concept of the covering radius of a code  $C$  over  $\mathfrak{R}$ . We begin by recalling the definition of the covering radius for a binary code. For a binary code  $C$ , the covering radius  $r(C)$  is defined as

$$r(C) = \max_{u \in \mathbb{Z}_2} \{ \min_{c \in C} d_H(u, c) \}.$$

This definition can also be interpreted as the smallest number  $r$  such that the spheres of radius  $r$  around the codewords cover  $\mathbb{Z}_2$ . When extending this definition to codes over  $\mathfrak{R}$ , the covering radius of a code  $C$  is the smallest number  $r$  such that the spheres of radius  $r$  around the codewords cover  $\mathfrak{R}$ . Therefore, the covering radius of a code  $C$  over  $\mathfrak{R}$ , with respect to the Lee and Euclidean distances, is given by

$$r_L(C) = \max_{u \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta} \{ \min_{c \in C} d_{L(E)}(u, c) \}.$$

It is evident that  $r_L(C)$  and  $r_E(C)$  represent the minimum values  $r_L$  and  $r_E$  such that  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta = \cup_{c \in C} S_{r_L}(c)$ , respectively, where  $S_{r_L}(u) = \{v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta : d_L(u, v) \leq r_L\}$ , and  $S_{r_E}(u) = \{v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta : d_E(u, v) \leq r_E\}$ , for  $u \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta$ .

The following result, given in [1], for codes over  $\mathbb{Z}_4$  also applies to codes over  $\mathfrak{R}$ . Its proof is based on the definition of the covering radius and the fact that the map  $\Phi$  is a weight-preserving map.

**Proposition 3.1.** *Let  $C$  be a code over  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_{2^s}^\beta$  and  $\Phi(C)$  be the Gray map image of  $C$ . Then,  $r_L(C) = r(\Phi(C))$ .*

Let  $C$  be a code over  $\mathfrak{R}$ , and define  $s(C^\perp) = |\{i : A_i(C^\perp) \neq 0, i \neq 0\}|$ , where  $A_i(C^\perp)$  denotes the number of codewords of weight  $i$  in  $C^\perp$ .

**Lemma 3.2.** *For a code  $C$  over  $\mathfrak{R}$ ,  $r_L(C) \leq r_E(C) \leq 2r_L(C)$ .*

**Proof.** The result is derived from the inequality  $d_L(x, y) \leq d_E(x, y) \leq 2d_L(x, y)$ , for any two vectors  $x$  and  $y$ .

In [16], it is demonstrated that the covering radius  $r(B)$  of a binary code  $B$  and the number of distinct nonzero weights in the distance distribution of the dual code  $B^\perp$ , denoted by  $s(B^\perp)$ , satisfy the following inequality, known as the Delsarte bound:

$$r(B) \leq s(B^\perp).$$

We now extend the Delsarte bound to codes over  $\mathfrak{R}$ .

**Theorem 3.3.** *Let  $C$  be a code over  $\mathfrak{R}$ . Then,  $r_L(C) \leq s(C^\perp)$  and  $r_E(C) \leq 2s(C^\perp)$ .*

## 4 Repetition Codes

Let  $\mathbb{F}_q = \{\alpha_0 = 0, \alpha_1 = 1, \dots, \alpha_{q-1}\}$  be a finite field. A  $q$ -ary repetition code  $C = \{\bar{\alpha} | \alpha \in \mathbb{F}_q\}$  is a  $(n, q, n)$  code over  $\mathbb{F}_q$ , where  $\bar{\alpha} = (\alpha, \dots, \alpha) \in \mathbb{F}_q^n$ . The covering radius of the repetition code  $C$  over  $\mathbb{F}_q$  is given by  $\lceil \frac{n(q-1)}{q} \rceil$ . In this context, we introduce several types of repetition codes over  $\mathfrak{R}$ .

### 4.1 Zero Divisor Repetition Codes

Let  $z$  be a zero divisor in the ring  $\mathbb{Z}_{2^s}$ . The code generated by the generator matrix  $[zz \dots z]$  is referred to as a *zero divisor repetition code*. In  $\mathbb{Z}_{2^s}$ , there are  $2^{s-1} - 1$  distinct zero divisors, which can be expressed in the form  $a_1 2^{s-1} + a_2 2^{s-2} + \dots + a_{s-1} 2$ , where each coefficient  $\alpha_i$  is an element of  $\{0, 1\}$ . This representation highlights the binary nature of the coefficients, indicating which powers of 2 contribute to the zero divisor. Initially, we focus on the zero divisors of order 2 within the ring  $\mathfrak{R}$ . For this specific case, we define the code  $C_2$  as follows:  $C_2 : [n, 1, \frac{n}{2}, n2^{s-2}]$ , which is generated by the generator matrix  $G_2 = [02^{s-1} \dots 02^{s-1}]$ . This structure allows us to investigate the properties and applications of zero divisor repetition codes in further detail, particularly in the context of their performance and efficiency in coding theory.

**Theorem 4.1.** *The covering radius of the code  $C_2$  over  $\mathfrak{R}$ , with respect to the Euclidean weight and the Lee weight, is expressed as follows:*

$$r_E(C_2) = \frac{n}{4}(2^{2s-2} + 1), \quad r_L(C_2) = \frac{n}{4} + 2^{s-3}n.$$

**Proof.** Let  $x = \overbrace{00 \dots 00}^{\frac{n}{4}} \overbrace{02^{s-1} \dots 02^{s-1}}^{\frac{n}{4}} \overbrace{10 \dots 10}^{\frac{n}{4}} \overbrace{12^{s-1} \dots 12^{s-1}}^{\frac{n}{4}} \in \mathfrak{R}^n$ . We can assume  $x = (x_1 x_2)$  and  $x_1 = \overbrace{0 \dots 0}^{\frac{n}{4}} \overbrace{1 \dots 1}^{\frac{n}{4}} \in \mathbb{Z}_2^{\frac{n}{2}}$  and  $x_2 = \overbrace{0 \dots 0}^{\frac{n}{4}} \overbrace{2^{s-1} \dots 2^{s-1}}^{\frac{n}{4}} \in \mathbb{Z}_2^{\frac{n}{2}}$ . Then

$$\begin{aligned} d_E(x, \overbrace{00 \dots 00}^n) &= d_H(x_1, \overbrace{0 \dots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{0 \dots 0}^{\frac{n}{2}}) \\ &= \frac{n}{4} + 2\left(\frac{n}{8}(2^{2(s-1)})\right) \\ d_E(x, \overbrace{00 \dots 00}^n) &= \frac{n}{4}(2^{2s-2} + 1) \\ d_E(x, \overbrace{00 \dots 00}^{\frac{n}{4}} \overbrace{02^{s-1} \dots 02^{s-1}}^{\frac{n}{4}}) \\ &= d_H(x_1, \overbrace{0 \dots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{0 \dots 0}^{\frac{n}{2}}) \\ &= \frac{n}{4} + 2\left(\frac{n}{8}(2^{2(s-1)})\right) \\ d_E(x, \overbrace{00 \dots 00}^{\frac{n}{4}} \overbrace{02^{s-1} \dots 02^{s-1}}^{\frac{n}{4}}) \\ &= d_H(x_1, \overbrace{0 \dots 0}^{\frac{n}{4}} \overbrace{12^{s-1} \dots 12^{s-1}}^{\frac{n}{4}}) + d_E(x_2, \overbrace{0 \dots 0}^{\frac{n}{4}} \overbrace{12^{s-1} \dots 12^{s-1}}^{\frac{n}{4}}) \\ &= \frac{n}{4} + 2\left(\frac{n}{8}(2^{2(s-1)})\right) \\ d_E(x, \overbrace{00 \dots 00}^{\frac{n}{4}} \overbrace{02^{s-1} \dots 02^{s-1}}^{\frac{n}{4}}) \\ &= d_H(x_1, \overbrace{0 \dots 0}^{\frac{n}{4}} \overbrace{12^{s-1} \dots 12^{s-1}}^{\frac{n}{4}}) + d_E(x_2, \overbrace{0 \dots 0}^{\frac{n}{4}} \overbrace{12^{s-1} \dots 12^{s-1}}^{\frac{n}{4}}) \\ &= \frac{n}{4}(2^{2s-2} + 1) \end{aligned}$$

Thus

$$r_E(C_2) \geq \frac{n}{4}(2^{2s-2} + 1). \quad (4.1)$$

Let  $x = (x_1x_2) \in \mathfrak{R}^n$  and let  $w_i$  be the number of  $i$  coordinates in  $x_2$  for  $0 \leq i \leq 2^{s+1} - 1$ . Then  $\sum_{i=0}^{2^{s+1}-1} w_i = \frac{n}{2}$ . Consider

$$\begin{aligned} d_E(x, \overbrace{00 \cdots 00}^n) &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) \\ &= \frac{n}{4} + 0(w_0 + w_{2^s}) + 1(w_1 + w_{2^s-1} + w_{2^s+1} + w_{2^{s+1}-1}) \\ &\quad + \cdots + 2^{2s-2}(w_{2^s-1} + w_{2^s+2^s-1}), \\ d_E(x, \overbrace{02^{s-1} \cdots 02^{s-1}}^n) &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_1, \overbrace{2^{s-1} \cdots 2^{s-1}}^{\frac{n}{2}}) \\ &= \frac{n}{4} + 0(w_{2^s-1} + w_{2^s+2^s-1}) + 1(w_{2^s-1-1} + w_{2^s-1+1} \\ &\quad + w_{2^s+2^s-1-1} + w_{2^s+2^s-1+1}) + \cdots + 2^{2s-2}(w_0 + w_{2^s}). \end{aligned}$$

The minimum is less than the average. So

$$r_E(C_2) \leq \frac{n}{4} + \frac{n}{4}(2^{2s-2}). \quad (4.2)$$

From equation (4.1) and (4.2), we have  $r_E(C_2) = \frac{n}{4}(2^{2s-2} + 1)$ . The above argument follows for Lee distance.

Now, we consider zero divisors of order  $i$  in  $\mathfrak{R}$ . Let  $C_{2^i} : [n, i, \frac{n}{2}, n2^{s-2}]$  be the code generated by the generator matrix  $G_{2^i} = [02^{s-i} \cdots 02^{s-i}]$ .

**Theorem 4.2.**  $r_E(C_{2^i}) = \frac{n}{4}(2^{2s-2} + 1)$ ,  $r_L(C_{2^i}) = \frac{n}{4} + 2^{s-3}n$ .

**Proof.** Let  $x = \overbrace{00 \cdots 00}^{\frac{n}{4}} \overbrace{02^{s-1} \cdots 02^{s-1}}^{\frac{n}{4}} \overbrace{10 \cdots 10}^{\frac{n}{4}} \overbrace{12^{s-1} \cdots 12^{s-1}}^{\frac{n}{4}} \in \mathfrak{R}^n$ . We can assume  $x = (x_1x_2)$  and  $x_1 = \overbrace{0 \cdots 0}^{\frac{n}{4}} \overbrace{1 \cdots 1}^{\frac{n}{4}} \in \mathbb{Z}_2^{\frac{n}{2}}$  and  $x_2 = \overbrace{0 \cdots 0}^{\frac{n}{4}} \overbrace{2^{s-1} \cdots 2^{s-1}}^{\frac{n}{4}} \in \mathbb{Z}_2^{\frac{n}{2}}$  and for  $i = 1, 2, \dots, s-$

1. We have

$$\begin{aligned}
d_E(x, \overbrace{00 \cdots 00}^n) &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{0 \cdots 0}^{\frac{n}{2}}) \\
&= \frac{n}{4} + 2\left(\frac{n}{8}(2^{2(s-1)})\right) \\
d_E(x, \overbrace{00 \cdots 00}^n) &= \frac{n}{4}(2^{2s-2} + 1) \\
d_E(x, \overbrace{00 \cdots 00}^{\frac{n}{4}} \overbrace{02^{s-i} \cdots 02^{s-i}}^{\frac{n}{4}}) & \\
\overbrace{10 \cdots 10}^{\frac{n}{4}} \overbrace{12^{s-i} \cdots 12^{s-i}}^{\frac{n}{4}} &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{0 \cdots 0}^{\frac{n}{2}}) \\
&= \frac{n}{4}(2^{2s-2} + 1) \\
&\vdots \\
d_E(x, \overbrace{00 \cdots 00}^{\frac{n}{4}} \overbrace{02^{s-1} \cdots 02^{s-1}}^{\frac{n}{4}}) & \\
\overbrace{10 \cdots 10}^{\frac{n}{4}} \overbrace{12^{s-1} \cdots 12^{s-1}}^{\frac{n}{4}} &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{0 \cdots 0}^{\frac{n}{2}}) \\
&= \frac{n}{4}(2^{2s-2} + 1) \\
&\vdots \\
d_E(x, \overbrace{00 \cdots 00}^{\frac{n}{4}} \overbrace{02^s - 2^{s-i} \cdots 02^s - 2^{s-i}}^{\frac{n}{4}}) & \\
\overbrace{10 \cdots 10}^{\frac{n}{4}} \overbrace{12^s - 2^{s-i} \cdots 12^s - 2^{s-i}}^{\frac{n}{4}} &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{0 \cdots 0}^{\frac{n}{2}}) \\
&= \frac{n}{4}(2^{2s-2} + 1).
\end{aligned}$$

So,  $r_E(C_{2^i}) \geq \frac{n}{4}(2^{2s-2} + 1)$ . Let  $x \in \mathfrak{R}^n$  and  $w_j$  be same as in Theorem 4.1. Then,

$\sum_{j=0}^{2^k-1} w_j = \frac{n}{2}$ . For  $i = 1, 2, \dots, s-1$ , we have

$$\begin{aligned}
d_E(x, \overbrace{00 \cdots 00}^n) &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{0 \cdots 0}^{\frac{n}{2}}) \\
&= \frac{n}{4} + 0(w_0 + w_{2^s}) + 1(w_1 + w_{2^s-1} + w_{2^s+1} + w_{2^s+1-1}) \\
&\quad + \cdots + 2^{2^s-2}(w_{2^s-1} + w_{2^s+2^s-1}), \\
d_E(x, \overbrace{02^{s-i} \cdots 02^{s-i}}^n) &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{2^{s-i} \cdots 2^{s-i}}^{\frac{n}{2}}) \\
&= \frac{n}{4} + 0(w_{2^s-2^{s-i}} + w_{2^{2^s+1-2^{s-i}}}) \\
&\quad + 1(w_{2^s-2^{s-i}-1} + w_{2^s-2^{s-i}+1} + w_{2^s+1-2^{s-i}-1} + w_{2^s+1-2^{s-i}+1}) \\
&\quad + \cdots + 2^{2^s-2}(w_{2^s-1-2^{s-i}} + w_{2^s+2^{s-1}-2^{s-i}}), \\
&\quad \vdots \\
d_E(x, \overbrace{02^{s-1} \cdots 02^{s-1}}^n) &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{2^{s-1} \cdots 2^{s-1}}^{\frac{n}{2}}) \\
&= \frac{n}{4} + 0(w_{2^s-1} + w_{2^s+2^s-1}) + 1(w_{2^s-1-1} + w_{2^s-1+1} \\
&\quad + w_{2^s+2^s-1-1} + w_{2^s+2^s-1+1}) + \cdots + 2^{2^s-2}(w_0 + w_{2^s}). \\
&\quad \vdots \\
d_E(x, \overbrace{02^s - 2^{s-i} \cdots 02^s - 2^{s-i}}^n) &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{2^s - 2^{s-i} \cdots 2^s - 2^{s-i}}^{\frac{n}{2}}) \\
&= \frac{n}{4} + 0(w_{2^s-i} + w_{2^s+2^s-i}) \\
&\quad + 1(w_{2^s-i-1} + w_{2^s-i+1} + w_{2^s+2^s-i-1} + w_{2^s+2^s-i+1}) \\
&\quad + \cdots + 2^{2^s-2}(w_{2^s-2^s-i} + w_{2^s+1-2^s-i}).
\end{aligned}$$

The minimum is less than the average. So,

$$\begin{aligned}
r_E(C_{2^i}) &\leq \frac{n}{4} + \frac{2^{2^s-2} + 2 \times 2^{2^s-2i}(1^2 + 2^2 + \cdots + (2^{s-1} - 1)^2)}{2^i} \\
&\leq \frac{n}{4} + \frac{1}{2^i}(2^{2^s-2} + 2 \times 2^{2^s-2i}(\frac{(2^{i-1} - 1)2^{i-1}(2^i - 1)}{6})) \\
&\leq \frac{n}{4} + \frac{n}{4}(2^{2^s-2}).
\end{aligned}$$

## 4.2 Unit Repetition Codes

Let  $\overbrace{11 \cdots 11}^n$  be a unit in  $\mathfrak{R}$ . Then the parameters of a code  $C_1 : [n, 2^s, n, n]$  is called *unit repetition code* generated by  $G_1 = \overbrace{[11 \cdots 11]}^n$ .

**Theorem 4.3.** *Let  $C_1$  be a code in  $\mathfrak{R}$ . The covering radius is  $r_E(C_1) = \frac{n}{3}(2^{2s-3}n+1)$ ,  $r_L(C_1) = \frac{n}{4} + n2^{s-3}$ .*

**Proof.** Let  $x = \overbrace{00 \cdots 00}^l \overbrace{01 \cdots 01}^l \cdots \overbrace{02^s - 1 \cdots 02^s - 1}^l \overbrace{10 \cdots 10}^l \cdots \overbrace{12^s - 1 \cdots 12^s - 1}^{n-(2^s-1)l} \in \mathfrak{R}^n$ . We can assume  $x = (x_1 x_2)$  and  $x_1 = \overbrace{00 \cdots 0}^{\frac{n}{2}} \in \mathbb{Z}_2^{\frac{n}{2}}$  and  $x_2 = \overbrace{012 \cdots 2^s - 1}^{\frac{n}{2}} \in \mathbb{Z}_{2^s}^{\frac{n}{2}}$ . Thus,

$$\begin{aligned} d_E(x, \overbrace{00 \cdots 00}^n) &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{0 \cdots 0}^{\frac{n}{2}}) \\ &= \frac{n}{4} + 2(2 \times \frac{2^{s-1}(2^{s-1}+1)(2^s+1)}{6} - 2^{2s-2}) \frac{l}{2} \\ &\quad - 1 \frac{l}{2} + \frac{n}{2} - (2^{s+1}-1) \frac{l}{2} \\ &= \frac{n}{3}(2^{2s-3}+1). \end{aligned}$$

$$\begin{aligned} d_E(x, \overbrace{01 \cdots 01}^n) &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{1 \cdots 1}^{\frac{n}{2}}) \\ &= \frac{n}{4} + 2(2 \times \frac{2^{s-1}(2^{s-1}+1)(2^s+1)}{6} - 2^{2s-2}) \frac{l}{2} \\ &= \frac{n}{3}(2^{2s-3}+1). \end{aligned}$$

⋮

$$\begin{aligned} d_E(x, \overbrace{02^s - 1 \cdots 02^s - 1}^n) &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{2^s - 1 \cdots 2^s - 1}^{\frac{n}{2}}) \\ &= \frac{n}{4} + 2(2 \times \frac{2^{s-1}(2^{s-1}+1)(2^s+1)}{6} - 2^{2s-2}) \frac{l}{2} \\ &= \frac{n}{3}(2^{2s-3}+1). \end{aligned}$$

We know,  $r_E(C_1) \geq d_E(x, C_1)$ , so  $r_E(C_1) \geq \frac{n}{3}(2^{2s-3}+1)$ . If  $x \in \mathfrak{R}^n$  and let  $w_i$  be the number

of  $i$  coordinates in  $x$  for  $0 \leq i \leq 2^{s+1} - 1$ . Then,  $\sum_{i=0}^{2^{s+1}-1} w_i = \frac{n}{2}$ . Consider

$$\begin{aligned}
 d_E(x, \overbrace{00 \cdots 00}^n) &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{0 \cdots 0}^{\frac{n}{2}}) \\
 &= \frac{n}{4} + 0(w_0 + w_{2^s}) + 1(w_1 + w_{2^s-1} + w_{2^s+1} + w_{2^s+1-1}) \\
 &\quad + \cdots + 2^{2^s-2}(w_{2^s-1} + w_{2^s+2^s-1}), \\
 &\quad \vdots \\
 d_E(x, \overbrace{02^{s-1} \cdots 02^{s-1}}^n) &= d_H(x_1, \overbrace{0 \cdots 0}^{\frac{n}{2}}) + d_E(x_2, \overbrace{2^{s-1} \cdots 2^{s-1}}^{\frac{n}{2}}) \\
 &= 0(w_{2^s-1} + w_{2^s+2^s-1}) + 1(w_{2^s-1-1} + w_{2^s-1+1} \\
 &\quad + w_{2^s+2^s-1-1} + w_{2^s+2^s-1+1}) + \cdots + 2^{2^s-2}(w_0 + w_{2^s}). \\
 d_E(x, \overbrace{10 \cdots 10}^n) &= d_H(x_1, \overbrace{1 \cdots 1}^{\frac{n}{2}}) + d_E(x_2, \overbrace{0 \cdots 0}^{\frac{n}{2}}) \\
 &= \frac{n}{4} + 0(w_0 + w_{2^s}) + 1(w_1 + w_{2^s-1} + w_{2^s+1} + w_{2^s+1-1}) \\
 &\quad + \cdots + 2^{2^s-2}(w_{2^s-1} + w_{2^s+2^s-1}), \\
 &\quad \vdots \\
 d_E(x, \overbrace{12^{s-1} \cdots 12^{s-1}}^n) &= d_H(x_1, \overbrace{1 \cdots 1}^{\frac{n}{2}}) + d_E(x_2, \overbrace{2^{s-1} \cdots 2^{s-1}}^{\frac{n}{2}}) \\
 &= \frac{n}{4} + 0(w_{2^s-1} + w_{2^s+2^s-1}) + 1(w_{2^s-1-1} + w_{2^s-1+1} \\
 &\quad + w_{2^s+2^s-1-1} + w_{2^s+2^s-1+1}) + \cdots + 2^{2^s-2}(w_0 + w_{2^s}).
 \end{aligned}$$

The minimum is less than the average. So,  $r_E(C_1) \leq \frac{n}{4} + \frac{n}{12}(2^{2^s-1} + 1)$ .

Now, we consider the covering radii of the block repetition codes over  $\mathfrak{R}$ . Let  $C^{n_{2^{s+1}-1}}$  be the block repetition code over  $\mathfrak{R}$ . It is an  $\mathfrak{R}$ -additive code of length  $n = \sum_{j=1}^{2^{s+1}-1} n_j$  with generator matrix

$$G = (\overbrace{01 \cdots 01}^{n_1} \cdots \overbrace{02^s - 1 \cdots 02^s - 1}^{n_{2^s-1}} \overbrace{10 \cdots 10}^{n_{2^s}} \cdots \overbrace{12^s - 1 \cdots 12^s - 1}^{n_{2^{s+1}-1}}).$$

We have the following result.

**Theorem 4.4.** *Let  $C^{n_{2^{s+1}-1}}$  be the block repetition code over  $\mathfrak{R}$ . Then*

$$\begin{aligned}
 r_E(C^{n_{2^{s+1}-1}}) &= \frac{1}{3}(2^{2^s-3}n + 1)(n_1 + n_3 + \cdots + n_{2^s-1}) + \frac{1}{4}(2^{2^s-2} + 1)(n_2 + n_4 + \cdots + n_{2^s-2}), \\
 r_L(C^{(2^{s+1}-1)n}) &= \frac{n}{4} + (2^{s-3})(n_1 + n_2 + \cdots + n_{2^{s+1}-1}).
 \end{aligned}$$

### 5 The covering radii of simplex codes over $\mathfrak{R}$

In this section, we consider the construction of simplex codes of types  $\alpha$  and  $\beta$  over  $\mathfrak{R}$ . Let  $m_{2,k}^\alpha$  be the generator matrix of  $S_{2,k}^\alpha$ , the binary simplex code of type  $\alpha$  is defined as

$$\left[ \begin{array}{c|c} 00 \cdots 0 & 11 \cdots 1 \\ \hline m_{2,k-1}^\alpha & m_{2,k-1}^\alpha \end{array} \right], \tag{5.1}$$

where

$$m_{2,1}^\alpha = \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{5.2}$$

The generator matrix  $G_{s,k}^\alpha$  of  $S_{s,k}^\alpha$  is

$$\left[ \begin{array}{c|c|c|c|c} 00 \cdots 0 & 11 \cdots 1 & 22 \cdots 2 & \cdots & 2^s - 12^s - 1 \cdots 2^s - 1 \\ \hline G_{s,k-1}^\alpha & G_{s,k-1}^\alpha & G_{s,k-1}^\alpha & \cdots & G_{s,k-1}^\alpha \end{array} \right], \tag{5.3}$$

where

$$G_{s,1}^\alpha = [0 \quad 1 \quad \cdots \quad 2^s - 1]. \tag{5.4}$$

We define the generator matrix of  $S_{s,k}^\alpha$ , the simplex code of type  $\alpha$  over  $\mathbb{Z}_2\mathbb{Z}_4$ , for  $k \geq 1$ , as the concatenation of  $2^{sk}$  copies of the generator matrix of  $S_{2,k}^\alpha$  and  $2^k$  copies of the generator matrix of  $S_{s,k}^\alpha$  is given by

$$\theta_k^\alpha = \left[ m_{2,k}^\alpha \mid m_{2,k}^\alpha \mid \cdots \mid m_{2,k}^\alpha \mid G_{s,k}^\alpha \mid \cdots \mid G_{s,k}^\alpha \mid \cdots \mid G_{s,k}^\alpha \right], \tag{5.5}$$

So, the length of  $S_{2,k}^\alpha$  is  $2^{(s+1)k+1}$ . The standard form of  $\theta_k^\alpha$ , for  $k \geq 2$  is given by

$$\Theta_k^\alpha = \left[ \begin{array}{c|c|c|c|c} 00 \cdots 00 & 01 \cdots 01 & \cdots & \cdots & 12^s - 112^s - 1 \cdots 12^s - 112^s - 1 \\ \hline \Theta_{s,k-1}^\alpha & \Theta_{s,k-1}^\alpha & \Theta_{s,k-1}^\alpha & \cdots & \Theta_{s,k-1}^\alpha \end{array} \right], \tag{5.6}$$

where

$$\Theta_1^\alpha = [00 \mid 01 \mid \cdots \mid 02^s - 1 \mid 10 \mid \cdots \mid 12^s - 1]. \tag{5.7}$$

Now, the following theorems provide bounds on the covering radii of simplex codes over  $\mathfrak{R}$ .

**Theorem 5.1.**  $r_L(S_{s,k}^\alpha) \leq (1 + 2^{s-1}) \times 2^{(s+1)(k-1)}$ .

**Proof.** By definition of  $\mathfrak{R}$ -simplex codes of type  $\alpha$ , it is the concatenation of  $2^{sk}$  copies of the generator matrix of  $S_{2,k}^\alpha$  and  $2^k$  copies of the generator matrix of  $S_{s,k}^\alpha$ . From [11] and Equation(5.3), we have

$$\begin{aligned} r_L(S_{s,k}^\alpha) &\leq r_L(2^{sk}S_{2,k}^\alpha) + r_L(2^kS_{s,k}^\alpha) \\ &\leq 2^{sk}r_L(S_{2,k}^\alpha) + 2^k r_L(S_{s,k}^\alpha) \\ &\leq 2^{sk}r_H(S_{2,k}^\alpha) + 2^k r_L(S_{s,k-1}^\alpha) + r_L(\langle \overbrace{11 \cdots 1}^{2^{s(k-1)}} \cdots \overbrace{2^s - 1 \cdots 2^s - 1}^{2^{s(k-1)}} \rangle) \\ &\leq 2^{sk}(2^{k-1}) + 2^k [(2^{s-2}(2^s - 1)(2^{s(k-1)} + \cdots + 2^{s \times 1})) + r_L(S_{s,1}^\alpha)] \\ &= 2^{(s+1)(k-1)} + 2^k [2^{s-2}(2^s - 1)(\frac{2^{sk} - 1}{2^s - 1} - 1) + 2^{s-2}] \\ &= 2^{(s+1)(k-1)} + 2^k (2^{s-2}2^{sk}) \\ &= 2^{(s+1)k} (2^{-1} + 2^{s-2}), \end{aligned}$$

and the result follows.

**Theorem 5.2.**  $r_E(S_{s,k}^\alpha) \leq 2^{(s+1)k} (\frac{5 \times 2^{3s-3} - 3 \times 2^{2s-2} + 2^{s-1} + 3(2^s - 1)}{6(2^s - 1)}) - 2^k (\frac{2^{2s-3}}{6})$ .

**Proof.** By the definition of  $\mathfrak{R}$ -simplex code of type  $\alpha$ , it is the concatenation of  $2^{sk}$  copies of the generator matrix of  $S_{2,k}^\alpha$  and  $2^k$  copies of the generator matrix of  $S_{s,k}^\alpha$ . From [11] and Equation

(5.3), we have

$$\begin{aligned}
 r_E(S_{s,k}^\alpha) &\leq r_H(2^{sk}S_{2,k}^\alpha) + r_E(2^k S_{s,k}^\alpha) \\
 &\leq 2^{sk}r_H(S_{2,k}^\alpha) + 2^k r_E(S_{s,k}^\alpha) \\
 &\leq 2^{sk}(2^{k-1}) + 2^k \left[ \left( \frac{1}{6}(2^{2s-1} + 1)(2^{s-1}) + 2^{2s-3}(2^{s-1} - 1) \right) (2^{s(k-1)}) \right. \\
 &\quad \left. + r_E(S_{s,k-1}^\alpha) \right] \\
 &\leq 2^{sk}(2^{k-1}) + 2^k \left[ \frac{1}{6}(5 \times 2^{3s-3} - 3 \times 2^{2s-2} + 2^{s-1})(2^{s(k-1)} + \dots + 2^{s \times 1}) \right. \\
 &\quad \left. + r_E(S_{s,1}^\alpha) \right] \\
 &\leq 2^{sk}(2^{k-1}) + 2^k \left[ \frac{1}{6}(5 \times 2^{3s-3} - 3 \times 2^{2s-2} + 2^{s-1}) \left( \frac{2^{sk} - 1}{2^s - 1} - 1 \right) \right. \\
 &\quad \left. + \frac{1}{6}(5 \times 2^{3s-3} + 2^{s-1}) \right] \\
 &\leq 2^{sk}(2^{k-1}) + 2^k \left[ \frac{1}{6}(5 \times 2^{3s-3} - 3 \times 2^{2s-2} + 2^{s-1}) \left( \frac{2^{sk} - 1}{2^s - 1} \right) + 2^{2s-3} \right] \\
 r_E(S_{s,k}^\alpha) &\leq 2^{(s+1)k} \left( \frac{5 \times 2^{3s-3} - 3 \times 2^{2s-2} + 2^{s-1} + 3(2^s - 1)}{6(2^s - 1)} \right) - 2^k \left( \frac{2^{2s-3}}{6} \right).
 \end{aligned}$$

We now focus on the  $\mathfrak{R}$ -simplex codes of type  $\beta$ . Since type  $\beta$  is a punctured version of  $S_k^\alpha$ , we define the generator matrix  $S_{s,k}^\beta$ , for  $k \geq 1$ , such that no two columns of  $G_k^\beta$  are multiple of each other. So  $G_k^\beta$  is the concatenation of  $2^{sk}$  copies of the generator matrix of  $S_{2,k}^\beta$  is denoted by  $m_{2,k}^\beta$  and  $2^k$  copies of the generator matrix of  $S_{s,k}^\beta$  is denoted by  $G_{s,k}^\alpha$  is given by

$$\Theta_k^\beta = \left[ m_{2,k}^\beta \mid m_{2,k}^\beta \mid \dots \mid m_{2,k}^\beta \mid G_{s,k}^\beta \mid \dots \mid G_{s,k}^\beta \right], \quad (5.8)$$

where

$$m_{2,2}^\beta = \left[ \begin{array}{c|c} 11 & 0 \\ \hline 01 & 1 \end{array} \right], \quad (5.9)$$

and  $m_{2,k}^\beta$ , for  $k \geq 3$  is given by

$$m_{2,k}^\beta = \left[ \begin{array}{c|c} 11 \dots 11 & 00 \dots 00 \\ \hline m_{2,k-1}^\alpha & m_{2,k-1}^\beta \end{array} \right], \quad (5.10)$$

Also the generator matrix  $G_{s,k}^\beta$ , for  $k \geq 3$  is

$$\left[ \begin{array}{c|c|c|c|c} 11 \dots 1 & 00 \dots 0 & 22 \dots 2 & \dots & 2^s - 2 \\ \hline G_{s,k-1}^\alpha & G_{s,k-1}^\beta & G_{s,k-1}^\beta & \dots & G_{s,k-1}^\beta \end{array} \right] \quad (5.11)$$

where

$$G_{s,2}^\beta = \left[ \begin{array}{c|c|c|c} 11 \dots 11 & 0 & 2 & \dots & 2^s - 2 \\ \hline 012 \dots 2^s - 1 & 1 & 1 & \dots & 1 \end{array} \right]. \quad (5.12)$$

**Theorem 5.3.** For  $\mathfrak{R}$ -simplex codes of types  $\beta$ , we have:

$$\begin{aligned}
 r_L(S_{s,k}^\beta) &\leq 2^{sk} \left( \frac{2^k - 1}{2} \right) + 2^{k-1} \cdot 2^{s-2} \left( \frac{(2^{s(k-2)} - 1)(2^{2s-1} + 2^{3s-2})}{2^s - 1} \right) \\
 &\quad + \frac{(2^{(s-1)(k-2)} - 1)(-2^{3s-4} + 2^{2s-3})}{2^{s-1} - 1} + 3 \times 2^{s-2} + 2^{2s-3}.
 \end{aligned} \quad (5.13)$$

**Proof.** According to the definition of  $\mathfrak{R}$ -simplex codes of type  $\beta$ , it is constructed by concatenating  $2^{sk}$  copies of the generator matrix of  $S_{2,k}^\beta$  with  $2^k$  copies of the generator matrix of  $S_{s,k}^\beta$ . As mentioned in [11] and presented in Equation (5.11), we have:

$$\begin{aligned}
r_L(S_k^\beta) &\leq r_L(2^{sk}S_{2,k}^\beta) + r_L(2^{k-1}S_{s,k}^\beta) \\
&\leq 2^{sk}r_L(S_{2,k}^\beta) + 2^{k-1}r_L(S_{s,k}^\beta) \\
&\leq 2^{sk}r_H(S_{2,k}^\beta) + 2^{k-1}r_L(S_{s,k}^\beta) \\
&\leq 2^{sk}r_H(S_{2,k}^\beta) + 2^{k-1}r_L(S_{s,k-1}^\alpha) \\
&\quad + r_L(\langle \underbrace{11 \cdots 1}_{2^{s(k-1)}} \quad \underbrace{22 \cdots 2}_{2^{(s-1)(k-1)-1}(2^{k-1}-1)} \quad \cdots \quad \underbrace{2^s - 2 \cdots 2^s - 2}_{2^{(s-1)(k-1)-1}(2^{k-1}-1)} \rangle) \\
&\leq 2^{sk} \left( \frac{2^k - 1}{2} \right) \\
&\quad + 2^{k-1} \cdot 2^{s-2} (2^{s(k-1)} + (2^{s-1} - 1)(2^{(s-1)(k-1)-1}(2^{k-1} - 1))) + r_L(S_{s,k-1}^\beta) \\
&= 2^{sk} \left( \frac{2^k - 1}{2} \right) \\
&\quad + 2^{k-1} \cdot 2^{s-2} (2^{s(k-1)} + (2^{s-1} - 1)(2^{(s-1)(k-1)-1}(2^{k-1} - 1))) + r_L(S_{s,k-1}^\beta) \\
&= 2^{sk} \left( \frac{2^k - 1}{2} \right) \\
&\quad + 2^{k-1} \cdot 2^{s-2} (2^{sk-s-1} + 2^{sk-2} - 2^{(s-1)k-1} + 2^{(s-1)k-s} + r_L(S_{s,k-1}^\beta))
\end{aligned}$$

Hence

$$\begin{aligned}
r_L(S_k^\beta) &\leq 2^{sk} \left( \frac{2^k - 1}{2} \right) \\
&\quad + 2^{k-1} \cdot 2^{s-2} \left( \frac{(2^{s(k-2)} - 1)(2^{2s-1} + 2^{3s-2})}{2^s - 1} + \frac{(2^{(s-1)(k-2)} - 1)(-2^{3s-4} + 2^{2s-3})}{2^{s-1} - 1} \right) \\
&\quad + 3 \times 2^{s-2} + 2^{2s-3}.
\end{aligned}$$

**Theorem 5.4.** For  $\mathfrak{R}$ -simplex codes of types  $\beta$ , we have:

$$\begin{aligned}
r_E(S_k^\beta) &\leq 2^{sk} \left( \frac{2^k - 1}{2} \right) + 2^{k-1} \left( \frac{2^{2s-1} + 1}{12} \right) \left( \frac{(2^{s(k-2)} - 1)(2^{2s-1} + 2^{3s-2})}{2^s - 1} \right) \\
&\quad + \frac{(2^{(s-1)(k-2)} - 1)(-2^{3s-4} + 2^{2s-3})}{2^{s-1} - 1} + 3 \times 2^{s-2} + 2^{2s-3}.
\end{aligned} \tag{5.14}$$

**Proof.** As stated in [11] and in Equation(5.11), we have:

$$\begin{aligned}
 r_E(S_k^\beta) &\leq r_E(2^{sk}S_{2,k}^\beta) + r_E(2^{k-1}S_{s,k}^\beta) \\
 &\leq 2^{sk}r_E(S_{2,k}^\beta) + 2^{k-1}r_E(S_{s,k}^\beta) \\
 &\leq 2^{sk}r_H(S_{2,k}^\beta) + 2^{k-1}r_E(S_{s,k}^\beta) \\
 &\leq 2^{sk}r_H(S_{2,k}^\beta) + 2^{k-1}[r_E(S_{s,k-1}^\alpha) \\
 &\quad + r_E(\langle \overbrace{11 \cdots 1}^{2^{s(k-1)}}, \overbrace{22 \cdots 2}^{2^{(s-1)(k-1)-1}(2^{k-1}-1)}, \dots, \overbrace{2^s - 2 \cdots 2^s - 2}^{2^{(s-1)(k-1)-1}(2^{k-1}-1)} \rangle)] \\
 &\leq 2^{sk}(\frac{2^k - 1}{2}) \\
 &\quad + 2^{k-1}[(\frac{2^{2s-1} + 1}{6})(2^{s(k-1)} + (2^{s-1} - 1)(2^{(s-1)(k-1)-1}(2^{k-1} - 1))) \\
 &\quad + (2^{2s-3})(2^{s-1} - 1)(2^{(s-1)(k-1)-1}(2^{k-1} - 1)) + r_E(S_{s,k-1}^\beta)] \\
 &= 2^{sk}(\frac{2^k - 1}{2}) + 2^{k-1}(\frac{2^{2s-1} + 1}{6})2^{s(k-1)} \\
 &\quad + 2^{k-1}(\frac{5 \times 2^{2s-2} + 1}{6})(2^{s-1} - 1)(2^{s(k-1)-1} - 2^{(s-1)(k-1)-1}) + r_E(S_{s,k-1}^\beta) \\
 &= 2^{sk}(\frac{2^k - 1}{2}) + 2^{k-1}(\frac{2^{2s-1} + 1}{6})(\frac{2^{sk} - 1}{2^s - 1} - 1) \\
 &\quad + 2^{k-1}(\frac{5 \times 2^{2s-2} + 1}{12})(2^{s-1} - 1)(\frac{2^{sk} - 1}{2^s - 1} - \frac{2^{(s-1)k} - 1}{2^{s-1} - 1}) + 2^{k-1}r_E(S_{s,2}^\beta).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 r_E(S_k^\beta) &= 2^{sk}(\frac{2^k - 1}{2}) + 2^{k-1}(\frac{2^{2s-1} + 1}{6})(\frac{2^{sk} - 1}{2^s - 1} - 1) \\
 &\quad + 2^{k-1}(\frac{5 \times 2^{2s-2} + 1}{12})(2^{s-1} - 1)(\frac{2^{sk} - 1}{2^s - 1} - \frac{2^{(s-1)k} - 1}{2^{s-1} - 1}) \\
 &\quad + 2^{k-1} \times 2^s(2^{3s-2} - 3 \times 2^{2s-2} + 2^{s-1} + 1).
 \end{aligned}$$

**Theorem 5.5.** Let  $S_k^{\alpha \perp}, S_k^{\beta \perp}$  be the dual of the simplex codes of types  $\alpha$  and  $\beta$ . Then,  $r_L(S_k^{\alpha \perp}) = r_L(S_k^{\beta \perp}) \leq 1$ , and  $r_E(S_k^{\alpha \perp}) = r_E(S_k^{\beta \perp}) \leq 1$ .

**Proof.** The Delsarte bound establishes that the covering radius of the dual codes  $S_k^{\alpha \perp}$  and  $S_k^{\beta \perp}$  satisfies the inequalities  $r_L(S_k^{\alpha \perp}) \leq 1$  and  $r_L(S_k^{\beta \perp}) \leq 1$ .

### 6 The MacDonal codes of Types $\alpha$ and $\beta$ and their covering radii

Let  $M_{k,u}(q)$  denote the MacDonal codes over the finite field  $F_q$ . The parameters of the code  $M_{k,u}(q)$  are given by  $[\frac{q^k - q^u}{q-1}, k, q^{k-1} - q^{u-1}]$ , where every nonzero codeword has a weight of either  $q^{k-1}$  or  $q^{k-1} - q^{u-1}$  [17]. Let  $m_{2,k}^\alpha$  represent the generator matrix of  $S_{2,k}^\alpha$  and  $m_{2,k}^\beta$  denote the generator matrix of  $S_{2,k}^\beta$ .

For  $1 \leq u \leq k - 1$ , we define  $m_{2,k,u}^\alpha$  (and similarly  $m_{2,k,u}^\beta$ ) by deleting the columns corresponding to  $m_{2,u}^\alpha$  and  $0_{2^u \times (k-u)}$  (or  $m_{2,u}^\beta$  and  $0_{(2^u-1) \times (k-u)}$ ), which form the generator matrix of  $M_{2,k,u}^\alpha$  (or  $M_{2,k,u}^\beta$ ), the binary MacDonal code of type  $\alpha$  (or  $\beta$ ). Thus, for  $k \geq 2$ , we have:

$$m_{2,k,u}^\alpha = \left[ \begin{array}{c} m_{2,k}^\alpha \setminus \frac{0_{2^u \times (k-u)}}{m_{2,u}^\alpha} \end{array} \right], \tag{6.1}$$

and for  $k \geq 3$ ,

$$m_{2,k,u}^\beta = \left[ m_{2,k}^\beta \setminus \frac{0_{(2^u-1) \times (k-u)}}{m_{2,u}^\beta} \right]. \tag{6.2}$$

In [12], the MacDonal codes of types  $\alpha$  and  $\beta$  over  $\mathfrak{R}$  were defined using the generator matrices of simplex codes of types  $\alpha$  and  $\beta$  over  $\mathfrak{R}$ . For  $1 \leq u \leq k-1$ , let  $G_{2^s,k,u}^\alpha$  (resp.,  $G_{2^s,k,u}^\beta$ ) be the generator matrix of  $M_{2^s,k,u}^\alpha$  (resp.,  $M_{2^s,k,u}^\beta$ ), which is derived from  $G_{2^s,k}^\alpha$  (resp.,  $G_{2^s,k}^\beta$ ) by deleting the columns corresponding to  $G_{2^s,u}^\alpha$  and  $0_{2^s u \times (k-u)}$  (resp.,  $G_{2^s,u}^\beta$  and  $0_{2^{(s-1)u-1} \times (k-u)}$ ). Therefore, for  $k \geq 2$ , we have

$$G_{2^s,k,u}^\alpha = \left[ G_{2^s,k}^\alpha \setminus \frac{0_{2^s u \times (k-u)}}{G_{2^s,u}^\alpha} \right], \tag{6.3}$$

and for  $k \geq 3$ ,

$$G_{2^{2s},k,u}^\alpha = \left[ G_{2^s,k}^\alpha \setminus \frac{0_{2^{(s-1)u-1} \times (k-u)}}{G_{2^s,u}^\alpha} \right]. \tag{6.4}$$

Next, we define  $M_{k,u}^\alpha$  and  $M_{k,u}^\beta$  as the MacDonal codes of type  $\alpha$  over  $\mathfrak{R}$ . Let  $\theta_{k,u}^\alpha$  for  $1 \leq u \leq k-1$ , be the generator matrix of the MacDonal code of type  $\alpha$  over  $\mathfrak{R}$ , which is obtained by concatenating  $2^{2k}$  copies of the generator matrix of  $M_{2^s,k,u}^\alpha$  and  $2^k$  copies of the generator matrix of  $M_{2^s,k,u}^\alpha$ . For  $k \geq 2$ , we have:

$$\Theta_{k,u}^\alpha = \left[ m_{2,k,u}^\alpha \mid \cdots \mid m_{2,k,u}^\alpha \mid G_{2^s,k,u}^\alpha \mid \cdots \mid G_{2^s,k,u}^\alpha \right]. \tag{6.5}$$

This takes the form:

$$\left[ \overbrace{m_{2,k}^\alpha \setminus \frac{0_{2^u \times (k-u)}}{m_{2,u}^\alpha} \cdots m_{2,k}^\alpha \setminus \frac{0_{2^u \times (k-u)}}{m_{2,u}^\alpha}}^{2^{2k}} \mid \overbrace{G_{2^s,k}^\alpha \setminus \frac{0_{2^s u \times (k-u)}}{G_{2^s,u}^\alpha} \cdots G_{2^s,k}^\alpha \setminus \frac{0_{2^s u \times (k-u)}}{G_{2^s,u}^\alpha}}^{2^k} \right]. \tag{6.6}$$

Rearranging gives

$$\left[ \overbrace{m_{2,k}^\alpha \cdots m_{2,k}^\alpha}^{2^{2k}} \overbrace{G_{2^s,k}^\alpha \cdots G_{2^s,k}^\alpha}^{2^k} \mid \overbrace{\left( \frac{0_{2^u \times (k-u)}}{m_{2,u}^\alpha} \cdots \frac{0_{2^u \times (k-u)}}{m_{2,u}^\alpha} \right)}^{2^{2k}} \overbrace{\left( \frac{0_{2^s u \times (k-u)}}{G_{2^s,u}^\alpha} \cdots \frac{0_{2^s u \times (k-u)}}{G_{2^s,u}^\alpha} \right)}^{2^k} \right]. \tag{6.7}$$

So, for  $k \geq 2$ , we decide

$$\Theta_{k,u}^\alpha = \left[ \theta_k^\alpha \mid \overbrace{\left( \frac{0_{2^u \times (k-u)}}{m_{2,u}^\alpha} \cdots \frac{0_{2^u \times (k-u)}}{m_{2,u}^\alpha} \right)}^{2^{2k}} \overbrace{\left( \frac{0_{2^s u \times (k-u)}}{G_{2^s,u}^\alpha} \cdots \frac{0_{2^s u \times (k-u)}}{G_{2^s,u}^\alpha} \right)}^{2^k} \right]. \tag{6.8}$$

Let  $\theta_{k,u}^\beta$ ,  $1 \leq u \leq k-1$ , be the generator matrix of the MacDonal code of type  $\beta$  over  $\mathfrak{R}$ . Obtained by the concatenation of  $2^{2k}$  copies of the generator matrix of  $M_{2^s,k,u}^\beta$  and  $2^{k-1}$  copies of the generator matrix of  $M_{2^s,k,u}^\beta$ . For  $k \geq 3$ , we have

$$\Theta_{k,u}^\beta = \left[ m_{2,k,u}^\beta \mid \cdots \mid m_{2,k,u}^\beta \mid G_{2^s,k,u}^\beta \mid \cdots \mid G_{2^s,k,u}^\beta \right]. \tag{6.9}$$

So, it takes the form

$$\left[ \overbrace{m_{2,k}^\beta \setminus \frac{0_{(2^u-1) \times (k-u)}}{m_{2,u}^\beta} \cdots m_{2,k}^\beta \setminus \frac{0_{(2^u-1) \times (k-u)}}{m_{2,u}^\beta}}^{2^{2k}} \mid \overbrace{G_{2^s,k}^\beta \setminus \frac{0_{2^{(s-1)u} \times (k-u)}}{G_{2^s,u}^\beta} \cdots G_{2^s,k}^\beta \setminus \frac{0_{2^{(s-1)u} \times (k-u)}}{G_{2^s,u}^\beta}}^{2^{k-1}} \right]. \tag{6.10}$$

Rearranging gives for  $k \geq 3$ ,

$$\Theta_{k,u}^\beta = \left[ \Theta_k^\beta \left| \overbrace{\left\{ \frac{0_{(2^u-1) \times (k-u)} \dots \left\{ \frac{0_{(2^u-1) \times (k-u)}}{m_{2,u}^\beta} \right\}}^{2^{sk}} \right\}} \overbrace{\left\{ \frac{0_{2^{(s-1)u}(2^u-1) \times (k-u)} \dots \left\{ \frac{0_{2^{(s-1)u}(2^u-1) \times (k-u)}}{G_{2^s,u}^\beta} \right\}}^{2^{k-1}} \right\}} \right. \right]. \quad (6.11)$$

We shall establish some bounds on the covering radii of MacDonal codes of types  $\alpha$  and  $\beta$ .

**Theorem 6.1.** *For  $u \leq r \leq k$ , the covering radii of the  $\mathfrak{R}$ -MacDonald codes of type  $\alpha$  are subject to the following upper bounds:*

$$\begin{aligned} r_L(M_{k,u}^\alpha) &\leq 2^{sk+k-1}(1+2^{s-1}) - 2^{k+s-2}(2^{sk} + 2^{sr}) \\ &\quad + 2^{sk}r_H(M_{2,r,u}^\alpha) + 2^k r_L(M_{2^s,r,u}^\alpha), \\ \text{and} & \end{aligned} \quad (6.12)$$

$$\begin{aligned} r_E(M_{k,u}^\alpha) &\leq 2^{sk+k-1}(1+2^{s-1}) - 2^k \left( \frac{1}{6} (5 \times 2^{3s-3} - 3 \times 2^{2s-2} + 2^{s-1}) \left( \frac{2^{sk} - 2^{sr}}{2^s - 1} \right) \right) \\ &\quad + 2^{sk}r_H(M_{2,r,u}^\alpha) + 2^k r_E(M_{2^s,r,u}^\alpha). \end{aligned}$$

**Proof.** For  $u \leq r \leq k$ , as stated in [11] and in Equation(6.9), we have:

$$\begin{aligned} r_L(M_{k,u}^\alpha) &\leq r_L(2^{sk}M_{2,k,u}^\alpha) + r_L(2^kM_{2^s,k,u}^\alpha) \\ &\leq 2^{sk}r_L(M_{2,k,u}^\alpha) + 2^k r_L(M_{2^s,k,u}^\alpha) \\ &\leq 2^{sk}r_H(M_{2,k,u}^\alpha) + 2^k r_L(M_{2^s,k,u}^\alpha) \\ &\leq 2^{sk}(2^{k-1} - 2^{r-1}) + 2^{sk}r_H(M_{2,r,u}^\alpha) + 2^k(2^{sk+s-2} - 2^{sr+s-2}) + 2^k r_L(M_{2^s,r,u}^\alpha) \\ r_L(M_{k,u}^\alpha) &\leq 2^{sk+k-1}(1+2^{s-1}) - 2^{k+s-2}(2^{sk} + 2^{sr}) + 2^{sk}r_H(M_{2,r,u}^\alpha) + 2^k r_L(M_{2^s,r,u}^\alpha). \end{aligned} \quad (6.13)$$

Similar arguments hold for  $r_E(M_{k,u}^\beta)$ .

**Theorem 6.2.** *For  $u \leq r \leq k$ , the covering radii of the  $\mathfrak{R}$ -MacDonald codes of type  $\beta$  are subject to the following upper bounds:*

$$\begin{aligned} r_L(M_{k,u}^\beta) &\leq 2^{sk}(2^{k-1} - 2^{r-1}) + 2^{sk}r_H(M_{2,r,u}^\beta) \\ &\quad + 2^{k-1}(2^{(s-1)k-1}(2^k - 1)) - 2^{k-1}(2^{(s-1)r-1}(2^r - 1)) + 2^{k-1}r_L(M_{2^s,r,u}^\beta), \end{aligned}$$

and

$$\begin{aligned} r_E(M_{k,u}^\beta) &\leq 2^{sk}(2^{k-1} - 2^{r-1}) + 2^{sk}r_H(M_{2,r,u}^\beta) \\ &\quad + \left( \frac{2^{2s-1} + 1}{6} \right) \left( \frac{2^{sk} - 2^{sr}}{2^s - 1} - 1 \right) \\ &\quad + 2^{k-1} \left( \frac{5 \times 2^{2s-2} + 1}{12} \right) (2^{s-1} - 1) \left( \frac{2^{sk} - 2^{sr}}{2^s - 1} - \frac{2^{(s-1)k} - 2^{(s-1)r}}{2^{s-1} - 1} \right) + 2^{k-1}r_E(M_{2^s,r,u}^\beta). \end{aligned}$$

**Proof.** For  $u \leq r \leq k$ , as stated in [11] and in Equation(6.9), we have:

$$\begin{aligned} r_L(M_{k,u}^\beta) &\leq r_L(2^{sk}M_{2,k,u}^\beta) + r_L(2^{k-1}M_{2^s,k,u}^\beta) \\ &\leq 2^{sk}r_L(M_{2,k,u}^\beta) + 2^{k-1}r_L(M_{2^s,k,u}^\beta) \\ &\leq 2^{sk}r_H(M_{2,k,u}^\beta) + 2^{k-1}r_L(M_{2^s,k,u}^\beta) \\ r_L(M_{k,u}^\beta) &\leq 2^{sk}(2^{k-1} - 2^{r-1}) + 2^{sk}r_H(M_{2,r,u}^\beta) \\ &\quad + 2^{k-1}(2^{(s-1)k-1}(2^k - 1)) - 2^{k-1}(2^{(s-1)r-1}(2^r - 1)) + 2^{k-1}r_L(M_{2^s,r,u}^\beta). \end{aligned} \quad (6.14)$$

Similar arguments hold for  $r_E(M_{k,u}^\beta)$ .

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## Author information

Abdollah Alhevaz, Faculty of Mathematical Sciences, Shahrood University of Technology, P.O. Box: 316-3619995161, Shahrood, Iran.

E-mail: a.alhevaz@gmail.com and a.alhevaz@shahroodut.ac.ir

Fariba Mahmoudi, Department of Mathematics, Islamic Azad University, Central Tehran Branch, Tehran, Iran.

E-mail: f.mahmoudi.algebra@gmail.com

P. Chella Pandian, Department of Mathematics, Vivekananda College(Autonomous), Tiruvedakam West, Madurai-625 234, Tamilnadu, India.

E-mail: chellapandianpc@gmail.com

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