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THE PARTIAL BELL POLYNOMIALS OF TYPE B

Meriem Moulay and Miloud Mihoubi

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Corresponding Author: M. Moulay

Abstract In this paper, we study some counting sequences on the $[\pm n]$-set partitions. Such sequences are generated from the partition polynomials, named, partial Bell polynomials of type B, which arise as particular cases of enumeration problems.

1 Introduction

A partition of the set $[n] = \{1, 2, \ldots, n\}$ is a collection of disjoint nonempty subsets $B_1, \ldots, B_k$ whose union is $[n]$. The subsets B_i are called blocks. In 1997, Reiner [15] introduced the notion of set partition of type B, defined as a partition π of the set

$$[\pm n] = \{-n, \ldots, -2, -1, 1, 2, \ldots, n\}$$

such that for any block B of π, $-B$ is also a block of π, and that π contains at most one block B satisfying $B = -B$. The block $B = -B$, if its exists, is called the zero-block B_0. We call the pair $\pm B$ of blocks a block pair of π if B is not the zero-block. For example, the following is a B_6-partition:

$$\begin{aligned} \pi &= \{\pm\{1,3\}, \{2,-2,6,-6\}, \pm\{5,-4\}\} \\ &= \{\{1,3\}, \{-1,-3\}, \{2,-2,6,-6\}, \{5,-4\}, \{-5,4\}\}. \end{aligned}$$

The block $\{2, -2, 6, -6\}$ is a zero block.

The (n, k)-th Stirling number of type B of the second kind $\left\{ {n \atop k} \right\}^B$ counts the number of signed partitions of $[\pm n]$ having exactly k pairs of nonzero blocks. Suter [16] proved that the exponential generating function of the Stirling numbers of type B of the second kind is

$$\sum_{n \geq k} \left\{ {n \atop k} \right\}^B \frac{t^n}{n!} = \frac{e^t}{k!} \left(\frac{e^{2t} - 1}{2} \right)^k. \tag{1.1}$$

The exponential Riordan arrays [2] are defined by a pair of power series $g(t)$ and $f(t)$ with $g(0) \neq 0, f(0) = 0$ and $f'(0) \neq 0$. The associated matrix $L = [d_{n,k}]_{n,k \geq 0}$ is the matrix whose k-th column has exponential generating function

$$\sum_{n \geq k} d_{n,k} \frac{t^n}{n!} = \frac{1}{k!} g(t) f^k(t), k = 0, 1, \ldots,$$

with general element

$$d_{n,k} = \frac{n!}{k!} [t^n] \, g(t) f^k(t),$$

where $[t^n]$ is the operator that extracts the coefficient of t^n in a power series, and we write this matrix as $L = \langle g(t), f(t) \rangle$. The product of two exponential Riordan arrays $\langle g(t), f(t) \rangle$ and

$\langle h(t), l(t)\rangle$ is defined by

$$\langle g(t), f(t)\rangle * \langle h(t), l(t)\rangle = \langle g(t)h(f(t)), l(f(t))\rangle .$$

The Riordan array $I = \langle 1, t\rangle$ is everywhere 0 except for the entries on the main diagonal, which are 1, I acts as an identity for the product $*$. Moreover, the inverse of a Riordan array is given by the formula

$$\langle g(t), f(t)\rangle^{-1} = \left\langle \frac{1}{g\left(\overline{f}\left(t\right)\right)}, \overline{f}\left(t\right) \right\rangle,$$

where $\overline{f}$ is the compositional inverse of f. In this way, the set of all exponential Riordan matrices is a group under the operator $*$.

For example, in [9] Mező and Ramírez defined the combinatorial array $S_2^B = \left[\left\{{n \atop k}\right\}^B\right]_{n,k\geq 0}$ and show that this matrix can be expressed as an exponential Riordan array given by

$$S_2^B = \left\langle e^t, \frac{e^{2t}-1}{2} \right\rangle .$$

Motivated by the articles cited above, we introduce and develop the corresponding sequences related to the partial Bell polynomials [13] from which we deduce new sequence related to the Lah numbers [3, 4, 12], similar to known sequence of the Stirling numbers of type B of the second kind.

The paper is organized as follows. In section 2, we introduce the partial Bell polynomials of type B, $B_{n,k}^B\left(\varphi;\psi\right), n \geq k \geq 0$. We derive an explicit formula, the generating function and give a combinatorial proof of the triangular recurrence relation for the Stirling numbers of type B of the second kind. In the third section, we introduce and study the sequence $\left(\left\lfloor {n \atop k} \right\rceil^B ; n\geq k\geq 0\right)$ called Lah numbers of the type B by giving their generating function, a triangular recurrence relation, and an explicit formula. We define the combinatorial array $L^B = \left[\left\lfloor {n \atop k} \right\rceil^B\right]_{n,k\geq 0}$ and show that this matrix can be expressed as an exponential Riordan array. Cross recurrence relation and combinatorial identities for the Lah numbers of type B are also established.

2 The partial Bell polynomials of type B

In this section, we introduce the partial Bell polynomials of type B, and, we start it with the following definition.

Definition 2.1. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 0}$ be two sequences of nonnegative integers with $b_0 = 1$ and let

$$\varphi\left(t\right) = \sum_{n\geq 1} a_n \frac{t^n}{n!} \quad \text{and} \quad \psi\left(t\right) = \sum_{n\geq 0} b_n \frac{t^n}{n!}. \tag{2.1}$$

The partial Bell number of type B is the number

$$B_{n,k}^B\left(a_1, a_2, \ldots; b_1, b_2, \ldots\right) := B_{n,k}^B\left(\varphi;\psi\right) \tag{2.2}$$

which counts the number of partitions of the $[\pm n]$ into k pairs of nonzero blocks and at most one zero block such that:

- the zero block B_0 of the length $2i$, is such that $B_0 = -B_0$, can be colored with b_i colors.
- if B is a nonzero block in such partition, then $-B$ is also a nonzero block in the same partition,
- any nonzero block of the length i can be colored with a_i colors,

We assume that any block with 0 color does not appear in partitions.

An explicit formula for $B_{n,k}^{B}(\varphi,\psi)$ is given by the following theorem.

Theorem 2.2. *We have*

$$B_{n,k}^{B}(\varphi,\psi)=\frac{1}{2^{k}k!}\sum_{j=k}^{n}\binom{n}{j}b_{n-j}2^{j}\sum_{n_1+\cdots+n_k=j,\ \ n_i\geq 1}\frac{j!}{n_1!\cdots n_k!}a_{n_1}\cdots a_{n_k}.$$

Proof. From Definition 2.1, we proceed as follows.
To construct a zero block B_0 of the length $2j$, there are $\binom{n}{j}$ ways to choose j elements with $0\leq j\leq n-k$ from the set $[n]$, we set B_0 the set of these j elements and their opposites. The number of colored zero blocks of length $2j$ must be $\binom{n}{j}b_j$.
Upon using the remaining $n-j$ elements, to construct the nonzero blocks $B_1,\ldots,B_k$ of lengths $n_1\geq 1,\ldots,n_k\geq 1$ (respectively) there are $\frac{1}{k!}\frac{(n-j)!}{n_1!\cdots n_k!}$ ways. From a nonzero block of length h, we can generate 2^{h-1} signed nonzero blocks without opposite blocks, then the number of the signed nonzero blocks is $\frac{2^{n_1+\cdots+n_k-k}}{k!}\frac{(n-j)!}{n_1!\cdots n_k!}=\frac{2^{n-j}}{2^k k!}\frac{(n-j)!}{n_1!\cdots n_k!}$. Hence, the number of colored signed nonzero blocks must be $\frac{2^{n-j}}{2^k k!}\frac{(n-j)!}{n_1!\cdots n_k!}a_{n_1}\cdots a_{n_k}$. So, the total number must be

$$B_{n,k}^{B}(\varphi,\psi)=\frac{1}{2^{k}k!}\sum_{j=0}^{n-k}\binom{n}{j}2^{n-j}b_{j}\sum_{n_1+\cdots+n_k=n-j,\ \ n_i\geq 1}\frac{(n-j)!}{n_1!\cdots n_k!}a_{n_1}\cdots a_{n_k}.$$

□

Theorem 2.2 is now used to produce the exponential generating function for $B_{n,k}^{B}(\varphi,\psi)$.

Theorem 2.3. *We have*

$$\sum_{n\geq k}B_{n,k}^{B}(\varphi,\psi)\frac{t^n}{n!}=\frac{1}{k!}\psi(t)\left(\frac{\varphi(2t)}{2}\right)^{k}. \tag{2.3}$$

Proof. From Theorem 2.2, we can write

$$\begin{aligned}\sum_{n\geq k}B_{n,k}^{B}(\varphi,\psi)\frac{t^n}{n!} &= \frac{1}{2^{k}k!}\sum_{n\geq k}\frac{t^n}{n!}\sum_{j=0}^{n-k}\binom{n}{j}b_j2^{n-j}\sum_{n_1+\cdots+n_k=n-j,\ \ n_i\geq 1}\frac{(n-j)!}{n_1!\cdots n_k!}a_{n_1}\cdots a_{n_k}\\ &= \frac{1}{2^{k}k!}\sum_{j\geq 0}\frac{b_jt^j}{j!}\sum_{n_1,\ldots,n_k\geq 1}\frac{(2t)^{n_1+\cdots+n_k}}{n_1!\cdots n_k!}a_{n_1}\cdots a_{n_k}\\ &= \frac{1}{2^{k}k!}\sum_{j\geq 0}\frac{b_jt^j}{j!}\left(\sum_{n\geq 1}a_n\frac{(2t)^n}{n!}\right)^{k}\\ &= \frac{1}{2^{k}k!}\psi(t)\left(\varphi(2t)\right)^{k}.\end{aligned}$$

□

Corollary 2.4. *For the choices $\psi(t)=e^t$ and $\varphi(t)=e^t-1$, we get*

$$B_{n,k}^{B}(\varphi,\psi)=\left\{ {n \atop k} \right\}^{B}.$$

From (2.3) and the definition of exponential Riordan array we obtain the following theorem.

Theorem 2.5. *The matrix $B:=\left[B_{n,k}^{B}(\varphi,\psi)\right]_{n,k\geq 0}$ is an exponential Riordan array given by*

$$B=\left\langle \psi(t),\frac{\varphi(2t)}{2}\right\rangle.$$

For later use, let $B_{n,k}(\varphi)$ be the (n,k)-th partial Bell polynomial [5, 7, 10, 11] , defined by

$$\sum_{n\geq k} B_{n,k}(\varphi)\frac{t^n}{n!}=\frac{1}{k!}(\varphi(t))^k.$$

The partial Bell polynomials of type B can be expressed by the partial Bell polynomials $B_{n,k}(\varphi)$ as follows.

Proposition 2.6. *The partial Bell polynomials of type B satisfy*

$$B^B_{n,k}(\varphi,\psi)=\sum_{i=k}^{n}\binom{n}{i}b_{n-i}2^{i-k}B_{i,k}(\varphi).$$

Proof. From the Riordan array multiplication we have

$$\begin{aligned}
B^B_{n,k}(\varphi,\psi) &= \left\langle \psi(t), \frac{\varphi(2t)}{2}\right\rangle \\
&= \langle \psi(t), t\rangle * \left\langle 1, \frac{\varphi(2t)}{2}\right\rangle \\
&= \langle \psi(t), t\rangle * \left(\langle 1, 2t\rangle * \langle 1, \varphi(t)\rangle * \left\langle 1, \frac{t}{2}\right\rangle\right) \\
&= \left[\binom{n}{k}b_{n-k}\right]_{n,k\geq 0} * \left[2^{n-k}B_{n,k}(\varphi)\right]_{n,k\geq 0}.
\end{aligned}$$

Hence the desired result follows by taking the (n,k)-th element of the product of matrices. □

For $\psi(t)=e^t$ and $\varphi(t)=e^t-1$ in Proposition 2.6, we get the known identity for the Stirling numbers of the second kind of type B [9]:

$$\left\{ n \atop k \right\}^B=\sum_{i=k}^{n}2^{i-k}\binom{n}{i}\left\{ i \atop k \right\}.$$

Moreover, the following corollary gives a combinatorial proof of a triangular recurrence relation for the Stirling numbers of type B of the second kind given in [9].

Corollary 2.7. *For $n\geq k\geq 1$, we have*

$$\left\{ n \atop k \right\}^B=\left\{ n-1 \atop k-1 \right\}^B+(2k+1)\left\{ n-1 \atop k \right\}^B. \tag{2.4}$$

with the initial conditions

$$\left\{ n \atop 0 \right\}^B=1 \text{ for } n\geq 0, \quad \text{and}, \quad \left\{ n \atop k \right\}^B=0 \text{ for } n,k<0. \tag{2.5}$$

Proof. We proceed by separating the elements $\pm n$. The total number of k-partitions $\left\{ n \atop k \right\}^B$ can be obtained as follows. If $n\in B_0$ (then also $-n\in B_0$), there are $\left\{ n-1 \atop k \right\}^B$ to construct k-partitions without $\pm n$. If $\{n\}$ forms a non-zero block (then $\{-n\}$ is also a non-zero block), there are $\left\{ n-1 \atop k-1 \right\}^B$ to construct $(k-1)$-partitions. If n is in a block having than one element (then $-n$ must also be in a block having than one element), then to partition the set $[\pm(n-1)]$ into k-partitions there are $\left\{ n-1 \atop k \right\}^B$, the elements n can be inserted in the $2k$ blocks $\pm B_1,\ldots,\pm B_k$ in $2k$ ways. Then we have

$$\left\{ n \atop k \right\}^B=\left\{ n-1 \atop k \right\}^B+\left\{ n-1 \atop k-1 \right\}^B+2k\left\{ n-1 \atop k \right\}^B.$$

□

Proposition 2.8. *Let B_n be the polynomial*

$$B_n(x) = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}^B x^k, \quad n \geq 0.$$

$B_n(x)$ has only real nonpositive zeros for $n \geq 1$, and, hence the Stirling numbers of the second kind of type B are log concave.

Proof. Using (2.4), it is easy to see that $B_n(x)$ can be expressed as:

$$B_n(x) = (x+1)B_{n-1}(x) + 2x\frac{\partial}{\partial x}B_{n-1}(x)$$

which leads to

$$B_n(x) = (x+1)^3 e^{-2x}\frac{\partial}{\partial x}\left(\frac{e^{2x}}{(x+1)^2}B_{n-1}(x)\right).$$

The induction on n proves that $B_n(x)$ has only real nonpositive zeros. □

The following result gives the ordinary generating function and an explicit formula for the Stirling numbers of the second kind of type B.

Proposition 2.9. *We have*

$$\sum_{n\geq 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}^B t^n = \frac{t^k}{(1-t)(1-3t)\cdots(1-(2k+1)t)}$$

and

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}^B = \frac{1}{2^k k!}\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j}(2j+1)^n.$$

Proof. For $k \geq 0$, let $f_k(t) = \sum_{n\geq 0} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}^B t^n$.
Multiplying the two sides of (2.4) by t^n and summing over all $n \geq 0$, leads to

$$f_k(t) = tf_{k-1}(t) + (2k+1)tf_k(t), \quad \text{i.e.} \quad f_k(t) = \frac{t}{1-(2k+1)t}f_{k-1}(t).$$

In view of $f_0(t) = \frac{1}{1-t}$, we obtain

$$f_k(t) = \frac{t^k}{(1-t)(1-3t)\cdots(1-(2k+1)t)} = \sum_{j=0}^{k}\frac{c_j}{1-(2j+1)t},$$

with $c_j = \frac{(-1)^{k-j}}{k!2^k}\binom{k}{j}$. We conclude that

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}^B = \sum_{j=0}^{k} c_j(2j+1)^n = \frac{1}{2^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(2j+1)^n.$$

□

3 The Lah numbers of type B

We start by introducing a sequence called the Lah numbers of type B. The (n,k)-th Lah number of type B, denoted by $\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^B$, counts the number of partitions of $[\pm n]$ having exactly k pairs of nonzero lists and at most one zero specific list. The specific list is an ordered block such that the pair of elements $\pm i$ must be successive. These numbers satisfy the recurrence relation given below.

Theorem 3.1. *For $n \geq k \geq 1$, we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil^{B} = \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rceil^{B} + (2n+2k) \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rceil^{B}. \tag{3.1}$$

with the initial conditions

$$\left\lfloor \begin{matrix} n \\ 0 \end{matrix} \right\rceil^{B} = 2^{n} n! \textit{ for } n \geq 0, \quad \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil^{B} = 0 \textit{ for } n,k < 0, \quad \textit{and,} \quad \left\lfloor \begin{matrix} n \\ n \end{matrix} \right\rceil^{B} = 1 \textit{ for } n \geq 0.$$

Proof. Let us consider k pairs of nonzero lists. If $\{n\}$ and $\{-n\}$ forms the first nonzero pair list then there are $\left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rceil^{B}$ ways to construct $(k-1)$-partitions. If $n \in B_0$ (then also $-n \in B_0$) or If n is in a nonzero list B having than one element (then $-n$ must also be in the opposite nonzero list $-B$ having than one element), then there are $\left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rceil^{B}$ to construct k-partitions without $\pm n$.

Let consider the $[\pm (n_1 + n_2)]$-set as union of two sets N_1 which contains $[\pm n_1]$ elements used to construct the k pairs of nonzero lists and N_2 which contains $[\pm n_2]$ elements used to construct the zero specific list $(n_1 + n_2 = n - 1)$. The element n can be inserted in the $2k$ nonzero lists $\pm B_1, \ldots, \pm B_k$ in $2n_1 + 2k$ ways (before any element of $[\pm n_1]$ or to the end of any nonzero lists $\pm B_i, 1 \leq i \leq k$), $-n$ must be inserted simultaneously in the opposites nonzero lists (before the same opposites elements or to the end of any nonzero lists). For the zero specific list the pair $\pm n$ can be inserted before any pair of $[\pm n_2]$ or to the end of the zero specific list after that we permute between n and $-n$ then the element n can be put to $2n_2 + 2$ places. Finally we have $2n_1 + 2k + 2n_2 + 2 = 2n + 2k$ places. Then we have

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil^{B} = \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rceil^{B} + 2(n+k) \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rceil^{B}.$$

□

Let us define an infinite matrix, named the infinite Lah matrix of type B, by

$$L^{B} := \left[\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil^{B} \right]_{n,k \geq 0}.$$

So, from the recurrence relation (3.1), the first few rows of L^B are given as follows:

$$\left[\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil^{B} \right]_{6 \geq n \geq k \geq 0} = \begin{pmatrix} 1 & & & & & & \\ 2 & 1 & & & & & \\ 8 & 8 & 1 & & & & \\ 48 & 72 & 18 & 1 & & & \\ 384 & 768 & 288 & 32 & 1 & & \\ 3840 & 9600 & 4800 & 800 & 50 & 1 & \\ 46080 & 138240 & 86400 & 19200 & 1800 & 72 & 1 \end{pmatrix}$$

Proposition 3.2. *For $n \geq 1$, the polynomial*

$$P_n(x) = \sum_{k=0}^{n} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil^{B} x^k$$

has only nonpositive real zeros, and, hence the Lah numbers of type B are log concave.

Proof. If we define $T_n(x) = x^{n+1} \exp\left(\frac{x}{2}\right) P_n(x)$ then, $T_n(x)$ has exactly the same finite roots as $P_n(x)$. Moreover, the recurrence relation 3.1 establishes that

$$T_n(x) = 2x^2 \tfrac{\partial}{\partial x} T_{n-1}(x).$$

By induction on n, it follows that $T_n(x)$ has only nonpositive real zeros.
Using Newton's inequality as presented by Hardy et al. [8, p. 52], we can conclude that the sequence $\left(\begin{bmatrix} n \\ k \end{bmatrix}^B ; n \geq k \geq 0 \right)$ is log-concave. □

According to Theorem 3.1, we deduce the exponential generating function of these numbers as follows.

Theorem 3.3. *The Lah numbers of type B have the following generating function*

$$\sum_{n \geq k} \begin{bmatrix} n \\ k \end{bmatrix}^B \frac{t^n}{n!} = \frac{1}{k!} \frac{1}{1-2t} \left(\frac{t}{1-2t} \right)^k. \tag{3.2}$$

Proof. Let $L_k(t) = \sum_{n \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}^B \frac{t^n}{n!}$. Then, by (3.1) we get

$$(1-2t) L_k(t) = \sum_{n \geq 1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^B \frac{t^n}{n!} + 2k \sum_{n \geq 1} \begin{bmatrix} n-1 \\ k \end{bmatrix}^B \frac{t^n}{n!},$$

from which we may state

$$(1-2t) \frac{d}{dt} L_k(t) = L_{k-1}(t) + 2(k+1) L_k(t)$$

which is equivalent to

$$\frac{d}{dt} \left((1-2t)^{k+1} L_k(t) \right) = (1-2t)^k L_{k-1}(t).$$

In view of $L_0(t) = \frac{1}{1-2t}$, we obtain by induction (3.2). □

Corollary 3.4. *We have*

$$\begin{bmatrix} n \\ k \end{bmatrix}^B := B^B_{n,k}(\varphi, \psi),$$

with $\psi(t) = \frac{1}{1-2t}$ *and* $\varphi(t) = \frac{t}{1-t}$.

Proposition 3.5. *For* $n \geq k \geq 0$, *we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}^B = \frac{n!}{k!} \binom{n}{k} 2^{n-k} = \frac{k+1}{n+1} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} 2^{n-k}, \tag{3.3}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ *is the usual* (n,k)*-th Lah number.*

Proof. Upon using Theorem 3.3, the desired explicit formula follows from generating function

$$\sum_{n \geq k} \begin{bmatrix} n \\ k \end{bmatrix}^B \frac{t^n}{n!} = \frac{1}{k!} \frac{1}{1-2t} \left(\frac{t}{1-2t} \right)^k = \frac{1}{k!} \sum_{n \geq k} 2^{n-k} \binom{n}{k} t^n.$$

□

Comparing to (3.1), another recurrence relation with rational coefficients can be derived from the explicit formula (3.3). Using Pascal's formula and relation (3.3), we obtain:

Proposition 3.6. *For $n \geq k \geq 1$, the Lah numbers of type B satisfy the recurrence relation*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil^B = \frac{n}{k} \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rceil^B + 2n \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rceil^B . \tag{3.4}$$

From the generating function of the Lah numbers of type B and the definition of exponential Riordan array we may state:

Corollary 3.7. *The matrix* $L^B := \left[\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil^B \right]_{n,k \geq 0}$ *is an exponential Riordan array given by*

$$L^B = \left\langle \frac{1}{1-2t}, \frac{t}{1-2t} \right\rangle .$$

4 Gross recurrence relation and combinatorial identities

The following proposition gives a cross recurrence relation for the Lah numbers of type B.

Proposition 4.1. *For any non negative integers $0 \leq k \leq n$, we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil^B = \sum_{j=0}^{n-k} 2^j \,(j+1)! \binom{n-1}{j} \left\lfloor \begin{matrix} n-j-1 \\ k-1 \end{matrix} \right\rceil^B . \tag{4.1}$$

Proof. Let us consider k lists. We suppose that the first list contains the first element "1" and j $(0 \leq j \leq n-k)$ elements chosen from the set $\{2, \cdots, n\}$. Thus, there are $\binom{n-1}{j}$ ways to choose the j elements, $(j+1)!$ ways to constitute the first list. We can generate 2^j signed nonzero lists without opposite lists and $\left\lfloor \begin{matrix} n-j-1 \\ k-1 \end{matrix} \right\rceil^B$ ways to distribute the remaining $n-j-1$ elements into $k-1$ lists and at most one zero list . We conclude by summing the product of the three term. □

Proposition 4.2. *For any nonnegative integers $0 \leq k \leq n$, we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil^B = \sum_{j=0}^{k} \sum_{s=j}^{n-k+j} 2^{s-j} \left\lfloor \begin{matrix} s \\ j \end{matrix} \right\rceil \left\lfloor \begin{matrix} n-s \\ k-j \end{matrix} \right\rceil^B .$$

Proof. We partition the n elements into two groups: The first group contains s elements, while the second group has $n-s$ elements. From the first group, we can form j $(0 \leq j \leq k)$ pairs of nonzero lists and with the second group we can constitute $k-j$ pairs of nonzero lists and at most one zero specific list. We have $\left\lfloor \begin{matrix} n-s \\ k-j \end{matrix} \right\rceil^B$ $(j \leq s \leq n-k+j)$ possibilities to constitute the $k-j$ pairs of nonzero lists and at most one zero specific list. What remains is to count how to form the remaining j pairs of nonzero lists. We have $\left\lfloor \begin{matrix} s \\ j \end{matrix} \right\rceil$ possibilities to constitute the j nonzero lists and we can generate 2^{s-j} signed nonzero lists without opposite, then it gives $2^{s-j} \left\lfloor \begin{matrix} s \\ j \end{matrix} \right\rceil$ possibilities. We conclude by summing. □

The Lah numbers of type B can be expressed by the classical Lah numbers $\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil$, as follows.

Corollary 4.3. *We have the combinatorial identity*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil^B = 2^{n-k} \sum_{i=k}^{n} \frac{n!}{i!} \left\lfloor \begin{matrix} i \\ k \end{matrix} \right\rceil . \tag{4.2}$$

Proof. This Corollary follows from Proposition 2.6 for the choices

$$\psi(t) = \frac{1}{1-2t}, \quad \varphi(t) = \frac{t}{1-t}.$$

It can also be proved combinatorially as follows.
Indeed, begin with placing j $(k \leq j \leq n)$ elements into k distinct nonzero lists.
There are $\binom{n}{j}$ ways to choose these elements, $\left\lfloor \begin{matrix} j \\ k \end{matrix} \right\rfloor$ to constitute the k nonzero lists and from a nonzero list of length h_i $\left(\sum_{i=0}^{k} h_i = j\right)$, we can generate 2^{h_i-1} signed nonzero lists without opposite lists, then the number of the signed nonzero lists is $\binom{n}{j}2^{j-k}\left\lfloor \begin{matrix} j \\ k \end{matrix} \right\rfloor$. The remaining $n-j$ elements have to be placed into the zero list, wish can be done in $2^{n-j}(n-j)!$ ways. We conclude by summing the product of the terms. □

Below, we give a connection between the Lah numbers of type B and the Stirling numbers of the second kind of type B and their inverses. To give such connection, recall that the inverse exponential Riordan array of S_2^B denoted by $S_1^B = \left[\begin{bmatrix} n \\ k \end{bmatrix}^B\right]_{n,k\geq 0}$ is an exponential Riordan array given in [9] by

$$S_1^B = \left\langle \frac{1}{\sqrt{1+2t}}, \frac{\ln(1+2t)}{2} \right\rangle.$$

Proposition 4.4. *For* $0 \leq k \leq n$, *we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^B = \sum_{i=k}^{n} (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}^B \begin{Bmatrix} i \\ k \end{Bmatrix}^B.$$

Proof. By Corollary 3.7 we have

$$\begin{aligned}
\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^B &= \left\langle \frac{1}{1-2t}, \frac{t}{1-2t} \right\rangle \\
&= \left\langle \frac{1}{\sqrt{1-2t}}, -\frac{\ln(1-2t)}{2} \right\rangle * \left\langle e^t, \frac{e^{2t}-1}{2} \right\rangle \\
&= \left[(-1)^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}^B\right]_{n,k\geq 0} * \left[\begin{Bmatrix} n \\ k \end{Bmatrix}^B\right]_{n,k\geq 0}.
\end{aligned}$$

Hence, the desired result follows from the product of matrices. □

The following corollary give the dual inverse formula of Proposition (4.4).

Corollary 4.5. *For* $0 \leq k \leq n$, *we have*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}^B = \sum_{i=k}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^B \left\lfloor \begin{matrix} i \\ k \end{matrix} \right\rfloor^B.$$

The next proposition give The number of the lists without zero block.

Proposition 4.6. *The number of the lists without zero block is*

$$2^{n-k}\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor = \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^B - 2n\left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor^B, \quad 1 \leq k \leq n.$$

Proof. From the generating function given by Theorem 3.3, we can write

$$(1-2t)\sum_{n\geq k}\left\lfloor\begin{matrix}n\\k\end{matrix}\right\rfloor^{B}\frac{t^{n}}{n!}=\sum_{n\geq k}2^{n-k}\left\lfloor\begin{matrix}n\\k\end{matrix}\right\rfloor\frac{t^{n}}{n!}$$

and the desired identity follows by identification of the coefficients of t^n. □

As an application of the Lah numbers of type B, we give a generalized identity considering rising and falling factorial of type B powers. This identity counts the number of ways to put n elements on x rails.

Remark 4.7. Let $\pi = \{B_0, B_1, -B_1, \ldots, B_k, -B_k\}$ be a signed partition, where B_0 is the zero block. In the next theorem we place the n positive elements on x rails (we may assume that $x \geq 2k+1$), where the first rail represent the zero block, and each pair of blocks B and $-B$ are adjacent. The negative elements are placed automatically: any positive element in block B occupies a single position, whereas in B_0 it occupies two. Moreover, the position of any positive element remains the same in B and $-B$.

Theorem 4.8. *For an non negative integers,* $0 \leq k \leq n$*, we have*

$$[x]_n^B=\sum_{k=0}^{n}\left\lfloor\begin{matrix}n\\k\end{matrix}\right\rfloor^{B}(x)_k^B\,,$$

where

$$[x]_n^B := (x+1)(x+3)\cdots(x+2n-1) \textit{ for } n\geq 1 \textit{ and } [x]_0^B := 1, \textit{ and,}$$
$$(x)_n^B := (x-1)(x-3)\cdots(x-2n+1) \textit{ for } n\geq 1 \textit{ and } (x)_0^B := 1.$$

Proof. Let $\pi = \{B_0, B_1, -B_1, \ldots, B_k, -B_k\}$ be a signed partition, where B_0 is the zero block. Now we must place the n elements on x rails. Thus we have $x+1$ ways to place the first element (2 places in the first rail and the remaining $x-1$ rails), $x+1+2 = x+3$ ways to place the second element, and so on $x+2n-1$ ways to place the n-th element. This gives $(x+1)(x+3)\cdots(x+2n-1) = [x]_n^B$ possibilities. For the right hand, for a given k, $(0 \leq k \leq n)$, we have to constitute k pairs of nonzero lists and at most one zero specific list with $\pm n$ elements. There are $\left\lfloor\begin{matrix}n\\k\end{matrix}\right\rfloor^{B}$ possibilities to constitute such lists, to affect all of them to x rails, we follow the same steps as in [1].

We consider an assignment of n elements into x rails as a function $f : [n] \longrightarrow [x]$. Let π represent the set of elements assignments, determined by the following procedure. For any positive $i \in B_0$, define: $f(i) = 1$.

$$B_0 = \{\pm i/f(i) = 1\}\,.$$

Choose a number p out of the $(x-1)$ remaining numbers $(2 \leq p \leq x)$ and send the positive elements of $B_1$to p_1 The absolute values of the negative elements of B_1 will be sent to the next number $p_1^{'}$ in cyclical order excluding the number 1. This can be done in $m-1$ different ways.

$$B_1 = \{i \in [n]\,/f(i) = p_1\} \cup \left\{-i/f(i) = p_1^{'}\right\}.$$

We pass to the pair of blocks B_2 and $-B_2$ Similarly, choose a new number P_2 out of the $m-3$ remaining numbers (the number 1 is occupied by the positive elements of the zero-block, and two additional numbers p_1 and $p_1^{'}$ are already occupied by the elements of the pair of blocks B_1 and $-B_1$), and send the positive elements of $B_2$to p_2 The absolute values of the negative elements of B_2 will be sent to the next number $p_2^{'}$ in cyclical order. This can be done in $m-3$ different ways.

$$B_2 = \{i \in [n]\,/f(i) = p_2\} \cup \left\{-i/f(i) = p_2^{'}\right\}.$$

Proceeding this way, we associate a set of $(x)_k^B$ functions from $[n]$ to $[x]$ to each signed partition having k pairs of nonzero blocks. □

5 Conclusion

Partition polynomials occupy a central position in a variety of applications, most notably in combinatorics, see [6, 14, 17, 18]. In particular, the partial Bell polynomials represent an important tool in combinatoris, analysis, probability, statistics, algebra and other fields. In this contribution, we introduced a new combinatorial interpretation of sequences of numbers related to these polynomials such the known Striling numbers of type B of two kinds [9] and we deduced new sequence related to Lah numbers. The results presented here open new avenues for research, including the exploration of generalizations and the discovery of further combinatorial interpretations.

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Author information

Meriem Moulay, USTHB, Faculty of Mathematics, RECITS Laboratory, BP 32, El-Alia, 16111, Algiers, Algeria.
E-mail: meriemro3@gmail.com, m.moulay@usthb.dz

Miloud Mihoubi, USTHB, Faculty of Mathematics, RECITS Laboratory, BP 32, El-Alia, 16111, Algiers, Algeria.
E-mail: mmihoubi@usthb.dz, miloudmihoubi@gmail.com

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