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QUASILINEAR ELLIPTIC SYSTEMS WITH PERTURBED GRADIENTS IN DEGENERATE FORM IN WEIGHTED SOBOLEV SPACES

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Abstract In this paper, we establish the existence of weak solutions for degenerate quasilinear elliptic systems given by

$$Gu(x) = f(x) \quad \text{in } \Omega, \quad \text{with } u = 0 \text{ on } \partial\Omega.$$

Here, G is the partial differential operator defined by

$$Gu(x) = -\operatorname{div}\left(\omega(x)\sigma(x, Du - \Upsilon(u))\right).$$

The weak solutions are obtained using Young measures and the Galerkin method within the framework of weighted Sobolev spaces.

The novelty of this work lies in the treatment of nonlinearities involving a perturbative term $\Upsilon(\varpi)$, which generalizes several existing models and captures more complex structural behaviors of the system. Furthermore, the presence of the weight function $\omega(x)$ introduces a degeneracy in the elliptic operator, making the analysis both broader and significantly more challenging, especially in establishing suitable conditions for convergence.

1 Introduction

The study of nonlinear partial differential equations (PDEs) characterized by degeneracy, singularity, or nonstandard growth conditions continues to attract considerable attention due to their theoretical richness and broad range of applications. Such equations frequently arise in areas including fluid mechanics, nonlinear elasticity, image processing, population dynamics, and materials science, where classical mathematical models are often insufficient to capture the complexity of physical phenomena. Addressing these problems requires analytical approaches that go beyond the standard Sobolev framework, employing functional structures more closely adapted to the irregular nature of the data, the nonlinearities of the operators, and the overall complexity of the models.

In recent years, considerable research attention has been devoted to partial differential equations (PDEs) governed by operators exhibiting nonstandard growth conditions, variable exponents, and fractional diffusion terms. These features often reflect intrinsic inhomogeneities in the medium or account for memory effects and long-range interactions. For example, various studies involving the fractional $p(x)$-Laplacian, double-phase functionals, or Kirchhoff-type operators with singular sources have employed variational methods, monotonicity techniques, and topological approaches.

For example, Akdim and Elmahi [4] demonstrated that renormalized solutions can be obtained for degenerate elliptic equations with L^1-data using truncation methods.

In [29], the authors combined variational methods with the Mountain Pass Theorem to investigate fractional Kirchhoff-type problems with variable exponents and indefinite weights.

In the context of fractional Laplacian problems, Aberqi and Ouaziz [1] used Morse theory and local linking techniques to compute critical groups and establish the existence of infinitely many solutions for a fractional $(p_1(x,\cdot), p_2(x,\cdot))$-Laplacian problem with a singular term on compact manifolds.

In subsequent work, Aberqi, Nguyen, Ouaziz, and Ragusa [2] studied the global dynamics of solutions for a new class of parabolic Kirchhoff-type equations governed by a double-phase operator with variable exponents. By combining the Faedo–Galerkin approximation method with Gronwall's inequality, they proved the existence and uniqueness of weak solutions, established conditions for finite-time blow-up or global existence based on the initial energy level, and demonstrated stabilization properties via Komornik's integral inequality.

In parallel with these developments, Daife et al. [15] examined a class of integro-differential elliptic problems governed by the operator $\mathcal{L}^p_K$, in which the nonlinearities include perturbative terms. By employing the Young measure approach to overcome challenges related to weak convergence and lack of compactness, they proved the existence of weak solutions and extended previous results on fractional p-Laplacian equations to broader nonlocal frameworks.

The analysis of operators in weighted spaces, as illustrated by Benyoub and Benaissa [7] in the context of fractional differential equations, provides a relevant conceptual framework for our study of degenerate elliptic systems. While their work focuses on fractional equations in weighted Banach spaces, we adapt this approach to weighted Sobolev spaces, specifically addressing the degeneracy induced by $\omega(x)$ and the nonlinear perturbation $\Upsilon(u)$.
The consideration of weighted Sobolev spaces and degenerate operators in our work is strongly motivated by physical applications involving heterogeneous media. Agarwal and Kumar [3] have shown that MHD flows of complex fluids in permeable beds naturally lead to degenerate differential operators, similar to our weighted operator $G(u) = -\mathrm{div}(\omega(x)\sigma(x, Du - \Upsilon(u)))$. Their analytical study underscores the importance of developing robust mathematical tools for such physically relevant problems.

In a closely related direction, Daife *et al.* [13, 14] used Galerkin approximation and Young measure theory to investigate quasilinear elliptic and parabolic systems with gradient perturbations. Their approach captures the weak convergence of nonlinear terms and extends previous results under relaxed monotonicity and growth assumptions. The Young measure framework provides a powerful tool for handling the weak convergence of nonlinearities, particularly in situations where the monotonicity conditions required by classical monotone operator theory—as developed by Višik [32], Brezis [8], Browder [11], Lions [27], and Minty [28]—are not satisfied.

In this paper, we investigate a new class of *degenerate quasilinear elliptic systems* governed by the boundary value problem:

$$Gu(x) := -\,\mathrm{div}\,\Big(\omega(x)\,\sigma\big(x, Du - \Upsilon(u)\big)\Big) = f(x), \quad x \in \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with a Lipschitz boundary. Here, $\omega : \Omega \to \mathbb{R}^+$ is a positive, measurable, and locally integrable weight; $\sigma : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function satisfying structural conditions to be specified later; and $\Upsilon : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a continuous mapping representing a gradient perturbation and satisfying appropriate growth and regularity assumptions.

The weak solutions are sought in the weighted Sobolev space $W^{1,p}_0(\Omega;\omega;\mathbb{R}^m)$, where the Jacobian matrix of u is $Du = (D_1u, \dots, D_nu)$ with $D_i = \partial/\partial x_i$. The source term f belongs to the dual space $W^{-1,p'}(\Omega;\omega^*,\mathbb{R}^m)$, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $\omega^* = \omega^{1-p'}$.

To ensure the rigorous definition of the functional setting, we impose the following conditions:

$$\omega \in L^1_{\mathrm{loc}}(\Omega), \tag{1.2}$$

$$\omega^{-1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega). \tag{1.3}$$

The main novelty of this work lies in the fact that, unlike the classical Leray–Lions framework where σ acts directly on Du, here the operator acts on the perturbed term $Du - \Upsilon(u)$. This formulation allows the analysis of a broader class of models incorporating internal feedback mechanisms, memory effects, and nonlocal interactions. Moreover, the weight function $\omega(x)$ introduces degeneracy or singularity, which justifies the use of weighted Sobolev spaces.

This study aims to extend several fundamental results from the classical Sobolev framework to the more general setting of weighted Sobolev spaces, as developed in [5], [12], and [24]. These functional spaces are particularly well suited for the rigorous analysis of partial differential equations defined in heterogeneous media or in domains exhibiting singular behavior, where the uniform assumptions of classical Sobolev spaces no longer hold. The weighted framework provides an effective means of handling degenerate or singular coefficients through suitable integrability conditions on the weight function, thereby relaxing the standard coercivity and growth conditions typically imposed in unweighted contexts.

Our contribution unifies and extends several existing frameworks by combining the theory of weighted Sobolev spaces with the Young measure approach. This integration allows us to manage the weak convergence of nonlinearities and to establish existence results for a broad class of quasilinear degenerate elliptic systems with perturbed gradient structures.

A concrete and representative example of the models considered in this study is the flow of a fluid through a porous medium, as discussed in [19]. This phenomenon can be modeled by the following nonlinear evolution equation:

$$\frac{\partial \theta}{\partial t} - \operatorname{div}\big(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e)\big) = 0,$$

where:

- $\theta = \theta(x,t)$ denotes the volumetric water content,
- $\varphi(\theta)$ is the hydrostatic potential,
- $K(\theta)$ is the hydraulic conductivity,
- and e is a unit vector in the upward vertical direction.

In the stationary regime, and under the assumption $\varphi(\theta) = \theta$, this equation reduces to a quasilinear elliptic problem of the form:

$$-\operatorname{div}\big(|\nabla\theta - K(\theta)e|^{p-2}(\nabla\theta - K(\theta)e)\big) = f(x),$$

subject to the homogeneous Dirichlet condition $\theta = 0$ on $\partial\Omega$, where $f(x)$ represents a source or sink term.

This model naturally fits within the general abstract framework studied in this paper, which considers nonlinear elliptic systems of the form:

$$\begin{cases} -\operatorname{div}\big(\sigma(x, Du(x) - \Upsilon(u(x)))\big) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where:

- $\sigma(x,\xi) = |\xi|^{p-2}\xi$ represents the p-Laplacian structure,
- $\Upsilon(u) = K(u)e$ models a nonlinear gradient perturbation.

In this particular case, setting $\omega(x) = 1$ removes the weight from the divergence operator, thereby simplifying the problem while retaining its nonlinear and degenerate character. This example clearly illustrates the type of problems addressed in this work, where the existence of weak solutions is established through the Galerkin method in combination with the Young measure approach.

This article is organized into five sections. The second section focuses on the functional spaces employed in the study, detailing their key properties and reviewing relevant results concerning Young measures. The third section presents the assumptions and main results regarding weak solutions of the problem under consideration. The fourth section constructs a sequence (u_j) of approximate solutions using the Galerkin method, establishes a priori estimates, and discusses essential properties and lemmas required to reach the main result. The final section concludes the paper by stating the principal theorem.

2 Mathematical Preliminaries

This section establishes the mathematical framework for our study. We first review the fundamental properties of weighted Sobolev spaces, namely $L^p(\Omega;\omega;\mathbb{R}^m)$ and $W^{1,p}(\Omega;\omega;\mathbb{R}^m)$. Subsequently, we provide a concise overview of Young measure theory, highlighting the aspects most pertinent to our analysis.

2.1 Functional Spaces $L^p(\Omega;\omega;\mathbb{R}^m)$ and $W^{1,p}(\Omega;\omega;\mathbb{R}^m)$

Weighted Sobolev spaces provide a natural functional framework for the analysis of degenerate or singular elliptic and parabolic systems. By incorporating a weight function ω, they extend classical Sobolev spaces to accommodate nonuniform coefficients and are fundamental for establishing the existence, uniqueness, and regularity of weak solutions.

For detailed discussions on their construction, embedding properties, and compactness results, we refer the reader to [10, 18, 22, 31].

Let $\Omega \subset \mathbb{R}^n$ be an open set with $n \geq 2$, and let ω be a weight function. For $1 < p < \infty$ and $m \geq 1$, we define the space $L^p(\Omega;\omega;\mathbb{R}^m)$ as the set of measurable functions $u = (u^1, \ldots, u^m)^t : \Omega \to \mathbb{R}^m$ such that each component u^i satisfies $|u^i|\,\omega^{1/p} \in L^p(\Omega)$ for $1 \leq i \leq m$.

The space is endowed with the norm

$$\|u\|_{p,\omega} = \left(\int_\Omega |u(x)|^p\,\omega(x)\,dx\right)^{1/p},$$

where $|u(x)|$ denotes the Euclidean norm of the vector $u(x)$. This space is a Banach space.

We denote by $(L^p(\Omega;\omega;\mathbb{R}^m))^*$ the dual of $L^p(\Omega;\omega;\mathbb{R}^m)$. Let p' be the conjugate exponent of p, satisfying $\frac{1}{p}+\frac{1}{p'}=1$. This dual space is isomorphic to $L^{p'}(\Omega,\omega^{-1/(p-1)};\mathbb{R}^m)$.

Remark 2.1. (1) Since $\omega^{-\frac{1}{p-1}}$ is locally integrable and $1 < p < \infty$, it follows that $L^p(\Omega;\omega;\mathbb{R}^m) \subset L^1_{\text{loc}}(\Omega;\mathbb{R}^m)$ for any open set Ω (see [26]). This inclusion allows us to consider the weak derivatives of functions in $L^p(\Omega;\omega;\mathbb{R}^m)$.

(2) When $\omega = 1$, the space $L^p(\Omega;\omega;\mathbb{R}^m)$ reduces to the classical Lebesgue space $L^p(\Omega;\mathbb{R}^m)$.

The weighted Sobolev space $W^{1,p}(\Omega;\omega;\mathbb{R}^m)$ consists of all functions $u \in L^p(\Omega;\omega;\mathbb{R}^m)$ such that the weak derivatives $D_i u$ (for $1 \leq i \leq m$) also belong to $L^p(\Omega;\omega;\mathbb{R}^m)$. This space is endowed with the norm

$$\|u\|_{1,p,\omega} = \left(\|u\|^p_{p,\omega} + \sum_{\substack{1\leq i\leq m\\1\leq j\leq n}}\int_\Omega \left|\frac{\partial u^i}{\partial x_j}\right|^p \omega(x)\,dx\right)^{1/p}.$$

The weighted Sobolev space $W^{1,p}_0(\Omega;\omega;\mathbb{R}^m)$ is defined as the closure of $C^\infty_c(\Omega;\mathbb{R}^m)$ in $W^{1,p}(\Omega;\omega;\mathbb{R}^m)$ with respect to the norm $\|\cdot\|_{1,p,\omega}$. This space is endowed with the norm $|||\cdot|||$ defined by

$$|||u||| = \left(\sum_{\substack{1\leq i\leq m\\1\leq j\leq n}}\int_\Omega \left|\frac{\partial u^i}{\partial x_j}\right|^p \omega(x)\,dx\right)^{1/p}.$$

The norms $\|\cdot\|_{1,p,\omega}$ and $|||\cdot|||$ are equivalent on $W^{1,p}_0(\Omega;\omega;\mathbb{R}^m)$ (see [30]).

Proposition 2.2. *The weighted Sobolev spaces $W^{1,p}(\Omega;\omega;\mathbb{R}^m)$ and $W^{1,p}_0(\Omega;\omega;\mathbb{R}^m)$ are Banach, separable, and reflexive spaces.*

Proof. The separability and reflexivity of $W^{1,p}(\Omega;\omega;\mathbb{R}^m)$ follow from [30, 18], and the reflexivity of $W^{1,p}_0(\Omega;\omega;\mathbb{R}^m)$ is also established in [18].

To show the separability of $W^{1,p}_0(\Omega;\omega;\mathbb{R}^m)$, consider the linear operator

$$T : W^{1,p}_0(\Omega;\omega;\mathbb{R}^m) \longrightarrow L^p(\Omega;\omega;\mathbb{M}^{m\times n}), \qquad T(u) = Du.$$

This operator is an isometric embedding since

$$|||u||| = \|T(u)\|_{L^p(\Omega;\omega;\mathbb{M}^{m\times n})}.$$

As $L^p(\Omega;\omega;\mathbb{R}^{m\times n})$ is separable, its closed subspace $T(W_0^{1,p}(\Omega;\omega;\mathbb{R}^m))$ is also separable. Therefore, $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$ is separable. □

It is worth noting that, as stated in [17], the following continuous embedding holds:

$$W_0^{1,p}(\Omega;\omega;\mathbb{R}^m) \hookrightarrow L^p(\Omega;\omega;\mathbb{R}^m). \tag{2.1}$$

2.2 Brief Reminder on Young Measures

In this subsection, we recall some fundamental properties of Young measures and summarize several established results. The Young measure framework serves as a powerful analytical tool to overcome challenges that arise from weak convergence, particularly in problems involving non-linear functionals and operators. Weak convergence remains one of the cornerstones of modern nonlinear analysis. For further details, see [6, 21, 23].

Let $C_0(\mathbb{R}^m)$ denote the space of real-valued continuous functions $\varphi : \mathbb{R}^m \to \mathbb{R}$ satisfying $\lim_{|\xi|\to\infty} \varphi(\xi) = 0$. Equipped with the L^∞-norm, this space is a Banach space. Its dual is identified with $\mathcal{M}(\mathbb{R}^m)$, the space of signed Radon measures with finite total variation.

The duality pairing between $C_0(\mathbb{R}^m)$ and $\mathcal{M}(\mathbb{R}^m)$ is defined by

$$\langle \nu, \varphi \rangle = \int_{\mathbb{R}^m} \varphi(\xi)\, d\nu(\xi),$$

for all $\varphi \in C_0(\mathbb{R}^m)$ and $\nu \in \mathcal{M}(\mathbb{R}^m)$.

The support of $\nu \in \mathcal{M}(\mathbb{R}^m)$ is given by

$$\operatorname{supp} \nu := \Big\{ \xi \in \mathbb{R}^m : \nu\big(B(\xi, r)\big) > 0 \text{ for all } r > 0 \Big\},$$

where $B(\xi, r)$ denotes the open ball of radius $r > 0$ centered at ξ.

The fundamental theorem of Young measures is stated in the following result:

Theorem 2.3 (**Young, Tartar, Ball** [16]).
Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable (not necessarily bounded), and let $z_j : \Omega \to \mathbb{R}^m$, $j = 1, 2, \ldots$, be a sequence of Lebesgue measurable functions. Then there exists a subsequence z_k and a family $\{\nu_x\}_{x\in\Omega}$ of non-negative Radon measures on $\mathbb{R}^m$ such that:

(i) $\|\nu_x\| := \int d\nu_x(\xi) \le 1$ *for almost every* $x \in \Omega$.

(ii) $\varphi(z_k) \rightharpoonup^* \overline{\varphi}$ *weakly* in* $L^\infty(\Omega)$ *for all* $\varphi \in C_0^0(\mathbb{R}^n)$, *where* $\overline{\varphi}(x) = \langle \nu_x, \varphi \rangle$.

(iii) If for all $R > 0$,

$$\lim_{L\to\infty} \sup_{k\in\mathbb{N}} \Big| \Big\{ x \in \Omega \cap B(0, R) : \ |z_k(x)| \ge L \Big\} \Big| = 0, \tag{2.2}$$

then

$$\|\nu_x\| = 1 \quad \text{for almost every } x \in \Omega, \tag{2.3}$$

and for all measurable $A \subset \Omega$, *the following holds:*

$$\left\{ \begin{array}{l} \varphi(z_k) \rightharpoonup \overline{\varphi} = \langle \nu_x, \varphi \rangle \text{ weakly in } L^1(A), \text{ for a continuous} \\ \text{function } \varphi : \mathbb{R}^m \to \mathbb{R}, \text{ provided that the sequence } \varphi(z_k) \\ \text{is weakly precompact in } L^1(A). \end{array} \right. \tag{2.4}$$

Here, $C_0^0(\mathbb{R}^m) = C_0(\mathbb{R}^m)$, *and* "$|.|$" *denotes the Lebesgue measure restricted to* Ω.

Remark 2.4. It is demonstrated in [25, Theorem 1.2] that condition (2.2) is required for (2.3) and (2.4) to hold. Therefore, (2.2), (2.3), and (2.4) are equivalent.

Lemma 2.5 ([21]). *Assume that the sequence (z_j) is bounded in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exists a subsequence (z_k) of (z_j) and, for almost every $x \in \Omega$, a Borel probability measure ν_x on $\mathbb{R}^m$ such that, for every $\varphi \in C(\mathbb{R}^m)$, we have*

$$\varphi(z_k) \overset{*}{\rightharpoonup} \overline{\varphi} \quad \textit{weakly}^* \textit{ in } L^\infty(\Omega),$$

where

$$\overline{\varphi}(x) = \int_{\mathbb{R}^m} \varphi(\xi)\, d\nu_x(\xi) \quad \textit{for a.e. } x \in \Omega.$$

Note that once the subsequence (z_k) of (z_j) is fixed, the associated family (ν_x) is uniquely determined. Furthermore, according to part (i) of Theorem 2.3, this family constitutes a sub-probability measure on $\mathbb{R}^m$, meaning that for almost every $x \in \Omega$, we have $|\nu_x| \le 1$.

Definition 2.6 ([21]). The family $(\nu_x)_{x\in\Omega}$ is called the Young measure generated by (z_k). Equivalently, it is said to be associated with (z_k).

In [6], it was established that, under hypothesis (2.2), for any measurable subset $\Omega' \subset \Omega$, we have

$$\varphi(\cdot, z_k) \rightharpoonup \langle \nu_x, \varphi(x,\cdot)\rangle = \int_{\mathbb{R}^m} \varphi(x,\xi)\, d\nu_x(\xi),$$

in $L^1(\Omega')$, for every Carathéodory function $\varphi : \Omega' \times \mathbb{R}^m \to \mathbb{R}$ such that the sequence $\{\varphi(\cdot, z_k)\}$ is sequentially weakly relatively compact in $L^1(\Omega')$.

This result is therefore equivalent to the conditions (2.2), (2.3), and (2.4).

Furthermore, Ball demonstrated that, for $\varphi \in L^1(\Omega; C_0(\mathbb{R}^m))$, we have

$$\lim_{k\to\infty} \int_\Omega \varphi(x, z_k(x))\, dx = \int_\Omega \langle \nu_x, \varphi(x,\cdot)\rangle\, dx,$$

whenever the sequence (z_k) generates the Young measure (ν_x).

Theorem 2.3 has several important applications that serve as fundamental tools in our subsequent analysis. Among the results that will be particularly useful in the remainder of this paper, we highlight the following.

Proposition 2.7 ([23]). *(i) If $|\Omega| < \infty$ and ν_x is the Young measure generated by the (whole) sequence (z_j), then the following equivalence holds:*

$$z_j \to z \textit{ in measure} \quad \textit{if and only if} \quad \nu_x = \delta_{z(x)} \quad \textit{for a.e. } x \in \Omega.$$

(ii) Let $|\Omega| < \infty$. If the sequences $z_j : \Omega \to \mathbb{R}^m$ and $z_j' : \Omega \to \mathbb{R}^d$ generate the Young measures $\delta_{z(x)}$ and ν_x, respectively, then the sequence (z_j, z_j') generates the Young measure $\delta_{z(x)} \otimes \nu_x$.

Lemma 2.8 ([24]). *Let $\Psi : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function, and let (u_j) be a sequence of measurable functions $u_j : \Omega \to \mathbb{R}^m$ such that Du_j generates the Young measure ν_x. Then*

$$\liminf_{j\to\infty} \int_\Omega \Psi(x, Du_j(x))\, dx \ge \int_\Omega \int_{\mathbb{M}^{m\times n}} \Psi(x,\xi)\, d\nu_x(\xi)\, dx, \tag{2.5}$$

provided that the negative part $\Psi^-(x, Du_j(x))$ is equiintegrable.

3 Main results

The functions Υ and σ are assumed to satisfy the following structural conditions.

(C1) Regularity of Υ: The function $\Upsilon : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is continuous, and there exists a constant $C_\Upsilon \geq 0$, depending on the exponent p and the diameter of Ω (denoted by $\operatorname{diam}(\Omega)$), such that

$$C_\Upsilon < \frac{1}{\operatorname{diam}(\Omega)} \left(\frac{1}{2}\right)^{\frac{1}{p}},$$

and

$$|\Upsilon(u) - \Upsilon(u')| \leq C_\Upsilon |u - u'|^\beta \quad \text{and} \quad \Upsilon(0) = 0, \tag{3.1}$$

for all $u, u' \in \mathbb{R}^m$ and some $0 < \beta < 1$.

(C2) Continuity of σ: The mapping $\sigma : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function; that is, $\sigma(\cdot, \xi)$ is measurable for every $\xi \in \mathbb{M}^{m\times n}$, and $\sigma(x, \cdot)$ is continuous for almost every $x \in \Omega$.

(C3) Growth and Coercivity: There exist functions $a_1 \in L^{p'}(\Omega;\omega)$, $b \in L^1(\Omega;\omega)$, and constants $c_1 \geq 0$, $c_2 > 0$ such that

$$|\sigma_{rs}(x, \xi - \Upsilon(\eta))| \leq a_1(x) + c_1 \sum_{i,j} |\xi_{ij} - \Upsilon(\eta)|^{p-1},$$

and

$$\sigma(x, \xi - \Upsilon(\eta)) : \xi \geq c_2 |\xi - \Upsilon(\eta)|^p - b(x),$$

where the colon denotes the standard inner product in $\mathbb{M}^{m\times n}$.

(C4) Monotonicity: The function σ satisfies one of the following conditions:

(a) For every $(x, \eta) \in \Omega \times \mathbb{R}^m$, the mapping $\xi \mapsto \sigma(x, \xi - \Upsilon(\eta))$ is of class C^1 and monotone, that is,
$$\big(\sigma(x, \xi - \Upsilon(\eta)) - \sigma(x, \xi' - \Upsilon(\eta))\big) : (\xi - \xi') \geq 0$$
for all $\xi, \xi' \in \mathbb{M}^{m\times n}$.

(b) There exists a function $W : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that for each $(x, \eta) \in \Omega \times \mathbb{R}^m$:

 (i) $\sigma(x, \xi - \Upsilon(\eta)) = D_\xi W(x, \xi - \Upsilon(\eta))$;

 (ii) the function $\xi \mapsto W(x, \xi - \Upsilon(\eta))$ is of class $\mathcal{C}^1$ and convex.

(c) For every $(x, \eta) \in \Omega \times \mathbb{R}^m$, the mapping $\xi \mapsto \sigma(x, \xi - \Upsilon(\eta))$ is strictly monotone, meaning that:

 (i) $\xi \mapsto \sigma(x, \xi - \Upsilon(\eta))$ is monotone;

 (ii) $\big(\sigma(x, \xi - \Upsilon(\eta)) - \sigma(x, \xi' - \Upsilon(\eta))\big) : (\xi - \xi') = 0$ if and only if $\xi = \xi'$ for all $\xi, \xi' \in \mathbb{M}^{m\times n}$.

(d) The function σ is strictly p-quasimonotone in its second variable; that is, for all $(x, \eta) \in \Omega \times \mathbb{R}^m$,
$$\int_{\mathbb{M}^{m\times n}} \big(\sigma(x, \xi - \Upsilon(\eta)) - \sigma(x, \overline{\xi} - \Upsilon(\eta))\big) : (\xi - \overline{\xi})\, d\nu(\xi) > 0,$$
where $\nu = \{\nu_x\}_{x\in\Omega}$ is a family of Young measures that does not reduce to a Dirac measure, generated by a sequence in $L^p(\Omega)$ for almost every $x \in \Omega$, and $\overline{\xi} = \langle \nu_x, \mathrm{id}\rangle$.

Definition 3.1. A measurable function $u \in W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$ is said to be a *weak solution* of problem (1.1) if

$$\int_\Omega \big(\omega(x)\, \sigma(x, Du - \Upsilon(u))\big) : D\xi \, dx = \langle f, \xi \rangle,$$

for all $\xi \in W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$ and its dual space $W^{-1,p'}(\Omega; w^*, \mathbb{R}^m)$.

The following theorem provides the main existence result of this paper.

Theorem 3.2. *Assume that σ satisfies conditions **(C2)–(C4)**. Then, for every Υ satisfying **(C1)** and every $f \in W^{-1,p'}(\Omega; w^*, \mathbb{R}^m)$, the Dirichlet problem (1.1) admits a weak solution*

$$u \in W_0^{1,p}(\Omega; \omega; \mathbb{R}^m).$$

Before proving the theorem, we introduce several essential components. We begin by defining the following operator:

$$\begin{aligned} \mathrm{F} : W_0^{1,p}(\Omega; \omega; \mathbb{R}^m) &\to W^{-1,p'}(\Omega; w^*, \mathbb{R}^m) \\ u &\mapsto \Big(v \mapsto \int_\Omega \big(\omega(x)\, \sigma(x, Du - \Upsilon(u))\big) : Dv \, dx - \langle f, v \rangle \Big). \end{aligned}$$

Throughout this article, C denotes a generic constant whose value may vary from line to line.

Lemma 3.3. *Assume that conditions **(C1)–(C5)** hold. Then, for any $u \in W_0^{1,p}(\Omega; \omega; \mathbb{R}^m)$ and any $f \in W^{-1,p'}(\Omega; w^*, \mathbb{R}^m)$, the following properties hold:*

(i) The functional $\mathrm{F}(u)$ is well-defined, linear, and bounded.

(ii) The restriction of F to any finite-dimensional linear subspace V of $W_0^{1,p}(\Omega; \omega; \mathbb{R}^m)$ is continuous.

Proof. (i) Let $u \in W_0^{1,p}(\Omega; \omega; \mathbb{R}^m)$ be arbitrary, and define

$$\mathrm{F}(u)(\xi) = \mathrm{F}_1(u)(\xi) - \mathrm{F}_2(u)(\xi),$$

where

$$\mathrm{F}_1(u)(\xi) = \int_\Omega \big(\omega(x)\, \sigma(x, Du - \Upsilon(u))\big) : D\xi \, dx, \quad \mathrm{F}_2(u)(\xi) = \langle f, \xi \rangle,$$

for all $\xi \in W_0^{1,p}(\Omega; \omega; \mathbb{R}^m)$.

Linearity of $\mathrm{F}(u)$ follows directly from the linearity of both $\mathrm{F}_1(u)$ and $\mathrm{F}_2(u)$, as well as that of the integral.

Define

$$I_{rs} = \int_\Omega \big(\omega(x)\, \sigma_{rs}(x, Du - \Upsilon(u))\big) : D_{r,s}\xi(x) \, dx.$$

By the continuity assumption **(C1)**, the growth condition **(C3)**, and the convexity of the mapping $x \mapsto x^{p'}$, combined with the equivalence of the norms $\|\cdot\|_{1,p,\omega}$ and $|||\cdot|||$ on $W_0^{1,p}(\Omega; \omega; \mathbb{R}^m)$, there exists a constant $C > 0$ (depending on c_1, C_Υ, p, and p') such that

$$\int_\Omega |\sigma_{rs}(x, Du - \Upsilon(u))|^{p'} \omega(x) \, dx \le C \left(\|a_1\|_{p',\omega}^{p'} + \|u\|_{1,p,\omega}^p \right) < \infty.$$

Therefore, $\sigma(x, Du - \Upsilon(u)) \in L^{p'}(\Omega; \omega; \mathbb{R}^m)$.

On the other hand, by Hölder's inequality and for a suitable constant $\tilde{c} > 0$, we obtain

$$\begin{aligned} |I_{rs}| &\le \int_\Omega |\omega(x)\, \sigma_{rs}(x, Du - \Upsilon(u))|\, |D_{r,s}\xi(x)| \, dx \\ &\le \int_\Omega \omega^{\frac{1}{p'}} a_1(x)\, |\omega^{\frac{1}{p}} D_{r,s}\xi(x)| \, dx + \int_\Omega c_1 \left[\sum_{i,j} \omega^{\frac{1}{p'}} |D_{ij}u(x) - \Upsilon(u)|^{p-1} \right] |\omega^{\frac{1}{p}} D_{r,s}\xi(x)| \, dx \\ &\le \|a_1\|_{p',\omega} \|\xi\|_{p,\omega} + \tilde{c}\, \|u\|_{1,p,\omega}^{p-1} \|\xi\|_{p,\omega}. \end{aligned}$$

By the equivalence of the norms $\|\cdot\|_{1,p,\omega}$ and $|||\cdot|||$ on $W_0^{1,p}(\Omega; \omega; \mathbb{R}^m)$, and using Hölder's inequality, we deduce that for every $\xi \in W_0^{1,p}(\Omega; \omega; \mathbb{R}^m)$, there exists a constant $C > 0$ such that

$$|\mathrm{F}(u)(\xi)| \le |\mathrm{F}_1(u)(\xi)| + |\mathrm{F}_2(u)(\xi)| \le C \|\xi\|_{1,p,w}.$$

Thus, $\mathrm{F}(u)$ is well-defined and bounded.

(ii) Let E be a finite-dimensional subspace of $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$ with a basis $(\psi_i)_{i=1,\ldots,d}$, where $d=\dim(E)$. Suppose that $u_j\to u$ in E, with $u_j=a_j^i\psi_i$ and $u=a^i\psi_i$ (using the Einstein summation convention). Then for each $1\le i\le d$, we have

$$a_j^i\to a^i \quad \text{in } \mathbb{R} \text{ as } j\to\infty,$$

which implies $u_j\to u$ and $Du_j\to Du$ almost everywhere.

By [9, Théorème IV.9], the sequences (u_j) and (Du_j) are uniformly bounded in $L^p(\Omega;w)$.

Using the continuity conditions **(C1)** and **(C2)**, we conclude that

$$\omega(x)\,\sigma(x,Du_j(x)-\Upsilon(u_j(x)))\to\omega(x)\,\sigma(x,Du(x)-\Upsilon(u(x))) \quad \text{a.e. in } \Omega.$$

Moreover, these sequences are equiintegrable due to **(C1)** and **(C2)**.

By Vitali's convergence theorem, for all $\xi\in W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$,

$$\|\mathrm{F}(u_j)-\mathrm{F}(u)\|_{-1,p',w}=\sup_{|||\xi|||=1}|\mathrm{F}(u_j)(\xi)-\mathrm{F}(u)(\xi)|\to 0 \quad \text{as } j\to\infty.$$

Hence, the restriction of F to E is continuous. □

4 Proof of the Main Result

In this section, we employ the Galerkin method to construct suitable approximations and establish several auxiliary lemmas, making use of Young measures to address potential issues of weak convergence.

4.1 Appropriate Approximations

We consider a sequence of finite-dimensional subspaces E_j forming an increasing (nested) sequence within $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$, such that the union $\bigcup_{j\in\mathbb{N}}E_j$ is dense in $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$. The existence of such a sequence (E_j) follows from the separability of $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$.

The approximate solutions to our problem are constructed using the Galerkin method, complemented by several auxiliary arguments.

Fix $j\in\mathbb{N}$, and assume that the dimension of E_j is d, with $(e_i)_{1\le i\le d}$ forming a basis of E_j. We then define the mapping G, associated with F, as follows:

$$\begin{array}{rccc}\mathrm{G}: & \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \\ & (a^1,\ldots,a^d)^t & \longmapsto & \big(\langle \mathrm{F}(a^ie_i),e_1\rangle,\ldots,\langle \mathrm{F}(a^ie_i),e_d\rangle\big)^t\end{array}$$

Proposition 4.1. *Under conditions **(C1)–(C4)**, the operator* G *satisfies the following properties:*

(i) G *is continuous.*

(ii) $\mathrm{G}(a)\cdot a\to\infty$ *as* $\|a\|_{\mathbb{R}^d}\to\infty$.

Proof. (i) The continuity of the restriction of F to E_j follows from Lemma 3.3; hence, G is also continuous.

(ii) Let $u=a^ie_i\in E_j$ (using the summation convention), where $a=(a^1,\ldots,a^d)^t\in\mathbb{R}^d$. Note that $\|a\|_{\mathbb{R}^d}\to\infty$ if and only if $\|u\|_{1,p,w}\to\infty$.

By the coercivity condition **(C3)**, we have

$$\begin{aligned}\langle \mathrm{F}(u),u\rangle &= \int_\Omega \omega(x)\,\sigma(x,Du-\Upsilon(u)):Du\,dx-\langle f,u\rangle\\ &\ge c_2\int_\Omega \omega(x)\,|Du-\Upsilon(u)|^p\,dx-\int_\Omega \omega(x)\,b(x)\,dx-\langle f,u\rangle.\end{aligned}$$

Since

$$\begin{aligned}|Du|^p &\le 2^{p-1}\big(|Du-\Upsilon(u)|^p+|\Upsilon(u)|^p\big)\\ &\le 2^{p-1}\big(|Du-\Upsilon(u)|^p+C_\Upsilon^p|u|^{\beta p}\big),\end{aligned}$$

it follows that

$$|Du - \Upsilon(u)|^p \geq \frac{1}{2^{p-1}}|Du|^p - C_\Upsilon^p |u|^{\beta p}.$$

Using the equivalence of the norms $\|\cdot\|_{1,p,\omega}$ and $|||\cdot|||$ on $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$, there exists a constant $C > 0$ such that

$$\begin{aligned}\langle \mathrm{F}(u), u\rangle &\geq \frac{c_2}{2^{p-1}}\int_\Omega \omega\,|Du|^p\,dx - c_2 C_\Upsilon^p \int_\Omega \omega\,|u|^{\beta p}\,dx - \int_\Omega \omega\, b(x)\,dx - \langle f, u\rangle \\ &\geq C\|u\|_{1,p,w}^p - C\|u\|_{1,p,w}^{\beta p} - \|b\|_{L^1(\Omega;\omega)} - \|f\|_{-1,p',\omega^*}\,\|u\|_{1,p,w}.\end{aligned}$$

Since $p > \max(1, \beta p)$, we obtain

$$\langle \mathrm{F}(u), u\rangle \to \infty \quad \text{as} \quad \|u\|_{1,p,w} \to \infty.$$

Finally, since $\mathrm{G}(a)\cdot a = \langle \mathrm{F}(a^i e_i), a^i e_i\rangle = \langle \mathrm{F}(u), u\rangle$, it follows that

$$\mathrm{G}(a)\cdot a \longrightarrow \infty \quad \text{as} \quad \|a\|_{\mathbb{R}^d} \to \infty.$$

□

Lemma 4.2. *(i) For every $j = 1, 2, \ldots$, there exists $u_j \in E_j$ such that*

$$\langle \mathrm{F}(u_j), \varphi\rangle = 0 \quad \textit{for all} \quad \varphi \in E_j. \tag{4.1}$$

(ii) The sequence (u_j) defined in (i) is uniformly bounded in $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$.

Proof. (i) By Lemma 4.1(ii), there exists $R > 0$ such that $\mathrm{G}(a)\cdot a > 0$ for all $a \in \partial B_R(0) \subset \mathbb{R}^d$. Applying a topological degree argument (see [34, Proposition 2.8]), we deduce that the equation $\mathrm{G}(x) = 0$ admits a solution in $B_R(0)$. Hence, equation (4.1) holds.

(ii) Note that $\langle \mathrm{F}(u), u\rangle = \mathrm{G}(a)\cdot a$ and that $|||u||| \to \infty$ if and only if $\|a\|_{\mathbb{R}^d} \to \infty$. From Lemma 4.1(ii), it follows that $\langle \mathrm{F}(u), u\rangle \to \infty$ as $|||u||| \to \infty$. Therefore, there exists $R > 0$ such that $\langle \mathrm{F}(u), u\rangle > 1$ whenever $|||u||| > R$.

However, the sequence (u_j) constructed in part (i) satisfies $\langle \mathrm{F}(u_j), u_j\rangle = 0$ for all j, which implies that $|||u_j||| \leq R$ for every j. Therefore, the sequence (u_j) is uniformly bounded in $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$. □

4.2 Properties of the sequences (u_j) and (Du_j)

Lemma 4.3. *Let (u_j) be the sequence introduced in Lemma 4.2. Then, the following properties hold:*

(i) For $p > 1$, the sequence (u_j) converges weakly in $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$ to an element denoted by u, up to a subsequence still denoted by (u_j).

(ii) The sequence (Du_j) is uniformly bounded in $L^p(\Omega;\omega;\mathbb{M}^{m\times n})$.

(iii) The sequence $\sigma(x, Du_j - \Upsilon(u_j))$ is uniformly bounded in $L^{p'}(\Omega;\omega;\mathbb{R}^m)$ and hence equi-integrable on Ω.

Proof. **(i)** The conclusion follows from Lemma 4.2(ii), together with the Eberlein–Šmulian theorem [9].

(ii) Since the sequence (u_j) is uniformly bounded in $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$, there exists a constant $C > 0$ such that

$$\int_\Omega \omega\,|Du_j|^p\,dx \leq C.$$

(iii) By using the same reasoning as in Lemma 3.3**(i)**, we infer that the sequence $\sigma(x, Du_j - \Upsilon(u_j))$ is uniformly bounded in $L^{p'}(\Omega;\omega;\mathbb{R}^m)$, and hence equiintegrable on Ω. □

Lemma 4.4. *Let (u_j) be the sequence constructed in Lemma 4.2, and assume that $u_j \rightharpoonup u$ in $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$. Then, there exists a sequence (v_j) converging strongly to u in $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$ such that*

$$\int_\Omega \omega\, \sigma(x, Du_j - \Upsilon(u_j)) : (Du_j - Dv_j)\, dx \longrightarrow 0 \quad \text{as } j \to \infty.$$

Proof. By Mazur's theorem, there exists a sequence $(v_j) \subset W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$, where each v_j is a convex linear combination of $\{u_1, \dots, u_j\}$, such that $v_j \to u$ in $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$ (see [33, Theorem 2]). It is worth noting that both v_j and u_j belong to the same subspace E_j.

By taking $u_j - v_j$ as a test function in (4.1) and using Hölder's inequality, we obtain, for some constant $C > 0$,

$$\begin{aligned}\Big| \int_\Omega \omega\, \sigma(x, Du_j - \Upsilon(u_j)) : (Du_j - Dv_j)\, dx \Big| &= |\langle f, u_j - v_j\rangle| \\ &\le C\|f\|_{-1,p',\omega^*}\, \|u_j - v_j\|_{1,p,\omega}.\end{aligned}$$

Since (v_j) converges strongly to u in $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$, we have

$$\|u_j - v_j\|_{1,p,\omega} \le \|u_j - u\|_{1,p,\omega} + \|v_j - u\|_{1,p,\omega} \longrightarrow 0 \quad \text{as } j \to \infty.$$

Therefore,

$$\int_\Omega \omega\, \sigma(x, Du_j - \Upsilon(u_j)) : (Du_j - Dv_j)\, dx \longrightarrow 0 \quad \text{as } j \to \infty.$$

□

We note that, by Lemma 4.3(i), for the sequence (u_j) constructed in Lemma 4.2(i), there exists a subsequence (still denoted by (u_j)) such that $u_j \in E_j$ for each $j = 1, 2, \dots$, and

$$u_j \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega;\omega;\mathbb{R}^m).$$

Moreover, since the embedding

$$W_0^{1,p}(\Omega;\omega;\mathbb{R}^m) \hookrightarrow L^p(\Omega;\omega;\mathbb{R}^m)$$

is continuous, the sequence (u_j) is bounded in $L^p(\Omega;\omega;\mathbb{R}^m)$. Therefore, by passing to a further subsequence if necessary, we can conclude that

$$u_j \to u \quad \text{in measure on } \Omega.$$

The gradient sequence (Du_j) generates the Young measure ν_x according to Theorem 2.3, and by Proposition 2.7, the pair (u_j, Du_j) generates the Young measure $\delta_{u(x)} \otimes \nu_x$.

In the remainder of this article, ν_x denotes the Young measure that is generated by the gradient sequence (Du_j).

4.3 Some auxiliary lemmas

The following lemma provides some facts about the Young measure generated by the gradient sequence (Du_j).

Lemma 4.5. *Let (u_j) be the sequence from Lemma 4.2. Then the Young measure ν_x generated by (Du_j) satisfies the following properties:*

(i) For almost every $x \in \Omega$, ν_x is a probability measure.

(ii) For almost every $x \in \Omega$, $\langle \nu_x, \mathrm{id}\rangle = Du(x)$.

Proof. **(i)** The sequence (Du_j) satisfies condition (2.2) of Theorem 2.3. It then follows that (Du_j) also satisfies condition (2.3) of the same theorem. Consequently, we have $\|\nu_x\| = 1$ for almost every $x \in \Omega$.

(ii) By Lemma 4.3(i), there exists a subsequence of (u_j), still denoted by (u_j), that converges weakly to a function u in $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$. Consequently, we have

$$Du_j \rightharpoonup Du \quad \text{in } L^p(\Omega;\omega;\mathbb{M}^{m\times n}).$$

It then follows that

$$Du_j \rightharpoonup Du \quad \text{in } L^1(\Omega;\omega;\mathbb{M}^{m\times n}),$$

since the embedding

$$L^p(\Omega;\omega;\mathbb{M}^{m\times n}) \hookrightarrow L^1(\Omega;\omega;\mathbb{M}^{m\times n})$$

is continuous for $p > 1$. By the uniqueness of weak limits and identity (2.4) in Theorem 2.3, we conclude that

$$\langle \nu_x, \mathrm{id}\rangle = Du(x) \quad \text{for almost every } x \in \Omega.$$

□

5 Proof of the Main Theorem

Before proceeding with the proof of the main theorem, we establish a few auxiliary lemmas.

Lemma 5.1. *Assume that conditions **(C1)–(C4)** are satisfied. Let ν_x be the Young measure associated with the gradient sequence (Du_j), where $u_j \rightharpoonup u$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. Then, the following inequality holds:*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \omega(x)\,\sigma\big(x,\xi-\Upsilon(u)\big) : \xi \,d\nu_x(\xi)\,dx \le \int_\Omega \int_{\mathbb{M}^{m\times n}} \omega(x)\,\sigma\big(x,\xi-\Upsilon(u)\big) : Du\,d\nu_x(\xi)\,dx.$$

Proof. The proof follows the same arguments as in Lemma 5 of [5], with minor modifications to account for the weighted setting. □

We now establish the following localization result for the support of the Young measure ν_x.

Lemma 5.2. *If the inequality stated in Lemma 5.1 holds, then*

$$\big(\sigma(x,\xi-\Upsilon(u)) - \sigma(x,Du-\Upsilon(u))\big) : (\xi - Du) = 0 \quad \text{on } \operatorname{supp}\nu_x, \quad \text{for almost every } x\in\Omega. \tag{5.1}$$

Proof. We first observe that

$$\begin{aligned}\liminf_{j\to\infty} &\int_\Omega \sigma(x,Du-\Upsilon(u)) : (Du_j - Du)dx \\ &= \int_\Omega \sigma(x,Du-\Upsilon(u)) : \Big(\int_{\mathbb{M}^{m\times n}} \xi d\nu_x(\xi) - Du\Big)dx \\ &= 0\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\int_\Omega\int_{\mathbb{M}^{m\times n}} \Big(\sigma\big(x,\xi-\Upsilon(u)\big) - \sigma\big(x,Du-\Upsilon(u)\big)\Big) : \Big(\xi - Du\Big)d\nu_x(\xi)dx \\ &\qquad \le \liminf_{j\to\infty}\int_\Omega \Big(\sigma\big(x,Du_j-\Upsilon(u_j)\big) - \sigma\big(x,Du-\Upsilon(u)\big)\Big) : \Big(Du_j - Du\Big)dx \\ &\qquad = \liminf_{j\to\infty}\int_\Omega \sigma\big(x,Du_j-\Upsilon(u_j)\big) : \Big(Du_j-Du\Big)dx \\ &\qquad \le 0.\end{aligned}$$

Since the operator σ is monotone, the integrand in the last inequality is nonnegative. Hence, it must vanish almost everywhere with respect to the product measure $d\nu_x(\xi)\otimes dx$. This establishes equality (5.1). □

At this stage, we have gathered all the essential components required to establish the proof of our main theorem. We now proceed to examine each of the cases specified in **(C4)**.

To summarize, according to Lemma 4.2, we have constructed a sequence (u_j) such that, for $j = 1, 2, \ldots$, $u_j \in E_j$, and the sequence is uniformly bounded in $W_0^{1,p}(\Omega; \omega; \mathbb{R}^m)$. Moreover, from Lemma 4.3 (i), we deduce the existence of a subsequence of (u_j) (which, for simplicity, we continue to denote by (u_j)) that converges weakly in $W_0^{1,p}(\Omega; \omega; \mathbb{R}^m)$ to an element u.

We now prove that the limit function u is indeed a weak solution to our problem. Henceforth, (u_j) will denote this weakly convergent subsequence.

Under assumptions **(C1)–(C4)**, and for each case described in **(C4)**, we show that for every $\xi \in W_0^{1,p}(\Omega; \omega; \mathbb{R}^m)$, the following convergence holds:

$$\int_\Omega \omega\, \sigma(x, Du_j - \Upsilon(u_j)) : D\xi\, dx \;\longrightarrow\; \int_\Omega \omega\, \sigma(x, Du - \Upsilon(u)) : D\xi\, dx, \quad \text{as } j \to \infty. \tag{5.2}$$

We begin with cases **(c)** and **(d)**, first proving that $Du_j \to Du$ in measure as $j \to \infty$. We then draw the corresponding conclusion based on this result.

Case (c): Since σ is strictly monotone, and using Lemma 5.2, we deduce that

$$\text{supp } \nu_x = \{Du(x)\}.$$

Consequently, $\nu_x = \delta_{Du(x)}$ for almost every $x \in \Omega$. Therefore, by part (i) of Proposition 2.7, it follows that

$$Du_j \to Du \quad \text{in measure as } j \to \infty.$$

Case (d): We now prove that ν_x cannot be a Dirac mass on a subset $\Omega' \subset \Omega$ of positive Lebesgue measure, where $x \in \Omega'$ and $\overline{\xi} = \langle \nu_x, \text{id} \rangle$. Moreover, by part (2) of Lemma 4.5, we have $\langle \nu_x, \text{id} \rangle = Du$.

Hence, we obtain

$$\begin{aligned}\int_{\mathbb{M}^{m\times n}} \omega\, \sigma(x, \overline{\xi} - \Upsilon(u)) : \xi\, d\nu_x(\xi) &= \omega\, \sigma(x, Du - \Upsilon(u)) : \int_{\mathbb{M}^{m\times n}} \xi\, d\nu_x(\xi) \\ &= \int_{\mathbb{M}^{m\times n}} \omega\, \sigma(x, \overline{\xi} - \Upsilon(u)) : \overline{\xi}\, d\nu_x(\xi).\end{aligned} \tag{5.3}$$

Combining this equality with the strict p-quasi-monotonicity of σ with respect to its second argument, as stated in **(C4)(d)**, we deduce that, for almost every $x \in \Omega'$,

$$\int_{\mathbb{M}^{m\times n}} \sigma(x, \xi - \Upsilon(u)) : \xi\, d\nu_x(\xi) > \int_{\mathbb{M}^{m\times n}} \sigma(x, \xi - \Upsilon(u)) : \overline{\xi}\, d\nu_x(\xi).$$

Integrating this inequality over Ω, we obtain

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, \xi - \Upsilon(u)) : \xi\, d\nu_x(\xi)\, dx > \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, \xi - \Upsilon(u)) : \overline{\xi}\, d\nu_x(\xi)\, dx.$$

However, from (5.3) and Lemma 5.1, we have

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, \xi - \Upsilon(u)) : \overline{\xi}\, d\nu_x(\xi)\, dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, \xi - \Upsilon(u)) : \xi\, d\nu_x(\xi)\, dx,$$

which leads to a contradiction when comparing the last two inequalities.

Consequently, there exists a measurable function κ such that $\nu_x = \delta_{\kappa(x)}$ for almost every $x \in \Omega$. Therefore,

$$\kappa(x) = \int_{\mathbb{M}^{m\times n}} \xi\, d\delta_{\kappa(x)}(\xi) = \int_{\mathbb{M}^{m\times n}} \xi\, d\nu_x(\xi) = Du(x).$$

We thus conclude that $\nu_x = \delta_{Du(x)}$ for almost every $x \in \Omega$. By Proposition 2.7 (i), it follows that $Du_j \to Du$ in measure on Ω as $j \to \infty$.

Now, for cases **(c)** and **(d)**, we establish that $Du_j \to Du$ in measure on Ω as $j \to \infty$.

To complete the proof, we already have $u_j \rightharpoonup u$ in $W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$. By the argument in the proof of Lemma 4.4 and the continuity of the embedding $W^{1,p}(\Omega;\omega;\mathbb{R}^m) \hookrightarrow L^p(\Omega;\omega;\mathbb{R}^m)$, it follows that, up to a subsequence, $u_j \to u$ in measure on Ω.

Since the sequences (u_j) and (Du_j) are bounded in $L^p(\Omega;\omega;\mathbb{R}^m)$ and $L^p(\Omega;\omega;\mathbb{M}^{m\times n})$, respectively, for $p>1$, they are equiintegrable. By Vitali's Convergence Theorem, this implies that $u_j \to u$ and $Du_j \to Du$ in $L^1(\Omega;\omega;\mathbb{R}^m)$ and $L^1(\Omega;\omega;\mathbb{M}^{m\times n})$, respectively, and there exist subsequences, again denoted by (u_j) and (Du_j), that converge almost everywhere in Ω.

Consequently, by the continuity of σ and Υ, for every $\xi \in W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$, we have

$$\omega\,\sigma(x, Du_j - \Upsilon(u_j)) : D\xi \;\to\; \omega\,\sigma(x, Du - \Upsilon(u)) : D\xi \quad \text{as } j\to\infty,$$

almost everywhere and in measure, since Ω has finite measure.

Moreover, under the boundedness assumptions **(C1)** and **(C2)**, and using Hölder's inequality, the sequence $\big(\omega\,\sigma(x, Du_j - \Upsilon(u_j)) : D\xi\big)$ is equiintegrable. Hence, by Vitali's convergence theorem, equality (5.2) holds for every $\xi \in W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$.

Case (a): First, due to the monotonicity of σ, using Lemma 5.2, and noting that the map $\xi \mapsto \sigma(x, \xi - \Upsilon(u))$ is C^1, we assert that the following equality holds on supp ν_x for all $(x,\zeta) \in \Omega \times \mathbb{M}^{m\times n}$:

$$\sigma(x,\xi-\Upsilon(u)) : \zeta = \sigma(x, Du - \Upsilon(u)) : \zeta + \big(\nabla_\xi \sigma(x, Du - \Upsilon(u))\zeta\big) : (Du - \xi), \qquad (**)$$

where ∇_ξ denotes the derivative of σ with respect to its second variable.

Since Υ is continuous, we have $\Upsilon(u_j) \to \Upsilon(u)$ almost everywhere and in $L^1(\Omega;\omega;\mathbb{M}^{m\times n})$ by Vitali's Convergence Theorem. Moreover, as $\sigma(x, Du_j - \Upsilon(u_j))$ is equiintegrable, it follows from Ball's theorem that its weak L^1-limit is given by

$$\overline{\sigma} = \int_{\text{supp }\nu_x} \sigma(x, \xi - \Upsilon(u))\, d\nu_x(\xi).$$

Using equality $(**)$ and the fact that

$$Du \int_{\text{supp }\nu_x} d\nu_x(\xi) - \int_{\text{supp }\nu_x} \xi\, d\nu_x(\xi) = 0,$$

we obtain

$$\overline{\sigma} = \sigma(x, Du - \Upsilon(u)).$$

By the Eberlein–Šmulian theorem [9], the sequence $\sigma(x, Du_j - \Upsilon(u_j))$ converges weakly in $L^{p'}(\Omega;\omega;\mathbb{M}^{m\times n})$, as it is bounded and $L^{p'}(\Omega;\omega;\mathbb{M}^{m\times n})$ is reflexive (for $p'>1$). Consequently, by uniqueness of the weak limit, $\sigma(x, Du - \Upsilon(u))$ is its weak $L^{p'}$-limit (for a subsequence).

Thus, equality (5.2) holds for every $\xi \in W_0^{1,p}(\Omega;\omega;\mathbb{R}^m)$.

Case (b): For $x \in \Omega$ and $u \in \mathbb{R}^m$, consider the set

$$\Lambda_x := \Big\{\xi \in \mathbb{M}^{m\times n} : W(x, \xi - \Upsilon(u)) = W(x, Du - \Upsilon(u)) + \sigma(x, Du - \Upsilon(u)) : (\xi - Du)\Big\}.$$

By the monotonicity of σ, Lemma 5.2, condition **(b)** of **(C4)**, and following a similar argument as in the proof of case **(b)** of Proposition 5.1 in [14], we obtain supp $\nu_x \subset \Lambda_x$ for almost every $x\in\Omega$.

Due to the convexity of W with respect to its second variable, for any $\xi \in \mathbb{M}^{m\times n}$, the inequality

$$W(x, \xi - \Upsilon(u)) \geq W(x, Du - \Upsilon(u)) + \sigma(x, Du - \Upsilon(u)) : (\xi - Du)$$

holds.

Now, for $\zeta \in \mathbb{M}^{m\times n}$, $\tau \in \mathbb{R}$, and $\xi \in \Lambda_x$, define

$$A(\xi) = W(x, \xi - \Upsilon(u)), \quad B(\xi) = A(\xi) + \sigma(x, Du - \Upsilon(u)) : (\xi - Du).$$

Since $\xi \mapsto W(x, \xi - \Upsilon(u))$ is C^1, it follows that

$$\frac{A(\xi + \tau\zeta) - A(\xi)}{\tau} \geq \frac{B(\xi+\tau\zeta) - B(\xi)}{\tau} \quad \text{for } \tau > 0,$$

and
$$\frac{A(\xi+\tau\zeta)-A(\xi)}{\tau} \leq \frac{B(\xi+\tau\zeta)-B(\xi)}{\tau} \quad \text{for } \tau < 0.$$

Hence, $D_\xi A = D_\xi B$, which implies that for all $\xi \in \Lambda_x \supset \text{supp } \nu_x$,

$$\sigma(x, \xi - \Upsilon(u)) = \sigma(x, Du - \Upsilon(u)). \tag{5.4}$$

Consequently,

$$\overline{\sigma}(x) = \int_{\text{supp } \nu_x} \sigma(x, Du - \Upsilon(u))\, d\nu_x(\xi) = \sigma(x, Du - \Upsilon(u)).$$

Next, define the function

$$\Phi(x, u, \xi) = |\sigma(x, \xi - \Upsilon(u)) - \overline{\sigma}(x)|,$$

which is a Carathéodory function, and consider the sequence $\Phi_j(x) = \Phi(x, u_j, Du_j)$.

Since $u_j \to u$ almost everywhere in Ω and Υ is continuous, it follows that $\Upsilon(u_j) \to \Upsilon(u)$ almost everywhere in Ω, and hence $\Upsilon(u_j) \to \Upsilon(u)$ in measure. Therefore, by Proposition (2.7), the pair $(\Upsilon(u_j), Du_j)$ generates the Young measure $\delta_{\Upsilon(u(x))} \otimes \nu_x$.

Moreover, the sequence $\Phi_j(x)$ is equiintegrable because $\sigma(x, Du_j - \Upsilon(u_j))$ is equiintegrable. Therefore,
$$\Phi_j \rightharpoonup \overline{\Phi} \quad \text{weakly in } L^1(\Omega),$$
where

$$\begin{aligned}
\overline{\Phi}(x) &= \int_{\mathbb{R}^m \times \mathbb{M}^{m\times n}} \Phi(x, \eta, \xi)\, d\delta_{\Upsilon(u(x))}(\eta) \otimes d\nu_x(\xi) \\
&= \int_{\text{supp } \nu_x} |\sigma(x, \xi - \Upsilon(u)) - \sigma(x, Du - \Upsilon(u))|\, d\nu_x(\xi) \\
&\overset{(5.4)}{=} 0.
\end{aligned}$$

Since $\Phi_j \geq 0$, it follows that $\Phi_j \to 0$ strongly in $L^1(\Omega)$. Hence, by Vitali's Convergence Theorem, equality (5.2) holds for any $\varphi \in W_0^{1,p}(\Omega; \omega; \mathbb{R}^m)$.

From the four cases discussed, for any $\xi \in \bigcup_{j \in \mathbb{N}} E_j$, we have

$$\int_\Omega \omega\, \sigma(x, Du_j - \Upsilon(u_j)) : D\xi\, dx \longrightarrow \int_\Omega \omega\, \sigma(x, Du - \Upsilon(u)) : D\xi\, dx \quad \text{as } j \to \infty.$$

Consequently,

$$\langle \mathrm{F}(u_j), \xi \rangle \longrightarrow \langle \mathrm{F}(u), \xi \rangle \quad \text{for any } \xi \in \bigcup_{j \in \mathbb{N}} E_j.$$

Since

$$\langle \mathrm{F}(u_j), \xi \rangle = 0 \quad \text{for any } \xi \in \bigcup_{j \in \mathbb{N}} E_j,$$

it follows that

$$\langle \mathrm{F}(u), \xi \rangle = 0 \quad \text{for any } \xi \in \bigcup_{j \in \mathbb{N}} E_j.$$

By the density of $\bigcup_{j \in \mathbb{N}} E_j$ in $W_0^{1,p}(\Omega; \omega; \mathbb{R}^m)$, we conclude that

$$\langle \mathrm{F}(u), \xi \rangle = 0 \quad \text{for all } \xi \in W_0^{1,p}(\Omega; \omega; \mathbb{R}^m).$$

Thus, the main theorem is proved.

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