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Solvability of n-Product For Nonlinear Weakly Singular Integral Equations

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Abstract This paper establishes the conditions necessary for the existence of solutions to a functional n-product nonlinear weakly singular integral equations. It effectively applies the Petryshyn's fixed theorem by analyzing a relevant measure of non-compactness on $C([0, a])$. Additionally, the paper provides examples to clarify the results.

1 Introduction

The idea of measures of noncompactness (M.N.C) goes back to the work of Kuratowski [1] which are functions that measure the degree of noncompactness of sets in Banach space. In 1955, Darbo presented a fixed point theorem (F.P.T) using this notion. Furthermore, several interesting papers have been shown on the solvability of various integral equations in Banach spaces utilizing Darbo's F.P.T. It became more popular among researchers [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. In 1971, W. V. Petryshyn introduced several fixed point theorems for condensing mapping [14]. This conception is directly to M.N.C. In 2016, Kazemi *et al.* in [15] utilized it for the solvability of integral equations ($\mathbb{IE}$s). This theorem has a key benefit over Darbo's and Schauder's theorem. In using it, one does not need to confirm that the used operator maps a closed convex set onto itself. This issue causes us to impose fewer conditions on the problem [16, 17, 18, 19, 20, 21].

Olaru in [22], utilized the F.P.T approach to establish the existence of $C([a, b])$-solutions of the following $\mathbb{IE}$,

$$z(t) = \prod_{i=1}^{n} \left(h_i(t) + \int_a^t g_i\big(t, s, z(s)\big)\, \mathrm{d}s \right), \qquad t \in [a, b].$$

In [23], Metwali presented an extension of Darbo F.P.T in Banach algebra to solve the q-$\mathbb{IE}$ is formulated by,

$$z(t) = \prod_{i=1}^{n} \left(h_i(t) + \tfrac{f_i(t,z(t))}{\Gamma_q(\alpha_i)} \int_a^t (t - qs)^{\alpha_i - 1} g_i\big(s, z(s)\big) \right) \mathrm{d}_q s, \qquad t \in [0, 1].$$

A generalization of Darbo fixed point theorem was used to investigate the existence results for the $\mathbb{IE}$ is given by,

$$z(t) = \prod_{i=1}^{n} \left(h_i(t) + \lambda_i \int_a^b k_i(t, s) f_i(s, z(s))\, \mathrm{d}s \right), \qquad t \in [a, b],$$

in ideal spaces (not be Banach algebras) in [24]. Alsaadi *et al.*, employed the presented F.P.T to solve the product of n-nonlinear Volterra $\mathbb{IE}$s, which are a generalization of the classical and quadratic $\mathbb{IE}$s of the form,

$$z(t) = \prod_{i=1}^{n} f_i\Big(t, z\big(\alpha_i(t)\big), z\big(\beta_i(t)\big), \int_0^{\phi_i(t)} k_i\big(t, s, z(\gamma_i(t))\big)\,\mathrm{d}s\Big), \qquad t \in I_a := [0, a],$$

for $n \geq 2$, in the Banach algebra $C(I_a)$ [25].

Motivated and inspired by the results mentioned and related in [15, 26], we present and prove an existence theorem for the product of n-nonlinear weakly singular integral equations (n-NWS$\mathbb{IE}$) as follows:

$$x(t) = \prod_{i=1}^{n} \Sigma_i x(t) = \prod_{i=1}^{n} \Lambda_i\Big(t, x\big(\theta_i(t)\big), \int_0^t \tfrac{\lambda^{\beta_i}}{(t-\lambda)^{\zeta_i}} k_i\big(t, \lambda, x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\Big), \tag{1.1}$$

where $\beta_i > 0$, $0 < \zeta_i < 1$ and $t \in I_a$.

We arrange our article just like that: in Section 2, we collect some definitions, lemmas, and theorems, which are essential to prove our main results. In Section 3, we establish and prove a new existence theorem using Petryshyn's F.P.T for the functional n-NWS$\mathbb{IE}$ (1.1). In Section 4, we also give some examples to support our main theorem. Finally, Section 5, concludes the paper.

2 Preliminaries

In this section, we review some definitions and theorems. We do this by stating some extra facts. Let

- $\mathbb{B}$: Banach space,
- E_ρ : A ball of radius α,
- $\bar{E}_\rho$: Sphere in $\mathbb{E}$ with radius ρ,
- $C(I_a)$: All real functions continuous on I_a,
- $B(x, y)$: the beta function which defined by

$$B(x, y) = \int_0^t t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

  It can see to that $\int_0^t \frac{s^\gamma}{(t-s)^\alpha}\,\mathrm{d}s = B(1-\alpha, 1+\gamma)t^{1+\gamma-\alpha}$,
- Fix(T): The set of fixed points of T in $\mathbb{E}$.

Let $H_1, H_2 \subset \mathbb{B}$ bounded, $\lambda \in \mathbb{R}$, a nonnegative function ν is called Sadovskii functional if it satisfies the following requirements:

(i) $\nu(H_1 \bigcup H_2) = \max\Big\{\nu(H_1), \nu(H_2)\Big\}$;

(ii) $\nu(H_1 + H_2) \leq \nu(H_1) + \nu(H_2)$;

(iii) $\nu(\lambda H_1) = |\lambda|\nu(H_1)$, where $\lambda H_1 = \Big\{\lambda h_1 :\ h_1 \in H_1\Big\}$;

(iv) $\nu(H_1) \leq \nu(H_2)$, for $H_1 \subset H_2$;

(v) $\nu(\bar{\text{co}} H_1) = \nu(H_1)$;

(vi) $\nu(H_1) = 0$ iff H_1 is precompact regular (Sadovskii property).

In what follows, we provide some important examples of a regular Sadovskii functional called M.N.C. that are utilized repeatedly in this paper. The first is "Set M.N.C",

$$\alpha(H) = \inf\Big\{\epsilon > 0,\ H \text{ may be covered by finitly many sets of diameter} \leq \epsilon\Big\}.$$

The second one is the "Ball M.N.C",

$$\nu(H) = \inf\Big\{\epsilon > 0, \text{ there exists a finitly } \epsilon\text{-net for } H \text{ in } \mathbb{B}\Big\}.$$

Recall that the module of continuity of a function $u \in C(I_a)$ is defined by

$$\partial(u,\epsilon) = \sup\Big\{|u(s) - u(t)| \,:\, |s-t| < \epsilon\Big\}. \tag{2.1}$$

Since $u \in C(I_a)$ is uniformly continuous, then $\partial(u,\epsilon) \to 0$ as $\epsilon \to 0$. Also,

$$\partial(G,\epsilon) = \sup\big\{\partial(u,\epsilon) : u \in G\big\}.$$

Theorem 2.1 ([15]). *The Ball* M.N.C *is similar to* $\nu(H) = \lim_{\epsilon\to 0}\big\{\sup \partial(u,\epsilon) \,:\, u \in H\big\}$*, for all bounded subset* $H \subset I_a$.

Let $\xi : \mathbb{B} \to \mathbb{B}$ be a continuous mapping of a Banach space $\mathbb{B}$. Then ξ is called a k-set contraction if for all $H \subset \mathbb{B}$ with H bounded, $\xi(H)$ is bounded and

$$\alpha\big(\xi(H)\big) \leq k\alpha(H), \qquad 0 \leq k < 1.$$

If $\alpha\big(\xi(H)\big) < \alpha(H)$ for all $\alpha(H) > 0$, then ξ is called densifying (or condensing) [27].

Theorem 2.2 ([14]). *If* $\xi : E_\rho \to \mathbb{B}$ *is a condensing mapping (and, in particular, a* k*-set contraction with* $k < 1$*) which satisfies the boundary condition* $\xi(x) = \lambda x$ *for some* $x \in \bar{E}_\rho, \lambda \leq 1$*, then* Fix$(\xi)$ *is nonempty and compact.*

Let ν be a M.N.C in $C(I_a)$. We say that ν satisfies condition (m) [28] if, for $Q, \bar{Q} \subset C(I_a)$ we have

$$\nu(Q\bar{Q}) \leq \|\bar{Q}\| \cdot \nu(Q) + \|Q\|\nu(\bar{Q}).$$

Theorem 2.3 ([25]). *Let* ν *denote a* M.N.C *in* $C(I_a)$ *fulfilling condition (m). For* $\big\{Q_i\big\}_{i\in\mathbb{N}}$*, be a finite sequence of bounded sets in* $C(I_a)$*,* $n \geq 2$*, then*

$$\nu\left(\prod_{i=1}^{n} Q_i\right) \leq \sum_{i=1}^{n}\ \prod_{j=1, j\neq i}^{n} \|Q_j\| \cdot \nu(Q_i).$$

Theorem 2.4 ([25]). *Let* $\Lambda_i : \bar{E}_\rho \to \mathbb{B}$*,* $i = 1,\dots,n$ *are satisfies the following properties:*

P1) Λ_i *are continuous,*

P2) *There exist* k_i *such that* $\nu\big(\Lambda_i(Q)\big) \leq k_i\nu(Q)$*,*

P3) $\mathfrak{m} = \sum_{i=1}^{n} k_i \prod_{j=1,j\neq i}^{n} \big\|\Lambda_i \bar{E}_\rho\big\| < \frac{1}{2}$*,*

P4) $\Lambda q = kq$ *for some* $q \in \partial\bar{E}_\rho$ *then* $k \leq 1$*,*

then Fix(λ) *is nonempty.*

3 An existence theorem for n-NWS$\mathbb{IE}$

In this section, we will study the existence of the n-NWS$\mathbb{IE}$ (1.1) for under the following assumptions:

N1) $\Lambda_i \in C\big(I_a \times \mathbb{R}^2\big)$, $k_i \in C\big(I_a^2 \times \mathbb{R}\big)$ and $\theta_i, \hat{\theta}_i : I_a \to I_a$ are continuous function for $i = 1,\dots,n$,

N2) For nonnegative constants $m_{i,1}, m_{i,2}$, such that $m_{i,1} < 1$,

$$\Big|\Lambda_i(t,\lambda,\gamma) - \Lambda_i(t,\hat{\lambda},\hat{\gamma})\Big| \leq m_{i,1}\big|\lambda - \hat{\lambda}\big| + m_{i,2}\big|\gamma - \hat{\gamma}\big|,$$

N3) There exists $\bar{M}_i \geq 0$ and $\alpha \geq 0$ such that

$$\sup\left\{\left|\prod_{i=1}^{n}\Lambda_i(t,\lambda,\gamma)\right| : t\in I_a,\ \lambda\in[-\rho,\rho],\ |\gamma|\leq\prod_{i=1}^{n}\bar{M}_i a^{1+\beta_i-\zeta_i}B\big(1-\zeta_i,1+\beta_i\big)\right\}\leq\rho,$$

where $\bar{M}_i = \sup\big\{|k_i(t,\lambda,\gamma)| : t,\lambda\in I_a,\ \gamma\in[-\rho,\rho]\big\}$.

Theorem 3.1. *Under the tacit assumption* (N1)-(N3)*, the functional* n*-NWS*$\mathbb{IE}$ (1.1) *has at least one solution in* $C(I_a)$*.*

Proof. We define the operator as follows

$$\begin{cases}\Sigma_i : E_\rho \to \mathbb{B},\\ \Sigma x(t) = \prod_{i=1}^{n}\Sigma_i x(t) = \prod_{i=1}^{n}\Lambda_i\Big(t, x\big(\theta_i(t)\big), \int_0^t \frac{\lambda^{\beta_i}}{(t-\lambda)^{\zeta_i}} k_i\big(t,\lambda,x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\Big),\end{cases}$$

where $E_\alpha = \big\{x\in C(I_a) : \|x\|\leq\alpha\big\}$.
`Step 1.` We show that Σ, Σ_i, $i=1,\ldots,n$ are continuous on E_ρ. Assume that $x,\hat{x}\in E_\rho$ and $\epsilon>0$ such that $|x-\hat{x}|\leq\epsilon$. We have

$$\begin{aligned}
&\Big|(\Sigma_i x)(t)-(\Sigma_i\hat{x})(t)\Big|\\
&\quad=\Big|\Lambda_i\Big(t,x\big(\theta_i(t)\big),\int_0^t\frac{\lambda^{\beta_i}}{(t^{\beta_i}-\lambda^{\beta_i})^{\zeta_i}}k_i\big(t,\lambda,x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\Big)\\
&\qquad-\Lambda_i\Big(t,\hat{x}\big(\theta_i(t)\big),\int_0^t\frac{\lambda^{\beta_i}}{(t^{\beta_i}-\lambda^{\beta_i})^{\zeta_i}}k_i\big(t,\lambda,\hat{x}(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\Big)\Big|\\
&\quad\leq\Big|\Lambda_i\Big(t,x\big(\theta_i(t)\big),\int_0^t\frac{\lambda^{\beta_i}}{(t-\lambda)^{\zeta_i}}k_i\big(t,\lambda,x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\Big)\\
&\qquad-\Lambda_i\Big(t,\hat{x}\big(\theta_i(t)\big),\int_0^t\frac{\lambda^{\beta_i}}{(t^{\beta_i}-\lambda^{\beta_i})^{\zeta_i}}k_i\big(t,\lambda,x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\Big)\Big|\\
&\qquad+\Big|\Lambda_i\Big(t,\hat{x}\big(\theta_i(t)\big),\int_0^t\frac{\lambda^{\beta_i}}{(t^{\beta_i}-\lambda^{\beta_i})^{\zeta_i}}k_i\big(t,\lambda,x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\Big)\\
&\qquad-\Lambda_i\Big(t,\hat{x}\big(\theta_i(t)\big),\int_0^t\frac{\lambda^{\beta_i}}{(t-\lambda)^{\zeta_i}}k_i\big(t^{\beta_i},\lambda^{\beta_i},\hat{x}(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\Big)\Big|\\
&\quad\leq m_{i,1}\Big|x\big(\theta_i(t)\big)-\hat{x}\big(\theta_i(t)\big)\Big|\epsilon B\left(1-\zeta_i,\tfrac{1+\beta_i}{\beta_i}\right)\frac{t^{\beta_i(1-\zeta_i)-1}}{\beta_i}\\
&\quad\leq m_{i,1}\big\|x-\hat{x}\big\|+m_{i,2}\partial(k_i,\epsilon)B\left(1-\zeta_i,\tfrac{1+\beta_i}{\beta_i}\right)\frac{a^{\beta_i(1-\zeta_i)-1}}{\beta_i},
\end{aligned}$$

where

$$\partial(k_i,\epsilon)=\sup\Big\{\big|k_i(t,\lambda,x)-k_i(t,\lambda,\hat{x})\big| : t,\lambda\in I_a,\ x,\hat{x}\in[-\rho,\rho],\ \big|x-\hat{x}\big|\leq\epsilon\Big\}.$$

Since $k_i = k_i(t,\lambda,x)$ are uniform continuity on the subset $I_a^2\times\mathbb{R}$. Consequently, the operator $\Sigma x = \prod_{i=1}^{n}\Sigma_i x$ is continuous on E_ρ.
`Step 2.` We will show that Σ is a condensing map. Let $\Gamma\subset\mathbb{B}$ to be a bounded set. For $\epsilon>0$

and $x \in \Gamma$ and $t_1, t_2 \in I_a$ with $t_2 - t_1 \leq \epsilon$. We get:

$$
\begin{aligned}
&\Big|(\Sigma_i x)(t_2) - (\Sigma_i x)(t_1)\Big| \\
&\quad = \bigg|\Lambda_i\bigg(t_2, x\big(\theta_i(t_2)\big), \int_0^{t_2} \tfrac{\lambda^{\beta_i}}{(t_2-\lambda)^{\zeta_i}} k_i\big(t_2, \lambda, x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\bigg) \\
&\quad\quad - \Lambda\bigg(t_1, x\big(\theta_i(t_1)\big), \int_0^{t_1} \tfrac{\lambda^{\beta_i}}{(t_1-\lambda)^{\zeta_i}} k\big(t_1, \lambda, x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\bigg)\bigg| \\
&\quad \leq \bigg|\Lambda\bigg(t_2, x\big(\theta_i(t_2)\big), \int_0^{t_2} \tfrac{\lambda^{\beta_i}}{(t_2-\lambda)^{\zeta_i}} k_i\big(t_2, \lambda, x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\bigg) \\
&\quad\quad - \Lambda\bigg(t_2, x\big(\theta_i(t_2)\big), \int_0^{t_1} \tfrac{\lambda^{\beta_i}}{(t_1-\lambda)^{\zeta_i}} k_i\big(t_1, \lambda, x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\bigg)\bigg| \\
&\quad\quad + \bigg|\Lambda\bigg(t_2, x\big(\theta_i(t_2)\big), \int_0^{t_1} \tfrac{\lambda^{\beta_i}}{(t_1-\lambda)^{\zeta_i}} k_i\big(t_1, \lambda, x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\bigg) \\
&\quad\quad - \Lambda\bigg(t_2, x\big(\theta_i(t_1)\big), \int_0^{t_1} \tfrac{\lambda^{\beta_i}}{(t_1-\lambda)^{\zeta_i}} k_i\big(t_1, \lambda, x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\bigg)\bigg| \\
&\quad\quad + \bigg|\Lambda\bigg(t_2, x\big(\theta_i(t_1)\big), \int_0^{t_1} \tfrac{\lambda^{\beta_i}}{(t_1-\lambda)^{\zeta_i}} k_i\big(t_1, \lambda, x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\bigg) \\
&\quad\quad - \Lambda\bigg(t_1, x\big(\theta_i(t_1)\big), \int_0^{t_1} \tfrac{\lambda^{\beta_i}}{(t_1-\lambda)^{\zeta_i}} k_i\big(t_1, \lambda, x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\bigg)\bigg| \\
&\quad \leq m_{i,2}\bigg|\int_0^{t_2} \tfrac{\lambda^{\beta_i}}{(t_2-\lambda)^{\zeta_i}} k_i\big(t_2, \lambda, x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda \\
&\quad\quad - \int_0^{t_1} \tfrac{\lambda^{\beta_i}}{(t_1-\lambda)^{\zeta_i}} k_i\big(t_1, \lambda, x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\bigg| \\
&\quad\quad + m_{i,1}\big|x\big(\theta_i(t_2)\big) - x\big(\theta_i(t_1)\big)\big| + \partial(\Lambda_i, \nu) \\
&\quad \leq m_{i,2}\bigg[\int_0^{t_1} \tfrac{\lambda^{\beta_i}}{(t_2-\lambda)^{\zeta_i}} \Big|k_i\big(t_2, \lambda, x(\hat{\theta}_i(\lambda))\big) - k_i\big(t_1, \lambda, x(\hat{\theta}_i(\lambda))\big)\Big|\,\mathrm{d}\lambda \\
&\quad\quad + \int_0^{t_1} \Big|\tfrac{\lambda^{\beta_i}}{(t_1-\lambda)^{\zeta_i}} k_i\big(t_1, \lambda, x(\hat{\theta}_i(\lambda))\big) - \tfrac{\lambda^{\beta_i}}{(t_1-\lambda)^{\zeta}} k\big(t_1, \lambda, x(\hat{\theta}_i(\lambda))\big)\Big|\,\mathrm{d}\lambda \\
&\quad\quad + \int_{t_1}^{t_2} \Big|\tfrac{\lambda^{\beta_i}}{(t_2-\lambda)^{\zeta_i}} k_i\big(t_2, \lambda, x(\hat{\theta}_i(\lambda))\big)\Big|\,\mathrm{d}\lambda\bigg] \\
&\quad\quad + m_{i,1}\Big|x\big(\theta_i(t_2)\big) - x\big(\theta_i(t_1)\big)\Big| + \partial(\Lambda_i, \nu) \\
&\quad \leq m_{i,2}\hat{\partial}(k_i, \nu) B\big(1-\zeta_i, 1+\beta_i\big) t_1^{1+\beta_i-\zeta_i} \\
&\quad\quad + m_{i,2}\bar{M}_i B(1-\zeta_i, 1+\beta_i)\Big[t_2^{1+\beta_i-\zeta_i} - t_1^{1+\beta_i-\zeta_i}\Big] \\
&\quad\quad + m_{i,1}\Big|x\big(\theta_i(t_2)\big) - x\big(\theta_i(t_1)\big)\Big| + \partial(\Lambda_i, \nu) \\
&\quad \leq m_{i,2}\hat{\partial}\big(k_i, \varrho\big) B\big(1-\zeta_i, 1+\beta_i\big) a^{1+\beta_i-\zeta_i} \\
&\quad\quad + 2m_{i,2}\bar{M}_i B\big(1-\zeta_i, 1+\beta_i\big) \nu^{1+\beta_i-\zeta_i} \\
&\quad\quad + m_{i,1}\Big|x\big(\theta_i(t_2)\big) - x\big(\theta_i(t_1)\big)\Big| + \partial(\Lambda_i, \nu) \\
&\quad \leq m_{i,2}\hat{\partial}(k_i, \epsilon) B\big(1-\zeta_i, 1+\beta_i\big) a^{1+\beta_i-\zeta_i} \\
&\quad\quad + 2m_{i,2}\bar{M}_i B\big(1-\zeta_i, 1+\beta_i\big) \nu^{1+\beta_i-\zeta_i} \\
&\quad\quad + m_{i,1}\partial\big(x, \partial(\theta_i, \nu)\big) + \partial(\Lambda_i, \nu),
\end{aligned}
$$

where

$$\hat{\partial}(k_i,\nu) = \sup\Big\{\big|k_i(t,\lambda,\gamma) - k_i(\hat{t},\lambda,\gamma)\big| \,:\, t,\hat{t},\lambda \in I_a,\, \gamma\in[-\rho,\rho],\, |t-\hat{t}|\le \nu\Big\},$$

$$\partial(\Lambda_i,\nu) = \sup\Big\{\big|\Lambda_i(t,\lambda,\gamma) - \Lambda_i(\hat{t},\lambda,\gamma)\big| \,:\, |\hat{t}-t|\le\nu,\, \lambda\in[-\rho,\rho],$$
$$|\gamma| \le Mt^{1+\beta_i-\zeta_i}B(1-\zeta_i,1+\beta_i)\Big\}.$$

Hence, we obtain

$$\begin{aligned}\Big|(\Sigma_i x)(t_2) - (\Sigma_i x)(t_1)\Big| &\le m_{i,2}\hat{\partial}(k_i,\epsilon)B\big(1-\zeta_i,1+\beta_i\big)a^{1+\beta_i-\zeta_i}\\ &\quad + 2m_{i,2}\bar{M}_i B\big(1-\zeta_i,1+\beta_i\big)\nu^{1+\beta_i-\zeta_i}\\ &\quad + m_{i,1}\partial\big(x,\partial(\theta_i,\nu)\big) + \partial(\Lambda_i,\nu).\end{aligned}$$

If $\nu\to 0$, we have $\partial(\Sigma_i x,\nu)\le m_{i,1}\partial(x,\nu)$. This implies that $\nu(\Sigma_i\Gamma)\le m_{i,1}\nu(\Gamma)$. Therefore

$$\nu(\Sigma\Gamma) = \nu\bigg(\prod_{i=1}^{n}\Sigma_i\Gamma\bigg) \le \bigg(\sum_{i=1}^{n} m_{i,1}\prod_{j=1,j\ne i}^{n}\|\Lambda_i\|\bigg)\nu(\Gamma).$$

This shows that Σ is a condensing map.
`Step 3.` Let $x\in\partial E_\rho$. If $\Sigma x = \hat{k}x$ then $\big\|\Sigma\pi\big\| = \hat{k}\,\big\|x\big\| = \hat{k}\alpha$. The condition (N1) implies that

$$\big|\Lambda x(t)\big| = \bigg|\prod_{i=1}^{n}\Lambda_i x(t)\bigg| = \bigg|\prod_{i=1}^{n}\Lambda_i\bigg(t, x\big(\theta_i(t)\big), \int_0^t \tfrac{\lambda^{\beta_i}}{(t-\lambda)^{\zeta_i}} k_i\big(t,\lambda,x(\hat{\theta}_i(\lambda))\big)\,\mathrm{d}\lambda\bigg)\bigg| \le \rho,$$

for any $t\in I_a$. Thus, $\big\|\Sigma x\big\|\le\rho$, so this show $\hat{k}\le 1$. This completes the proof. □

4 Examples

In this section, we will present examples using Maple software to support Theorem 3.1, based on the approach explained earlier.

Example 4.1. Consider the following 2-NWS𝕀𝔼 in $C(I_a)$, $a=1$, as follows,

$$\begin{aligned}x(t) &= \bigg(\tfrac{t+2}{9} + \tfrac{t^2+x(\sin t)}{t^2+18} + \int_0^t \tfrac{\lambda^5}{(t-\lambda)^{1/10}}\bigg(\tfrac{\sqrt[3]{x(t)}+\cos(x(t))}{5+\lambda^2}\bigg)\,\mathrm{d}\lambda\bigg)\\ &\quad\times\bigg[\tfrac{tx(\sqrt{t})}{\sin t+6} + \tfrac{te^t}{15} + \int_0^t \tfrac{\lambda^{1/2}}{(t-\lambda)^{1/16}}\bigg(\tfrac{\cos(\sqrt{t})+\ln(1+|x(t)|)}{3+\lambda t^2}\bigg)\,\mathrm{d}\lambda\bigg].\end{aligned}$$

Clearly, $\beta_1 = 5$, $\theta_1(t)=\sin t$, $\zeta_1(t)=\frac{1}{10}$, $\beta_2=\frac{1}{2}$, $\theta_2(t)=\sqrt{t}$, $\zeta_2(t)=\frac{1}{16}$,

$$\Lambda_1\big(t,x(\theta_1(t)),\nabla_1\big) = \tfrac{t+2}{7} + \tfrac{t^2+x(\sin t)}{t^2+18} + \nabla_1,$$
$$\Lambda_2\big(t,x(\theta_2(t)),\nabla_2\big) = \tfrac{tx(\sqrt{t})}{\sin t+6} + \tfrac{te^t}{12} + \nabla_2,$$

with

$$\nabla_1 = \int_0^t \tfrac{\lambda^5}{(t-\lambda)^{1/10}}\bigg(\tfrac{\sqrt[3]{x(t)}+\cos(x(t))}{1+\lambda^2}\bigg)\,\mathrm{d}\lambda,$$
$$\nabla_2 = \int_0^t \tfrac{\lambda^{1/2}}{(t-\lambda)^{1/16}}\bigg(\tfrac{\cos(\sqrt{t})+\ln(1+|x(t)|)}{9+\lambda t^2}\bigg)\,\mathrm{d}\lambda.$$

One can easily see that,

$$\big|\Lambda_1(t,Z_1,Z_2) - \Lambda_1(t,\bar{Z}_1,\bar{Z}_2)\big| \le m_{1,1}\big|Z_1-\bar{Z}_1\big| + m_{1,2}\big|Z_2-\bar{Z}_2\big|,$$
$$\big|\Lambda_2(t,Z_1,Z_2) - \Lambda_2(t,\bar{Z}_1,\bar{Z}_2)\big| \le m_{2,1}\big|Z_1-\bar{Z}_1\big| + m_{2,2}\big|Z_2-\bar{Z}_2\big|,$$

in which, $m_{1,1} = \frac{1}{18}$, $m_{1,2} = \frac{1}{4}$, $m_{2,1} = \frac{1}{6}$ and $m_{2,2} = \frac{4}{5}$. To verify assumption (N3) observe that the inequality appearing in this assumption has the form

$$\left(\tfrac{1}{3} + \tfrac{1+\rho}{18} + \tfrac{1+\sqrt[3]{\rho}}{20}\right)\left[\tfrac{1}{15} + \tfrac{\rho}{6} + \tfrac{4}{15}\left[1 + \ln(1+|\rho|)\right]\right] \leq \rho.$$

It is easy to verify that the number $\rho \leq 87.7464$ satisfies the above inequality. For $\rho \in [0, 18.5241] \subset [0, 87.7464]$, we have,

$$\mathfrak{m} = m_{1,1}\|\Lambda_2\| + m_{2,1}\|\Lambda_1\| < \tfrac{1}{2}.$$

Example 4.2. Consider the following 3-NWS$\mathbb{IE}$ in $C(I_a)$, $a = 1$, as follows

$$\begin{aligned} x(t) = &\left(\frac{\sin t}{8(e^{t^2}+4\cos\sqrt{t})} + \frac{t^2 x(\sqrt{\sin t})}{9t^4+9} + \int_0^t \frac{\lambda^{9/4}}{(t-\lambda)^{1/7}}\left(\frac{1+\cos\sqrt{\lambda}+x(\sqrt{\lambda})}{3+\lambda t^2+\ln t}\right)\mathrm{d}\lambda\right)\\ &\times\left[\frac{t}{6} + \frac{\sin\left(\frac{t}{2}\right)x(\sqrt{\lambda})}{4} + \int_0^t \frac{\lambda^{5/2}}{(t-\lambda)^{1/8}}\left(\frac{\lambda e^{-2t}\cos x(\sqrt{\lambda})}{4+\left|\sin x(\sqrt{\lambda})\right|}\right)\mathrm{d}\lambda\right]\\ &\times\left[\frac{\sin t e^{-\sin t}}{8} + \frac{t x(\sqrt{\lambda})}{10\left(1+x(\sqrt{\lambda})\right)} + \int_0^t \frac{\lambda^{10/3}}{(t-\lambda)^{2/9}}\left(\frac{\sqrt[3]{1+2x(\lambda)}}{1+\lambda t+\ln(1+|\sin(\lambda^2)|)}\right)\mathrm{d}\lambda\right]. \end{aligned}$$

In view of n-NWS$\mathbb{IE}$ (1.1), we consider, $\beta_1 = \frac{1}{4}$, $\beta_2 = \frac{5}{2}$, $\beta_3 = \frac{4}{3}$, $\zeta_1(t) = \frac{3}{4}$, $\zeta_2(t) = \frac{1}{8}$, $\zeta_3(t) = \frac{2}{9}$, $\theta_1(t) = \sqrt{t}$, $\theta_2(t) = t$, $\theta_3(t) = \sqrt{t}$, and

$$\begin{aligned} \Lambda_1\left(t, x(\theta_1(t)), \nabla_1\right) &= \frac{\sin t}{8\left(e^{t^2}+4\cos\sqrt{t}\right)} + \frac{t^2 x\left(\sqrt{\sin t}\right)}{9t^4+9} + \nabla_1,\\ \Lambda_2\left(t, x(\theta_2(t)), \nabla_2\right) &= \frac{t}{6} + \frac{\sin\left(\frac{t}{2}\right)x(\sqrt{\lambda})}{4} + \nabla_2,\\ \Lambda_3\left(t, x(\theta_2(t)), \nabla_3\right) &= \frac{\sin t e^{-\sin t}}{8} + \frac{t x(\sqrt{\lambda})}{10\left(1+x(\sqrt{\lambda})\right)} + \nabla_3, \end{aligned}$$

where

$$\begin{aligned} \nabla_1 &= \int_0^t \frac{\lambda^{9/4}}{(t-\lambda)^{1/7}}\left(\frac{1+\cos\sqrt{\lambda}+x(\sqrt{\lambda})}{3+\lambda t^2+lnt}\right)\mathrm{d}\lambda,\\ \nabla_2 &= \int_0^t \frac{\lambda^{10/3}}{(t-\lambda)^{2/9}}\left(\frac{\lambda e^{-2t}\cos x(\sqrt{\lambda})}{4+\left|\sin x(\sqrt{\lambda})\right|}\right)\mathrm{d}\lambda,\\ \nabla_3 &= \int_0^t \frac{\lambda^{4/3}}{(t-\lambda)^{2/9}}\left(\frac{\sqrt[3]{1+2x(\lambda)}}{1+\lambda t+\ln(1+|\sin(\lambda^2)|)}\right)\mathrm{d}\lambda. \end{aligned}$$

By these data, we obtain,

$$\begin{aligned} \left|\Lambda_1(t, Z_1, Z_2) - \Lambda_1(t, \bar{Z}_1, \bar{Z}_2)\right| &\leq m_{1,1}\left|Z_1 - \bar{Z}_1\right| + m_{1,2}\left|Z_2 - \bar{Z}_2\right|,\\ \left|\Lambda_2(t, Z_1, Z_2) - \Lambda_2(t, \bar{Z}_1, \bar{Z}_2)\right| &\leq m_{2,1}\left|Z_1 - \bar{Z}_1\right| + m_{2,2}\left|Z_2 - \bar{Z}_2\right|,\\ \left|\Lambda_3(t, Z_1, Z_2) - \Lambda_3(t, \bar{Z}_1, \bar{Z}_2)\right| &\leq m_{3,1}\left|Z_1 - \bar{Z}_1\right| + m_{3,2}\left|Z_2 - \bar{Z}_2\right|, \end{aligned}$$

where $m_{1,1} = \frac{1}{9}$, $m_{1,2} = \frac{1}{5}$, $m_{2,1} = \frac{1}{4}$, $m_{2,2} = \frac{9}{100}$, $m_{3,1} = \frac{1}{10}$ and $m_{3,2} = \frac{4}{10}$. To verify assumption (N3) observe that the inequality appearing in this assumption has the form,

$$\left(\tfrac{1}{8} + \tfrac{\rho}{9} + \tfrac{4(2+\rho)}{10}\right)\left[\tfrac{1}{4} + \tfrac{\rho}{4} + \tfrac{922\rho}{1000}\right]\left[\tfrac{1}{8} + \tfrac{\rho}{4} + \tfrac{387}{1000}\sqrt[3]{1+2\rho}\right] \leq \rho.$$

It is easy to verify that the number $\rho \in [0.1819955, 2.34109]$ satisfies the above inequality. For $\rho \in [0.1819955, 1.26056] \subset [0.1819955, 2.34109]$, we have

$$\mathfrak{m} = m_{1,1}\|\Lambda_1\| + m_{2,1}\|\Lambda_2\| + m_{3,1}\|\Lambda_3\| < \tfrac{1}{2}.$$

5 Conclusion

Due to the unsolvability of many problems associated with n-NWS$\mathbb{IE}$, establishing the existence of solutions for these equations is crucial. Consequently, numerous researchers have published papers in this area, detailing their techniques and findings. In this paper, the authors introduce a novel approach that leverages M.N.C and Petryshyn's F.P.T. This method presents several advantages over similar approaches, including fewer required conditions and the elimination of the necessity to verify that the operator maps a closed convex subset onto itself.

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