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Investigations on the fractional SEIAR epidemic model utilizing the Caputo derivative

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Abstract. This paper examines a SEIAR-type epidemic model, utilizing the Caputo derivative to characterize the dynamics of the disease. We propose a reproduction number and present the equilibrium points pertinent to the model. Moreover, both local and global stability analyses of the SEIAR epidemic model are conducted. A numerical scheme is developed to enable the simulation of the model, accompanied by a series of graphics that illustrate the results of these simulations. Furthermore, we propose various applications of the model and elucidate the influence of the order of the fractional derivative on the dynamics of the system.

1 Introduction

The fractional operator is a crucial component in modeling epidemic models. It has been noted in the literature that the order of the fractional operator can significantly influence the spread of the disease under consideration. Therefore, the study being discussed in the paper examines an epidemic model of the SEIAR type, which has broad applications in epidemic modeling

$$\begin{aligned} D^\alpha S &= \Lambda - \mu S - nS\left(I + \psi A\right), \\ D^\alpha E &= nS\left(I + \psi A\right) - \left[\left(1-\theta\right)\omega + \theta\rho + \mu\right]E, \\ D^\alpha I &= \left(1-\theta\right)\omega E - \left(\tau + \mu\right)I, \\ D^\alpha A &= \theta\rho E - \left(\gamma + \mu\right)A, \\ D^\alpha R &= \tau I + \gamma A - \mu R. \end{aligned} \tag{1.1}$$

In epidemic models, many factors can explain the spread of a disease. In some cases, it is difficult to determine which parameter in the model can explain the extinction of a disease. For example, various models can be used for the novel coronavirus, making it difficult to determine the most appropriate one. Therefore, the order of the fractional derivative can be effectively used to attempt to stop the spread of disease by admitting properties to accelerate or decelerate the dynamics of the solution. In this paper, we aim to investigate the influence of the order of the Caputo derivative in the SEIAR epidemic model and provide potential applications of this modeling. In fractional calculus, there are many fractional operators, such as the Riemann-Liouville derivative, the Caputo derivative, the Caputo-Fabrizio derivative, the Atangana-Baleanu derivative, and others. The significance of this field lies in the existence of numerous fractional operators, each with its advantages and disadvantages. For example, the Riemann-Liouville derivative has a problem with the derivative of a constant function that is not zero, while the

Caputo derivative has a singularity. Despite these inconveniences, these derivatives have many advantages in explaining the memory dynamics in partial differential equations. Nonsingular derivatives like the Atangana-Baleanu and Caputo-Fabrizio derivatives have the advantage of removing the inconvenience of the Riemann and Caputo derivatives. However, these derivatives are often complex and difficult to manipulate in certain types of problems. In this paper, we prefer the Caputo derivative because the derivative of a constant function is zero, and it is suitable for use with initial conditions. As will be evident, the influence of the order of the fractional derivative will help us in the extinction of disease by decelerating the fast spread of the disease.

Before further investigations, we recall some recent and previous epidemiological model studies using fractional operators. In [2], the authors proposed a domain decomposition method to find the approximate solution for a fractional Seir epidemic model under consideration. In [6], the authors investigated the fractional order spatiotemporal SEIR model, proposed an investigation related to the stability of the equilibrium point, and gave some predictions. In [7], the author proposed a numerical scheme to find an approximate solution to a fractional epidemic model. In [13], the authors used a derivative with a nonsingular derivative to model the SEIR and Blood Coagulation systems. In [14], the authors proposed the stability analysis and an optimal control strategy of a fractional-order generalized SEIR model and applied it to the COVID-19 pandemic. In [1], the authors proposed an approximation solution of the SEIR epidemic model described by a fractional order derivative. The authors particularly used the measles and Fibonacci wavelets operational matrix approach to get the solutions and propose the illustrative graphics for the dynamics. For the SEIR epidemic model, where the solution has been proposed via a domain decomposition method, see the following paper [2]. In [3], the authors model the SEIR epidemic using time delays. In [4], the authors proposed an investigation related to the SEIR epidemic model with infectivity in the latent period and a general nonlinear incidence rate. There exist in the literature many papers related in general to epidemic models with fractional or integer derivatives [21, 22, 23], see these papers for information [6, 8, 15, 16, 17, 20]. For the applications of the numerical procedure of solving differential equations, see [24], and the use of the Mittag-Leffler function is seen in [25].

The paper will be of interest to the literature due to the novel idea discussed in the present paper. The first novelty is modeling the epidemic model using the fractional operator. The second novelty is the role played by the order of the fractional derivative because it serves to control the spread of the considered disease. In general, it will serve to stop the spread of the disease. Another point is the numerical scheme used to get the approximate solutions of the considered model and to draw the graphics.

The paper will be structured as follows. In the first section, we will introduce the fractional operator needed for our research and review some relevant properties. Then, in the second section, we will calculate the reproduction number and assess the stability of the equilibrium points. Moving on to the third section, we will outline the numerical scheme and present graphics depicting the dynamics of the solutions in the epidemic model. Finally, we will conclude with our remarks in the last section.

2 Some operators in fractional calculus

This section serves to review the operators, lemmas, and theorems required for our research related to the epidemic model under consideration. We will review the Caputo derivative, as it will be the fractional operator used in the investigation. Additionally, we will review the fractional integral, as it will be used in the discretization process and with other fractional operators.

Definition 1. *[11, 12] Let the function defined as the form* $u : [0, +\infty[\longrightarrow \mathbb{R}$*, then we represents the called Riemann-Liouville integral utilized in fractional calculus as the form that*

$$\left(I^{\alpha} u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \tag{2.1}$$

where $\Gamma(...)$ is the Gamma Euler function and the order is represented by α and respects the $\alpha > 0$ condition.

Definition 2. *[11, 12] Let the function defined as the form $u : [0, +\infty[\longrightarrow \mathbb{R}$, then we represents the called Riemann-Liouville derivative utilized in fractional calculus as the form that*

$$D^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t u(s)(t-s)^{-\alpha} ds, \tag{2.2}$$

where $t > 0$, and the order of the fractional operator describes the condition that $\alpha \in (0, 1)$.

Definition 3. *[11, 12] Let the function defined as the form $u : [0, +\infty[\longrightarrow \mathbb{R}$, then we represents the called Caputo derivative utilized in fractional calculus as the form that*

$$D^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds, \tag{2.3}$$

where $t > 0$, and the order of the fractional operator describes the condition that $\alpha \in (0, 1)$.

This derivative is more complete than the Riemann-Liouville derivative. Its advantage is that the derivative of a constant function will be zero, which can be seen directly in the form of Eq. (2.3).

It is important to review the Laplace transform as it will be essential for representing the analytical solution of the particular fractional differential equation. The Laplace transform of the Caputo derivative, when the order is within the range of $\alpha \in (0, 1)$, can be represented in the following form:

$$\mathcal{L}\{(D_c^{\alpha}u)(t)\} = s^{\alpha}\mathcal{L}\{u(t)\} - s^{\alpha-1}u(0). \tag{2.4}$$

3 Main results

We propose the existence of the considered model using the resolvent operator, and we combine it with the Krasnoselkii fixed-point theorem to help us establish the existence and uniqueness of the proposed solutions. Let the model under consideration be described by the following equations

$$D^{\alpha}y = G[t, x], \tag{3.1}$$

Here, the function G is described in the following forms

$$G[t,x] = \begin{cases} G_1[t,x] = \Lambda - \mu S - nS(I + \psi A) \\ G_2[t,x] = nS(I + \psi A) - [(1-\theta)\omega + \theta\rho + \mu]E, \\ G_3[t,x] = (1-\theta)\omega E - (\tau + \mu) I, \\ G_4[t,x] = \theta\rho E - (\gamma + \mu) A, \\ G_5[t,x] = \tau I + \gamma A - \mu R, \end{cases} \tag{3.2}$$

where $x = [S, E, A, I, R]$, the beginning of the existence section is to provide that all the previous functions are continuous and Lipschitz. Let the function G_5 according to the variable R, we have the following calculations

$$\begin{aligned} \|G_5(t, R_1) - G_5(t, R_2)\| &= \|\tau I + \gamma A - \mu R_1 - \tau I - \gamma A + \mu R_2\| \\ &\leq \mu \|R_1 - R_2\|. \end{aligned} \tag{3.3}$$

Thus, by Eq. (3.3), we observe that the function G_5 is Lipschitz continuous. We continue the same sketch of proof with the function defined by G_4 and we consider the variable A, we have the following calculations

$$\begin{aligned} \|G_4(t, A_1) - G_4(t, A_2)\| &= \|\theta\rho E - (\gamma + \mu) A_1 - \theta\rho E + (\gamma + \mu) A_2\| \\ &\leq (\gamma + \mu) \|A_1 - A_2\|. \end{aligned}$$

We continue the investigations with the function G_3, considering the variable denoted by I, the sketch of the proof does not change, and we have the calculations

$$\begin{aligned}\left\|G_3\left(t,I_1\right)-G_3\left(t,I_2\right)\right\| &= \left\|(1-\theta)\,\omega E-(\tau+\mu)\,I_1-(1-\theta)\,\omega E+(\tau+\mu)\,I_2\right\| \\ &\leq (\tau+\mu)\left\|I_1-I_2\right\|.\end{aligned}$$

Now we proceed with the second function denoted by the function G_2 and we consider the variable E, the same procedure is applied, and we get the following results after the calculations

$$\begin{aligned}\left\|G_2\left(t,E_1\right)-G_2\left(t,E_2\right)\right\| &= \left\|nS\left(I+\psi A\right)-\left[(1-\theta)\,\omega+\theta\rho+\mu\right]E_1-nS\left(I+\psi A\right)+\left[(1-\theta)\,\omega+\theta\right.\right. \\ &\leq \left[(1-\theta)\,\omega+\theta\rho+\mu\right]\left\|E_1-E_2\right\|.\end{aligned}$$

We finish the proof of the Lipschitz continuous function, by the function G_1 and we consider the variable S, we have the following calculations

$$\begin{aligned}\left\|G_1\left(t,S_1\right)-G_1\left(t,S_2\right)\right\| &= \left\|\Lambda-\mu S_1-nS_1\left(I+\psi A\right)-\Lambda+\mu S_2+nS_2\left(I+\psi A\right)\right\| \\ &\leq \mu\left\|S_1-S_2\right\|+n\left\|(I+\psi A)\right\|\left\|S_1-S_2\right\| \\ &\leq \mu\left\|S_1-S_2\right\|+n\kappa\left\|S_1-S_2\right\|.\end{aligned}$$

where the $\|(I+\psi A)\|\leq\kappa$ because the variables I and A are bounded as in the assumption of the model under consideration. We conclude that the function G is Lipschitz continuous. Let the form of the solution. To get the solution, we apply the Laplace transform on the fractional differential equation defined by Eq. (3.1), we have the form that

$$\begin{aligned}s^{\alpha}\bar{y}-s^{\alpha-1}y(0) &= \bar{G}, \\ \bar{y} &= \frac{y\left(0\right)}{s}+s^{-\alpha}\bar{G}.\end{aligned} \tag{3.4}$$

To get the analytical form of the solution, it is required to apply the inverse of the Laplace transform on Eq. (3.4), we get that

$$y\left(t\right)=R_{\alpha}\left(t\right)y\left(0\right)+\int_0^t T_{\alpha}\left(t-s\right)G\left(s\right)ds. \tag{3.5}$$

We define the resolvent operators using the analytical solution from Eq. (3.5), and we get the form defined by

$$R_{\alpha}\left(t\right) = \frac{1}{2\pi i}\int_{\Omega}e^{st}s^{-1}ds \tag{3.6}$$

$$T_{\alpha}\left(t\right) = \frac{1}{2\pi i}\int_{\Omega}e^{st}s^{-\alpha}ds. \tag{3.7}$$

The operators R and T are referred to in the literature as resolvent operators. We now continue the proof of existence using the Krasnoselskii fixed-point theorem. Let the operators be defined by the forms

$$\psi_1 y\left(t\right) = R_{\alpha}\left(t\right)\left[y_0\right] \tag{3.8}$$

$$\psi_2 y = \int_0^t T_{\alpha}\left(t-s\right)G\left(s\right)ds. \tag{3.9}$$

We consider the Banach space defined in this part by $\mathcal{C}$ as a Banach space $C(U,Y)$, which will be utilized for the rest of the investigation is given by the form that

$$\left\|y\right\|_{\mathcal{C}}=\sup_{\sigma\in U}\left\|y\left(\sigma\right)\right\|,y\in\mathcal{C}. \tag{3.10}$$

We take the operator ψ_1 and we let the ball defined by the following relationship

$$B_r = \{y \in Y : \|y\|_Y \leq r\}. \tag{3.11}$$

The ball defined in Eq. (3.11) is a closed, bounded, and convex set in Y. For the application of the fixed point theorem recalled previously, we prove that the operator ψ_1 is bound, let for any $u \in B_r$, we have the following form that $\|Au\| \leq r$. We have the following relationships

$$|\psi_1 u(t)| = |R_\alpha [u_0]| \leq M_\alpha |u|. \tag{3.12}$$

The same procedure is done with the second resolvent operator, in other words, we try to prove the operator ψ_2 is also bounded, as well, we have the following

$$\begin{aligned} |\psi_2 v| &\leq \int_0^t |T_\alpha (t-s) G(s, v(s))| \, ds \\ &\leq M_\alpha \int_0^t |G(s, v(s))| \, ds \\ &\leq M_\alpha \int_0^t |G(s, v(s)) - G(s, 0)| \, ds + M_\alpha \int_0^t |G(s, 0)| \, ds \\ &\leq M_\alpha k |v| + M_\alpha |G(t, 0)| \, a. \end{aligned} \tag{3.13}$$

Let the form that $\mu = \sup_{t \in I} |G(t, 0)|$. Thus, Eq. (3.13) becomes represented by the form that, we have following form

$$\|\psi_2 v\| \leq M_\alpha k \|v\| + M_\alpha \mu a. \tag{3.14}$$

Let any $u, v \in B_r$, we should prove that $\|\psi_1 u + \phi_2 v\| \leq r$. We get the following calculations

$$\begin{aligned} \|\psi_1 u + \phi_2 v\| &\leq \|\psi_1 u\| + \|\psi_2 v\| \\ &\leq M_\alpha |u| + M_\alpha k \|v\| + M_\alpha \mu a \\ &\leq r (M_\alpha + M_\alpha k) + \omega_\alpha + M_\alpha \mu a. \end{aligned} \tag{3.15}$$

Thus, by Eq. (3.15), we can notice that the following assumption $\|\phi_1 u + \phi_2 v\| \leq r$ is satisfied when we have the following relationship

$$r \geq \frac{\omega_\alpha + M_\alpha \mu a}{1 - (M_\alpha + M_\alpha k)}. \tag{3.16}$$

The second step of the proof is to prove that the operator defined by ψ_2 is a contraction. Let that for $u, v \in B_r$, we have the following forms

$$\begin{aligned} |\psi_2 u - \psi_2 v| &\leq \int_0^t |T_\alpha (t-s) [G(s, u(s)) - G(s, v(s))]| \, ds \\ &\leq M_\alpha \int_0^t |G(s, u(s)) - G(s, v(s))| \, ds \\ &\leq M_\alpha |u - v| \int_0^t k ds = M_\alpha k a |u - v|. \end{aligned}$$

We have now verified that the operator ψ_2 is a contraction.

Let's prove the resolvent operator defined by ψ_1 is completely continuous. We consider that $t_1, t_2 \in I$, we get the form expressed as

$$|\psi_1 u(t_1) - \psi_1 u(t_2)| \leq |u_0| |R_\alpha (t_1) - R_\alpha (t_2)|. \tag{3.17}$$

We assume that the operator defined at the beginning of the proof known as the resolvent operator is considered to be uniformly continuous on the considered set B_r, we observe that when $t_2 \to t_1$ then $|\phi_1 u(t_1) - \phi_1 u(t_2)|$ goes to zero. Then we conclude that the operator ψ_1 is equicontinuos, thus ψ_1 is compact on the set B_r. Then operator ψ_1 is completely continuous. By the theorem known as the Krasnoselskii fixed point theorem, we conclude that our problem has at least one mild solution.

4 Stability analysis

This section will be dedicated to determining the different equilibrium points, trivial and endemic, and their local stability. It will also propose the reproduction number of the considered epidemic model, which is very important to analyze and qualify the spread of the considered epidemic model. The reproduction number is also necessary to see if the disease is epidemic.

We begin with the equilibrium points of the considered model. To obtain them, the standard method is to solve the equations

$$F(t,y)=0 \tag{4.1}$$

where the function $F=(F_1,F_2,F_3,F_4,F_5)$ is given by the following sub representations

$$\begin{aligned} F_1(t,y_1) &= \Lambda-\mu S-nS(I+\psi A),\\ F_2(t,y_1) &= nS(I+\psi A)-[(1-\theta)\omega+\theta\rho+\mu]E,\\ F_3(t,y_1) &= (1-\theta)\omega E-(\tau+\mu)I,\\ F_4(t,y_1) &= \theta\rho E-(\gamma+\mu)A,\\ F_5(t,y_1) &= \tau I+\gamma A-\mu R. \end{aligned} \tag{4.2}$$

In other words, we have to solve the following fractional differential equation

$$\begin{aligned} 0 &= \Lambda-\mu S-nS(I+\psi A),\\ 0 &= nS(I+\psi A)-[(1-\theta)\omega+\theta\rho+\mu]E,\\ 0 &= (1-\theta)\omega E-(\tau+\mu)I,\\ 0 &= \theta\rho E-(\gamma+\mu)A,\\ 0 &= \tau I+\gamma A-\mu R. \end{aligned} \tag{4.3}$$

We get two solutions: the trivial equilibrium points at first, and the endemic equilibrium points given respectively by the trivial equilibrium point as $A_0=\left(\frac{\lambda}{\mu},0,0,0,0\right)$ and the endemic equilibrium point as $A_1=(S^*,E^*,I^*,A^*,R^*)$ where the detail of S^* is given as

$$\begin{aligned} S^* &= \frac{\mu^2[\gamma+\omega+\tau+\mu-\omega\theta+\rho\theta]+\gamma[\mu\omega+\mu\tau+\omega\tau]+\mu[\omega\tau-\gamma\omega\theta+\gamma\rho\theta]}{\gamma n\omega+\mu n\omega-\gamma n\omega\theta-\mu n\omega\theta+\mu n\psi\rho\theta+n\psi\rho\tau\theta}\\ &+ \frac{\gamma[-\omega\tau\theta+\rho\tau\theta]+\mu[-\omega\tau\theta+\rho\tau\theta]}{\gamma n\omega+\mu n\omega-\gamma n\omega\theta-\mu n\omega\theta+\mu n\psi\rho\theta+n\psi\rho\tau\theta}. \end{aligned} \tag{4.4}$$

The rest can be obtained using the MATLAB program; we omitted them due to space restrictions. The equilibrium point can be used to evaluate the local stability of the equilibrium points. Due to the complexity of the calculation, we try in this part to explain the process of getting local stability when the Caputo derivative is used. The first step consists of recalling the Jacobian matrix represented for our model by the following

$$J=\begin{pmatrix} -\mu-n(I+A\psi) & 0 & -Sn & -Sn\psi & 0\\ n(I+A\psi) & \omega(\theta-1)-\rho\theta-\mu & Sn & Sn\psi & 0\\ 0 & -\omega(\theta-1) & -\mu-\tau & 0 & 0\\ 0 & \rho\theta & 0 & -\gamma-\mu & 0\\ 0 & 0 & \tau & \gamma & -\mu. \end{pmatrix} \tag{4.5}$$

After the Jacobian matrix, the second step is to get the Jacobian matrix to the trivial equilibrium points and the endemic equilibrium points. The following results are obtained: at the trivial equilibrium point, the Jacobian matrix is given by the form that

$$J = \begin{pmatrix} -\mu & 0 & -\frac{\lambda}{\mu}n & -\frac{\lambda}{\mu}n\psi & 0 \\ 0 & \omega(\theta-1)-\rho\theta-\mu & \frac{\lambda}{\mu}n & \frac{\lambda}{\mu}n\psi & 0 \\ 0 & -\omega(\theta-1) & -\mu-\tau & 0 & 0 \\ 0 & \rho\theta & 0 & -\gamma-\mu & 0 \\ 0 & 0 & \tau & \gamma & -\mu, \end{pmatrix} \tag{4.6}$$

and at the endemic equilibrium point, the Jacobian matrix sketch is described by the form that

$$J = \begin{pmatrix} -\mu-n(I^*+A^*\psi) & 0 & -S^*n & -S^*n\psi & 0 \\ n(I^*+A^*\psi) & \omega(\theta-1)-\rho\theta-\mu & S^*n & S^*n\psi & 0 \\ 0 & -\omega(\theta-1) & -\mu-\tau & 0 & 0 \\ 0 & \rho\theta & 0 & -\gamma-\mu & 0 \\ 0 & 0 & \tau & \gamma & -\mu. \end{pmatrix} \tag{4.7}$$

The last step is to determine the eigenvalues of the Jacobian matrix defined by the equations and to verify when they satisfy the Matignon criterion described by the form that

$$|\arg(\lambda(J))| > \alpha\pi/2. \tag{4.8}$$

For example, with the Jacobian matrix of the trivial equilibrium points we get five equilibrium points which all are real and negative, which they satisfy directly the condition proposed by Matignon in Eq. (4.8). It follows from the calculation of the real part of the eigenvalue because we can get complex values, satisfies Eq. (4.8) thus the trivial equilibrium point is local stable. The same procedure is repeated with the endemic equilibrium point. A second example to illustrate the main results in this section, we consider our epidemic model described in Eq. (1.1) with the parameter the transmissibility multiplier $\psi = 0$ and $\theta = 0$, in other words, we consider the SEIAR epidemic model, the equilibrium points of the new considered model is given by the form that $B_0 = \left(\frac{\lambda}{\mu}, 0, 0, 0\right)$ and the endemic equilibrium point as $B_1 = (S^*, E^*, I^*, R^*)$ where the detail of the solution are is given as

$$\begin{aligned} S^* &= \frac{\mu\omega+\mu\tau+\omega\tau+\mu^2}{n\omega} \\ E^* &= -\frac{\mu^2\omega+\mu^2\tau+\mu^3-\Lambda n\omega+\mu\omega\tau}{n\omega^2+\mu n\omega} \\ I^* &= -\frac{\mu^2\omega+\mu^2\tau+\mu^3-\Lambda n\omega+\mu\omega\tau}{\mu^2 n+\mu n\omega+\mu n\tau+n\omega\tau} \\ R^* &= -\frac{\mu^3\tau+\mu^2\tau^2+\omega\mu^2\tau+\omega\mu\tau^2-\Lambda n\omega\tau}{\mu^3 n+\mu^2 n\omega+\mu^2 n\tau+\mu n\omega\tau}. \end{aligned}$$

In general, it is very difficult to get the explicit form of the endemic equilibrium point; the contribution of this part is that we have the explicit form. For the local stability of the first equilibrium point, B_0 the trivial equilibrium points have a related Jacobian matrix given by the form that

$$J = \begin{pmatrix} -\mu & 0 & -\frac{\Lambda n}{\mu} & 0 \\ 0 & -\mu-\omega & \frac{\Lambda n}{\mu} & 0 \\ 0 & \omega & -\mu-\tau & 0 \\ 0 & 0 & \tau & -\mu. \end{pmatrix} \tag{4.9}$$

The eigenvalues $\lambda_{1,2} = -\mu$ respect the condition (4.8) and

$$\begin{aligned}\lambda_3 &= -\mu - \omega/2 - \tau/2 - \frac{1}{2}\sqrt{\frac{\mu\omega^2 - 2\mu\omega\tau + 4\Lambda n\omega + \mu\tau^2}{\mu}} \\ \lambda_4 &= \frac{1}{2}\sqrt{\frac{\mu\omega^2 - 2\mu\omega\tau + 4\Lambda n\omega + \mu\tau^2}{\mu}} - \omega/2 - \tau/2 - \mu,\end{aligned}$$

respect the condition (4.8), if the following condition is satisfied, that is, as well

$$\frac{n\Lambda\omega}{\mu^2(\omega + \tau + \mu)} < 1. \tag{4.10}$$

From this condition, we can deduce that the reproduction number is described by the form that this number can also be obtained using the regeneration method, we have that

$$\mathcal{R}_0 = \frac{n\Delta\omega}{\mu^2(\omega + \tau + \mu)}. \tag{4.11}$$

As reported in the literature, his role is to observe that if there is less than one, then the infection dies out; if there is not, it signifies that the infection will persist as well. The same manipulation can be done with the endemic equilibrium point, and it will be observed that when $\mathcal{R}_0 < 1$, the endemic equilibrium point is locally stable, but if not the point should be unstable.

5 Numerical scheme and solutions approximation

In this part, we will produce the graphics of the models in 2D. To arrive at our end, we have many methods to do it; we can use the analytical solutions using the Laplace transform, we can use the approximate solutions using the Homotopy methods, and we can use the numerical approximation using the numerical schemes. In the present version, it is more adequate for us to propose the numerical scheme, the method under consideration comes from Garrapa. We will describe the procedure, and the novelty will be how the method can be applied in our considered model. Let us SEIAR epidemic model (1.1), and we first suggest the analytical solution using the Riemann-Liouville derivative, we have the form

$$S(t) = S(0) + I^\alpha F_1(t, y_1), \tag{5.1}$$

$$E(t) = E(0) + I^\alpha F_2(t, y_1), \tag{5.2}$$

$$I(t) = I(0) + I^\alpha F_3(t, y_1), \tag{5.3}$$

$$A(t) = A(0) + I^\alpha F_4(t, y_1), \tag{5.4}$$

$$R(t) = R(0) + I^\alpha F_5(t, y_1), \tag{5.5}$$

where the functions utilized are defined using our model (1.1), we mean the function that F_1, F_2 F_3, F_4 and F_5 expressed in our present context by the forms

$$\begin{aligned}F_1(t, y_1) &= \Lambda - \mu S - nS(I + \psi A), \\ F_2(t, y_1) &= nS(I + \psi A) - [(1-\theta)\omega + \theta\rho + \mu]E, \\ F_3(t, y_1) &= (1-\theta)\omega E - (\tau + \mu)I, \\ F_4(t, y_1) &= \theta\rho E - (\gamma + \mu)A, \\ F_5(t, y_1) &= \tau I + \gamma A - \mu R.\end{aligned} \tag{5.6}$$

We begin the numerical procedure, we consider the discrete point t_n and we apply it to the analytical solution described in Eqs (5.1)-(5.5), we get the forms that

$$\begin{aligned} S(t_n) &= S(0) + I^{\alpha} F_1 (t_n, y_1), & (5.7)\\ E(t_n) &= E(0) + I^{\alpha} F_2 (t_n, y_1), & (5.8)\\ I(t_n) &= I(0) + I^{\alpha} F_3 (t_n, y_1), & (5.9)\\ A(t_n) &= A(0) + I^{\alpha} F_3 (t_n, y_1), & (5.10)\\ R(t_n) &= R(0) + I^{\alpha} F_4 (t_n, y_1). & (5.11) \end{aligned}$$

Here, the variable in the discretized equation is supposed to be the form $y_1 = (S, E, I, A, R)$. Now using the numerical scheme proposed by Garrapa in[18, 19], we get the numerical discretization, let the form that $t_n = t_0 + h$ where h represents the step size, note that the initial time $t_0 = 0$, and then we obtain the forms

$$S(t_n) = S(0) + h^{\alpha} \left[\bar{\kappa}_n^{(\alpha)} F_1(0) + \sum_{j=1}^{n-1} \kappa_{n-j}^{(\alpha)} F_1(t_j, y_{1j}) + \kappa_0^{(\alpha)} F_1\left(t, y_1^P\right) \right], \quad (5.12)$$

$$E(t_n) = E(0) + h^{\alpha} \left[\bar{\kappa}_n^{(\alpha)} F_2(0) + \sum_{j=1}^{n-1} \kappa_{n-j}^{(\alpha)} F_2(t_j, y_{1j}) + \kappa_0^{(\alpha)} F_2\left(t, y_1^P\right) \right], \quad (5.13)$$

$$I(t_n) = I(0) + h^{\alpha} \left[\bar{\kappa}_n^{(\alpha)} F_3(0) + \sum_{j=1}^{n-1} \kappa_{n-j}^{(\alpha)} F_3(t_j, y_{1j}) + \kappa_0^{(\alpha)} F_3\left(t, y_1^P\right) \right], \quad (5.14)$$

$$A(t_n) = A(0) + h^{\alpha} \left[\bar{\kappa}_n^{(\alpha)} F_3(0) + \sum_{j=1}^{n-1} \kappa_{n-j}^{(\alpha)} F_3(t_j, y_{1j}) + \kappa_0^{(\alpha)} F_3\left(t, y_1^P\right) \right], \quad (5.15)$$

$$R(t_n) = R(0) + h^{\alpha} \left[\bar{\kappa}_n^{(\alpha)} F_4(0) + \sum_{j=1}^{n-1} \kappa_{n-j}^{(\alpha)} F_4(t_j, y_{1j}) + \kappa_0^{(\alpha)} F_4\left(t, y_1^P\right) \right], \quad (5.16)$$

where the following form can represent the predictor for all components of the model (1.1) can be represented by the form

$$\begin{aligned} S^P(t_n) &= S(0) + h^{\alpha} \sum_{j=1}^{n-1} \kappa_{n-j-1}^{(\alpha)} F_1(t_j, y_{1j},), & (5.17)\\ E^P(t_n) &= E(0) + h^{\alpha} \sum_{j=1}^{n-1} \kappa_{n-j-1}^{(\alpha)} F_2(t_j, y_{1j},), & (5.18)\\ I^P(t_n) &= I(0) + h^{\alpha} \sum_{j=1}^{n-1} \kappa_{n-j-1}^{(\alpha)} F_3(t_j, y_{1j}), & (5.19)\\ A^P(t_n) &= A(0) + h^{\alpha} \sum_{j=1}^{n-1} \kappa_{n-j-1}^{(\alpha)} F_4(t_j, y_{1j}). & (5.20)\\ R^P(t_n) &= R(0) + h^{\alpha} \sum_{j=1}^{n-1} \kappa_{n-j-1}^{(\alpha)} F_4(t_j, y_{1j}). & (5.21) \end{aligned}$$

In the predictor functions, the values of the parameters used are symbolized by the forms

$$\bar{\kappa}_n^{(\alpha)} = \frac{(n-1)^{\alpha} - n^{\alpha}(n - \alpha - 1)}{\Gamma(2+\alpha)}, \quad (5.22)$$

with n describing the set that $1, 2, ...$, and the rest of the parameters can be represented by the forms

$$\kappa_0^{(\alpha)} = \frac{1}{\Gamma(2+\alpha)} \text{ and } \kappa_n^{(\alpha)} = \frac{(n-1)^{\alpha+1} - 2n^{\alpha+1} + (n+1)^{\alpha+1}}{\Gamma(2+\alpha)}. \tag{5.23}$$

We finish the description of the numerical scheme proposed by Garrapa by giving the discretization of the functions under consideration in our epidemic model (1.1), we have the form that

$$\begin{aligned}
F_1(t_j, y_{1j}) &= \Lambda - \mu S_j - n S_j (I_j + \psi A_j), \\
F_2(t_j, y_{1j}) &= n S_j (I_j + \psi A_j) - [(1-\theta)\omega + \theta\rho + \mu] E_j, \\
F_3(t_j, y_{1j}) &= (1-\theta)\omega E_j - (\tau + \mu) I_j, \\
F_4(t_j, y_{1j}) &= \theta\rho E_j - (\gamma + \mu) A_j, \\
F_5(t_j, y_{1j}) &= \tau I_j + \gamma A_j - \mu R_j.
\end{aligned} \tag{5.24}$$

We continue by giving short remarks concerning the convergence and the stability of the proposed numerical scheme in the present part. Let consider that $S(t_n), E(t_n), I(t_n)$ and $R(t_n)$ be the approximate solutions of the system under consideration (1.1) and S_n, E_n, I_n and R_n be the exact solutions of the SEIAR model described in Eq. (1.1). The residual functions are described as

$$|S(t_n) - S_n| = \mathcal{O}\left(h^{\min\{\alpha+1,2\}}\right), \tag{5.25}$$

$$|E(t_n) - E_n| = \mathcal{O}\left(h^{\min\{\alpha+1,2\}}\right), \tag{5.26}$$

$$|I(t_n) - I_n| = \mathcal{O}\left(h^{\min\{\alpha+1,2\}}\right), \tag{5.27}$$

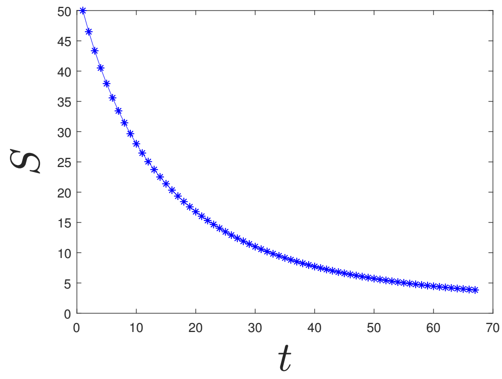
$$|A(t_n) - A_n| = \mathcal{O}\left(h^{\min\{\alpha+1,2\}}\right), \tag{5.28}$$

$$|R(t_n) - R_n| = \mathcal{O}\left(h^{\min\{\alpha+1,2\}}\right). \tag{5.29}$$

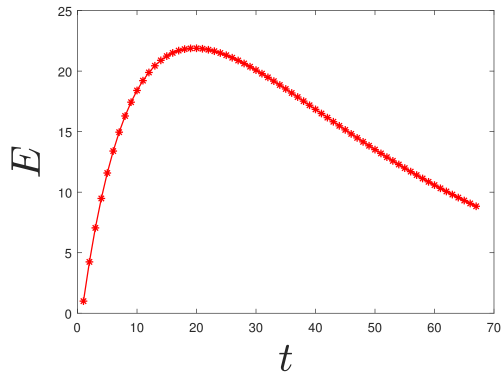
We get the convergence when the step size h considered in the discretization converges to zero. Finally, we can affirm that after a high number of iterations, the numerical solution will converge to the model's exact solution.

6 Analysis of the solution model

In this section, we illustrate our numerical scheme proposed in the previous section. We give several graphics by varying the order of the fractional operator, which we mean the order of the Caputo derivative. The applications of the SEIAR epidemic model will be applied in this paper for modeling the novel coronavirus. The calibration of the data is important, for the rest of the section and the paper, we let the following values $\Lambda = 0.0079$ denote the birth rate considered to be proportional to the natural mortality rate. The parameter $n = 0.004$ represents the contact rate transmission coefficient. The parameter $\psi = 0.8$ is the transmission multiplier of the novel coronavirus disease. The coefficient $\omega = 0.1$ represents the incubation period parameter; in practice, it signifies that an infected person takes 10 days to present the first signs of the novel coronavirus. The coefficient $\rho = 1/10$ has the same significance as the previous parameter. The parameter$\theta = 0.6$ denotes the proportion of asymptomatic infection for certain countries, the parameter asks how many persons are asymptomatic with the novel coronavirus. Let that $\mu = 0.0079$ be the natural mortality rate. The parameter considered to be $\gamma = 1/9$ is the removal or recovery rate of the variable A, which means that an asymptomatic person takes 9 days in general to recover when they receive treatment. We conclude that the parameter $\tau = 1/9$ is the removal or recovery rate of I; it signifies that an infected person takes 9 days in general to recover when he receives specific treatment related to the novel coronavirus therapy. The previous calibration corresponds to the following graphics1a,1b,2a,2b,3a.

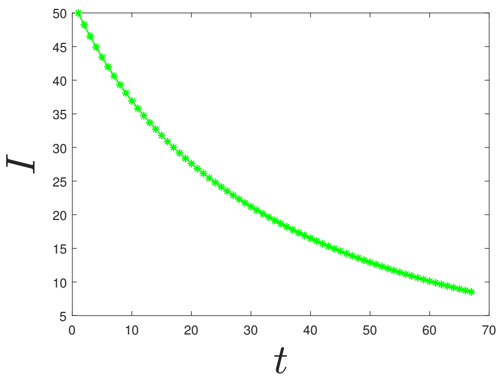


(a)

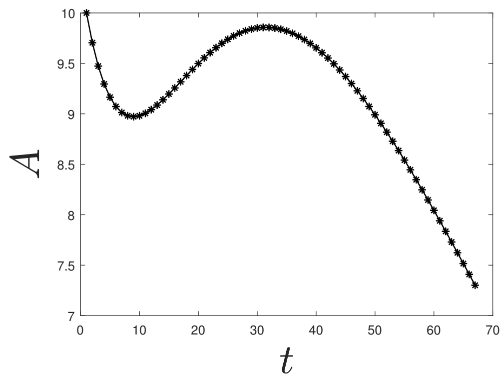


(b)

Figure 1: SEIAR model S $\alpha = 0.95$ (a). SEIAR model E $\alpha = 0.95$ (b).

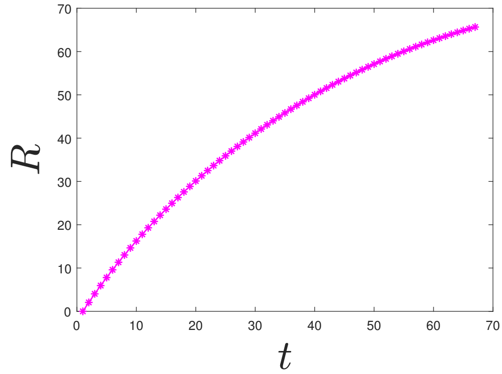


(a)



(b)

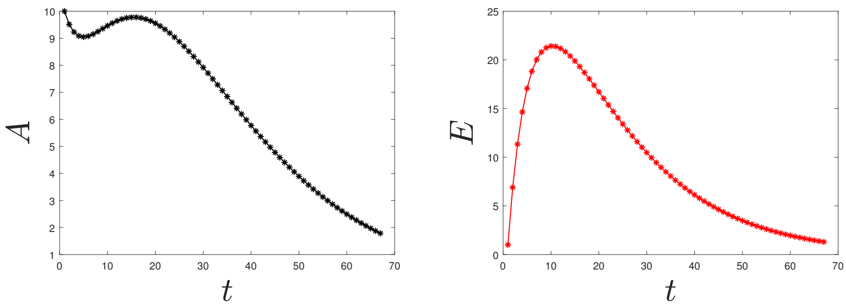
Figure 2: SEIAR model I $\alpha = 0.95$ (a). SEIAR model A $\alpha = 0.95$ (b).



(a)

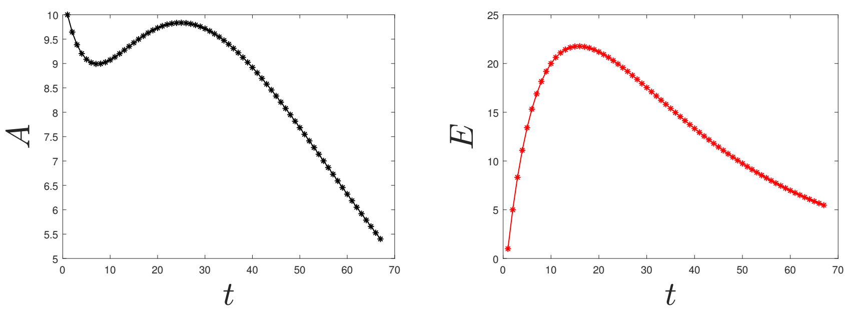
Figure 3: SEIAR model R $\alpha = 0.95$ (a).

We now continue by analyzing the impact of the order of the Caputo derivative on the epidemic model, which will inform us of the utility of the use of the fractional operator in general in modeling the epidemic. We consider the asymptomatic and exposed individuals denoted by A and E in the model. The previous calibration corresponds to the following graphics4a,4b,5a,5b,6a,6b.



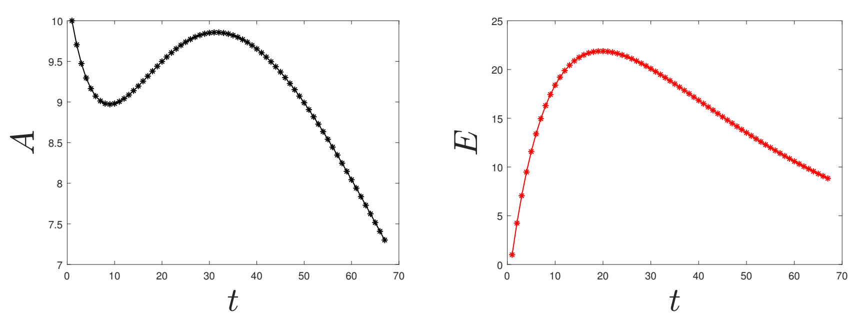
(a) (b)

Figure 4: SEIAR model A $\alpha = 0.35$ (a). SEIAR model E $\alpha = 0.35$ (b).



(a) (b)

Figure 5: SEIAR model A $\alpha = 0.75$ (a). SEIAR model E $\alpha = 0.75$ (b).



(a) (b)

Figure 6: SEIAR model A $\alpha = 0.95$ (a). SEIAR model E $\alpha = 0.95$ (b).

The role of the order of the fractional derivative in the asymptomatic component is to reduce the number of asymptomatic persons when the order of the derivative increases. In other words, the order role to stop in time the spread of the number of asymptomatic individuals. The number of exposed individuals increases with the increase in the order of the fractional derivative. This can be explained by the fact that the number of exposed persons in practice is very important when we have no isolation of the individuals. The general conclusion is that the order of the fractional derivative can be manipulated to to the spread of a disease.

7 Conclusion remarks

The fractional SEIAR epidemic model has been thoroughly examined in this study. We introduce a numerical scheme specifically tailored for the fractional model, which enables the acquisition of various results presented in this investigation. An explicit formulation of both trivial and endemic equilibria is provided, accompanied by an analysis of local stability utilizing the Matignon criterion. Furthermore, we derive the reproduction number based on the conditions of local stability outlined herein. For future research, we recommend exploring the same model with a fractional derivative characterized by a non-singular kernel, accompanied by a proposed numerical discretization approach to obtain solutions. Additionally, a focus on global stability utilizing the Lyapunov function is warranted.

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