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Generating functions of triple product of Tribonacci and Padovan numbers with square well-known Gaussian numbers

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Abstract *This article introduces new generating functions for the triple product of Tribonacci and Padovan numbers with some square well-known Gaussian numbers. These generating functions are constructed by applying the symmetrizing endomorphism operator* $\delta^k_{b_1b_2}\delta^h_{c_1c_2}$ *to the formal series* $\sum\limits_{n=0}^{+\infty} S_n(A)b_1^n c_1^n z^n$.

1 Introduction

The authors in [9, 11, 18, 19, 20] defined and studied the Tribonacci $\{T_n\}_{n=0}^{+\infty}$ *and Padovan* $\{P_n\}_{n=0}^{+\infty}$ *sequences, providing their generating functions, explicit formulas, and Q-matrices. The Padovan sequence* $\{P_n\}_{n=0}^{+\infty}$ *appears to have been first discovered in 1924 by the French architecture student Gérard Cordonnier and was independently rediscovered by Dom Hans van der Laan [18]. These sequences are defined, respectively, by the following third-order recurrence relations:*

$$T_n = T_{n-1} + T_{n-2} + T_{n-3},\ \text{for } n \geq 3,$$

with initial values $T_0 = T_1 = 1$ *and* $T_2 = 2$.

$$P_n = P_{n-2} + P_{n-3},\ \text{for } n \geq 3,$$

with initial values $P_0 = P_1 = P_2 = 1$.

In the papers [4, 3], a second-order linear recurrence sequence $H_n(a, b, p, q)$ *or briefly* $(H_n)_{n\geq 0}$ *is defined by*

$$H_{n+2} = pH_{n+1} + qH_n,$$

with the initial conditions, $H_0 = a$ *and* $H_1 = b$*, where* $a, b \in \mathbb{C}$ *and* $p, q \in \mathbb{Z}$.

This sequence was introduced by Horadam in 1965 [12] and generalizes many well-known sequences [13]. Several authors have studied particular cases of this sequence through its generating functions (see [4, 5, 6, 7, 8, 10, 16, 17]), including sequences of Gaussian numbers such as :

- *The Gaussian Fibonacci number sequence* $(GF_n)_{n\geq 0}$*, with parameters* $b = q = p = 1,\ a = i$.
- *The Gaussian Pell number sequence* $(GP_n)_{n\geq 0}$*, with parameters* $b = q = 1,\ p = 2,\ a = i$.
- *The Gaussian Jacobsthal number sequence* $(GJ_n)_{n\geq 0}$*, with parameters* $b = p = 1,\ q = 2,\ a = \frac{i}{2}$.

- *The Gaussian Lucas numbers sequence* $(GL_n)_{n\geq 0}$, *with parameters* $p = q = 1,\ a = 2 - i,\ b = 1 + 2i$.

2 Definitions and properties

In this section, we introduce a new symmetric function and establishes its fundamental properties. We also recall essential definitions from the literature [5, 1, 6, 3] that will be used in subsequent sections.

Definition 2.1. [15] Let n and m be positive integers and let $\{a_1, a_2, ..., a_m\}$ be a set of variables. The n-th elementary symmetric function, denoted $e_n(a_1, a_2, ..., a_m)$, is defined by

$$e_n(a_1, a_2..., a_m) = \sum_{i_1+i_2+...+i_m=n} a_1^{i_1} a_2^{i_2} ... a_m^{i_m}, \quad (n \geq 0),$$

with $i_1, i_2, ..., i_m = 0$ or 1.

Definition 2.2. [15] Let n and m be positive integers and let $\{a_1, a_2, ..., a_m\}$ be a set of variables. The n-th complete homogeneous symmetric function, denoted $c_n(a_1, a_2, ..., a_m)$, is defined by

$$c_n(a_1, a_2, ..., a_m) = \sum_{i_1+i_2+...+i_m=n} a_1^{i_1} a_2^{i_2} ... a_m^{i_m}, \quad (n \geq 0),$$

with $i_1, i_2, ..., i_m \geq 0$.

Remark 2.3. By usual convention, we set

$$e_0(a_1, a_2, \ldots, a_n) = c_0(a_1, a_2, \ldots, a_n) = 1.$$

For $n < 0$, we set

$$e_n(a_1, a_2, \ldots, a_m) = c_n(a_1, a_2, \ldots, a_m) = 0.$$

Furthermore, for $n > m$, we have $e_n(a_1, a_2, \ldots, a_m) = 0$.

Definition 2.4. [2] Let A and B be any two alphabets. We define $S_n(A - B)$ by the following expression:

$$\frac{\prod_{b\in B}(1 - bz)}{\prod_{a\in A}(1 - az)} = \sum_{n=0}^{\infty} S_n(A - B)z^n,$$

with the condition that $S_n(A - B) = 0$ for $n < 0$.

Definition 2.5. [2] For any alphabet B, the sequence $\{S_n(-B)\}_{n\geq 0}$ is defined via its generating function:

$$\sum_{n=0}^{\infty} S_n(-B)z^n = \frac{1}{\sum_{n=0}^{\infty} S_n(B)z^n} = \prod_{b\in B}(1 - bz).$$

Remark 2.6. The following notations and identities hold:

- For the alphabet $B = \{b_1, b_2\}$, where $S_n(B)$ denotes $S_n(b_1 + b_2)$:

$$S_n(B) = c_n(b_1, b_2) = \frac{b_1^{n+1} - b_2^{n+1}}{b_1 - b_2}, \quad S_n(-B) = (-1)^n e_n(B), \quad n \in \mathbb{N}.$$

- For the alphabet $A = \{a_1, a_2, a_3\}$, where $S_n(A)$ denotes $S_n(a_1 + a_2 + a_3)$:

$$S_n(A) = c_n(a_1, a_2, a_3), \quad S_n(-A) = (-1)^n e_n(A), \quad n \in \mathbb{N}.$$

Definition 2.7. [3] Let $B = \{b_1, b_2\}$ be an alphabet. The *symmetrizing operator* $\delta^k_{b_1 b_2}$ *is defined by*

$$\delta^k_{b_1 b_2} f(b_1) = \frac{b_1^k f(b_1) - b_2^k f(b_2)}{b_1 - b_2}, \textit{ for } k \in \mathbb{N}$$

3 Theorems and main results

In this section, we establish a new theorem on the generating functions for the triple Hadamard product of complete symmetric functions.

Proposition 3.1. *Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be two alphabets. The symmetrizing operator $\delta^k_{b_1b_2}\delta^h_{c_1c_2}$ is defined by*

$$\delta^k_{b_1b_2}\delta^h_{c_1c_2} f(b_1,c_1) = \frac{b_1^k c_1^h f(b_1,c_1) - b_1^k c_2^h f(b_1,c_2) - b_2^k c_1^h f(b_2,c_1) + b_2^k c_2^h f(b_2,c_2)}{(b_1-b_2)(c_1-c_2)}, \text{ for all } k,\ h \in \mathbb{N}.$$

Proof. Since [3] we obtain

$$\begin{aligned}
\delta^k_{b_1b_2}\delta^h_{c_1c_2}(f(b_1,c_1) &= \delta^k_{b_1b_2}\left(\delta^h_{c_1c_2}(f(b_1,c_1))\right)\\
&= \delta^k_{b_1b_2}\left(\frac{c_1^h f(b_1,c_1) - c_2^h f(b_1,c_2)}{(c_1-c_2)}\right)\\
&= \frac{b_1^k\left(\frac{c_1^h f(b_1,c_1)-c_2^h f(b_1,c_2)}{(c_1-c_2)}\right) - b_2^k\left(\frac{c_1^h f(b_2,c_1)-c_2^h f(b_2,c_2)}{(c_1-c_2)}\right)}{(b_1-b_2)}\\
&= \frac{b_1^k c_1^h f(b_1,c_1) - b_1^k c_2^h f(b_1,c_2) - b_2^k c_1^h f(b_2,c_1) + b_2^k c_2^h f(b_2,c_2)}{(b_1-b_2)(c_1-c_2)}.
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$, and $C = \{c_1, c_2\}$. Then ,we have*

$$\frac{\sum\limits_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)S_{n+h-1}(C)z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^nc_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^nc_2^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_2^nc_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_2^nc_2^nz^n\right)}$$
$$= \frac{c_1^hc_2^h}{(b_1-b_2)}\left(\begin{array}{l}\sum\limits_{n=0}^{\infty} S_n(-A)S_{n-h-1}(C)b_2^{n+k}z^n\sum\limits_{n=0}^{\infty} S_n(-A)b_1^nc_1^nz^n\sum\limits_{n=0}^{\infty} S_n(-A)b_1^nc_2^nz^n\\ -\sum\limits_{n=0}^{\infty} S_n(-A)S_{n-h-1}(C)b_1^{n+k}z^n\sum\limits_{n=0}^{\infty} S_n(-A)b_2^nc_1^nz^n\sum\limits_{n=0}^{\infty} S_n(-A)b_2^nc_2^nz^n\end{array}\right) \quad (3.1)$$

Proof. let $\sum\limits_{n=0}^{\infty} S_n(A)b_1^nc_1^nz^n$ and $\sum\limits_{n=0}^{\infty} S_n(-A)b_1^nc_1^nz^n$ be two sequences such that $\left(\sum\limits_{n=0}^{\infty} S_n(A)b_1^nc_1^nz^n\right)\times\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^nc_1^nz^n\right) = 1$. On one hand, since $f(b_1,c_1) = \sum\limits_{n=0}^{\infty} S_n(A)b_1^nc_1^nz^n$, we have

$$\begin{aligned}
&\delta^k_{b_1b_2}\delta^h_{c_1c_2} f(b_1,c_1)\\
&= \delta^k_{b_1b_2}\left(\frac{\sum\limits_{n=0}^{\infty} S_n(A)c_1^{n+h}b_1^nz^n - \sum\limits_{n=0}^{\infty} S_n(A)c_2^{n+h}b_1^nz^n}{c_1-c_2}\right)\\
&= \delta^k_{b_1b_2}\left(\sum_{n=0}^{\infty} S_n(A)\frac{c_1^{n+h}-c_2^{n+h}}{c_1-c_2}b_1^nz^n\right)\\
&= \delta^k_{b_1b_2}\left(\sum_{n=0}^{\infty} S_n(A)S_{n+h-1}(C)b_1^nz^n\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sum\limits_{n=0}^{\infty} S_n(A)S_{n+h-1}(C)b_1^{n+k}z^n - \sum\limits_{n=0}^{\infty} S_n(A)S_{n+h-1}(C)b_2^{n+k}z^n}{b_1-b_2} \\
&= \sum_{n=0}^{\infty} S_n(A)S_{n+h-1}(C)\frac{b_1^{n+k}-b_2^{n+k}}{b_1-b_2}z^n \\
&= \sum_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)S_{n+h-1}(C)z^n.
\end{aligned}
$$

On the other hand, given that $f(b_1,c_1) = \frac{1}{\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_1^n z^n}$, we get

$$
\begin{aligned}
\delta_{b_1b_2}^{k}\delta_{c_1c_2}^{h} f(b_1,c_1) &= \delta_{b_1b_2}^{k}\delta_{c_1c_2}^{h}\left(\frac{1}{\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_1^n z^n}\right) \\
&= \delta_{b_1b_2}^{k}\left(\frac{1}{c_1-c_2}\frac{\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_1^h c_2^n z^n - \sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_1^n c_2^h z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_2^n z^n\right)}\right) \\
&= \delta_{b_1b_2}^{k}\left(\frac{-c_1^h c_2^h \sum\limits_{n=0}^{\infty} S_n(-A)b_1^n \frac{c_1^{n-h}-c_2^{n-h}}{c_1-c_2} z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_2^n z^n\right)}\right) \\
&= \delta_{b_1b_2}^{k}\left(\frac{-c_1^h c_2^h \sum\limits_{n=0}^{\infty} S_n(-A)S_{n-h-1}(C)b_1^n z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_2^n z^n\right)}\right) \\
&= \frac{1}{b_1-b_2}\left(\begin{array}{c}\frac{-b_1^k c_1^h c_2^h \sum\limits_{n=0}^{\infty} S_n(-A)S_{n-h-1}(C)b_1^n z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_2^n z^n\right)} \\ +\frac{b_2^k c_1^h c_2^h \sum\limits_{n=0}^{\infty} S_n(-A)S_{n-h-1}(C)b_2^n z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_2^n c_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_2^n c_2^n z^n\right)}\end{array}\right) \\
&= \frac{c_1^h c_2^h}{b_1-b_2}\left(\begin{array}{c}\frac{\sum\limits_{n=0}^{\infty} S_n(-A)S_{n-h-1}(C)b_2^{n+k} z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_2^n c_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_2^n c_2^n z^n\right)} - \\ \frac{\sum\limits_{n=0}^{\infty} S_n(-A)S_{n-h-1}(C)b_1^{n+k} z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_2^n z^n\right)}\end{array}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{\sum\limits_{n=0}^{\infty} S_n(A)S_{n+k-1}(B)S_{n+h-1}(C)z^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_2^n c_1^n z^n \times \sum\limits_{n=0}^{\infty} S_n(-A)b_2^n c_2^n z^n \times \sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_1^n z^n \times \sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_2^n z^n\right)} \\
&= \frac{c_1^h c_2^h}{b_1-b_2}\left(\begin{array}{c}\sum\limits_{n=0}^{\infty} S_n(-E)S_{n-h-1}(C)b_2^{n+k}z^n\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_1^n z^n \times \sum\limits_{n=0}^{\infty} S_n(-A)b_1^n c_2^n z^n\right) \\ -\sum\limits_{n=0}^{\infty} S_n(-E)S_{n-h-1}(C)b_1^{n+k}z^n\left(\sum\limits_{n=0}^{\infty} S_n(-A)b_2^n c_1^n z^n \times \sum\limits_{n=0}^{\infty} S_n(-A)b_2^n c_2^n z^n\right)\end{array}\right).
\end{aligned}
$$

which completes the proof. □

Substituting $k = h = 1$ into Formula (3.1) yields the following corollary.

Corollary 3.3. *Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$. Then, we have*

$$\sum_{n=0}^{\infty} S_n(A)S_n(b_1 + b_2)S_n(c_1 + c_2)z^n = \frac{c_1c_2}{(b_1 - b_2)} \times \frac{\phi_1}{\phi_2}, \tag{3.2}$$

where

$$\phi_1 = \begin{pmatrix} \prod_{a\epsilon A}(1 - ab_1c_1z) \prod_{a\epsilon A}(1 - ab_1c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-2}(c_1 + c_2)b_2^{n+1}z^n \\ - \prod_{a\epsilon A}(1 - ab_2c_1z) \prod_{a\epsilon A}(1 - ab_2c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-2}(c_1 + c_2)b_1^{n+1}z^n \end{pmatrix},$$

$$\phi_2 = \left(\sum_{n=0}^{\infty}(-1)^n e_n(A)b_1^n c_1^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(A)b_1^n c_2^n z^n\right)\left(\sum_{n=0}^{\infty}(-1)^n e_n(A)b_2^n c_1^n z^n\right) \times \left(\sum_{n=0}^{\infty}(-1)^n e_n(A)b_2^n c_2^n z^n\right).$$

Remark 3.4. Setting $a_3 = 0$ in (3.2) of corollary 3.3 yeilds the Formula given in [8].

Similarly, from (3.2) we obtain

$$\sum_{n=0}^{\infty} S_{n-1}(A)S_{n-1}(b_1 + b_2)S_{n-1}(c_1 + c_2)z^n = \frac{c_1c_2}{(b_1 - b_2)} \times \frac{\phi_3}{\phi_2}, \tag{3.3}$$

where

$$\phi_3 = \begin{pmatrix} \prod_{a\epsilon A}(1 - ab_1c_1z) \prod_{a\epsilon A}(1 - ab_1c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-2}(c_1 + c_2)b_2^{n+1}z^{n+1} \\ - \prod_{a\epsilon A}(1 - ab_2c_1z) \prod_{a\epsilon A}(1 - ab_2c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-2}(c_1 + c_2)b_1^{n+1}z^{n+1} \end{pmatrix}.$$

Substituting $k = h = 0$ into Formula (3.1) yields the following corollary.

Corollary 3.5. *Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$, and $C = \{c_1, c_2\}$. Then, we have*

$$\sum_{n=0}^{\infty} S_n(A)S_{n-1}(b_1 + b_2)S_{n-1}(c_1 + c_2)z^n = \frac{1}{(b_1 - b_2)} \times \frac{\phi_4}{\phi_2}, \tag{3.4}$$

where

$$\phi_4 = \begin{pmatrix} \prod_{a\epsilon A}(1 - ab_1c_1z) \prod_{a\epsilon A}(1 - ab_1c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-1}(C)b_2^n z^n \\ - \prod_{a\epsilon A}(1 - ab_2c_1z) \prod_{a\epsilon A}(1 - ab_2c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-1}(C)b_1^n z^n \end{pmatrix}.$$

Remark 3.6. Setting $a_3 = 0$ in (3.4) of corollary 3.5 yields the Formula given in [8].

Substituting $k = 0$ and $h = 1$, into Formula (3.1), we obtain the following corollary.

Corollary 3.7. *Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$, and $C = \{c_1, c_2\}$. Then, we have*

$$\sum_{n=0}^{\infty} S_n(A)S_n(c_1 + c_2)S_{n-1}(b_1 + b_2)z^n = \frac{c_1c_2}{(b_1 - b_2)} \times \frac{\phi_5}{\phi_2}, \tag{3.5}$$

where

$$\phi_5 = \begin{pmatrix} \prod_{a\epsilon A}(1 - ab_1c_1z) \prod_{a\epsilon A}(1 - ab_1c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-2}(C)b_2^n z^n \\ - \prod_{a\epsilon A}(1 - ab_2c_1z) \prod_{a\epsilon A}(1 - ab_2c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-2}(C)b_1^n z^n \end{pmatrix}.$$

Remark 3.8. Setting $a_3 = 0$, in (3.5) of corollary 3.7, yields the Formula given in [8].

Substituting $k = h = 2$, into Formula (3.1), we obtain the following result.

Corollary 3.9. *Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$, and $C = \{c_1, c_2\}$. Then, we have*

$$\sum_{n=0}^{\infty} S_n(A)S_{n+1}(b_1 + b_2)S_{n+1}(c_1 + c_2)z^n = \frac{c_1^2 c_2^2}{(b_1 - b_2)} \times \frac{\phi_6}{\phi_2}, \tag{3.6}$$

where

$$\phi_6 = \left(\begin{array}{c} \prod_{a\epsilon A}(1 - ab_1c_1z) \prod_{a\epsilon A}(1 - ab_1c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-3}(C)b_2^{n+2}z^n \\ - \prod_{a\epsilon A}(1 - ab_2c_1z) \prod_{a\epsilon A}(1 - ab_2c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-3}(C)b_1^{n+2}z^n \end{array} \right).$$

Similarly, from (3.6) we obtain

$$\sum_{n=0}^{\infty} S_{n-1}(A)S_n(b_1 + b_2)S_n(c_1 + c_2)z^n = \frac{c_1^2 c_2^2}{(b_1 - b_2)} \times \frac{\phi_7}{\phi_2}, \tag{3.7}$$

where

$$\phi_7 = \left(\begin{array}{c} \prod_{a\epsilon A}(1 - ab_1c_1z) \prod_{a\epsilon A}(1 - ab_1c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-3}(C)b_2^{n+2}z^{n+1} \\ - \prod_{a\epsilon A}(1 - ab_2c_1z) \prod_{a\epsilon A}(1 - ab_2c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-3}(C)b_1^{n+2}z^{n+1} \end{array} \right).$$

Substituting $k = 2$ and $h = 1$ into formula (3.1), we obtain the following corollary.

Corollary 3.10. *Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$, and $C = \{c_1, c_2\}$. Then, we have*

$$\sum_{n=0}^{\infty} S_n(A)S_{n+1}(b_1 + b_2)S_n(c_1 + c_2)z^n = \frac{c_1 c_2}{(b_1 - b_2)} \times \frac{\phi_8}{\phi_2}, \tag{3.8}$$

where

$$\phi_8 = \left(\begin{array}{c} \prod_{a\epsilon A}(1 - ab_1c_1z) \prod_{a\epsilon A}(1 - ab_1c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-2}(C)b_2^{n+2}z^n \\ - \prod_{a\epsilon A}(1 - ab_2c_1z) \prod_{a\epsilon A}(1 - ab_2c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-2}(C)b_1^{n+2}z^n \end{array} \right).$$

Similarly, from(3.8) we obtain

$$\sum_{n=0}^{\infty} S_{n-1}(A)S_n(b_1 + b_2)S_{n-1}(c_1 + c_2)z^n = \frac{c_1 c_2}{(b_1 - b_2)} \times \frac{\phi_9}{\phi_2}, \tag{3.9}$$

where

$$\phi_9 = \left(\begin{array}{c} \prod_{a\epsilon A}(1 - ab_1c_1z) \prod_{a\epsilon A}(1 - ab_1c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-2}(C)b_2^{n+2}z^{n+1} \\ - \prod_{a\epsilon A}(1 - ab_2c_1z) \prod_{a\epsilon A}(1 - ab_2c_2z) \times \sum_{n=0}^{\infty}(-1)^n e_n(A)S_{n-2}(C)b_1^{n+2}z^{n+1} \end{array} \right).$$

4 Applications

4.1 Generating Functions for Products of Tribonacci and Square Gaussian Numbers.

In this part, we derive new generating functions for products of Tribonacci numbers with certain well-known square Gaussian numbers. Replacing b_2 by $(-b_2)$ and c_2 by $(-c_2)$ in relationships (3.2), (3.4), and (3.5), we observe three related cases.

First step. *Using the substitutions* $\begin{cases} S_1(-A) = -1 \\ S_2(-A) = -1 \\ S_3(-A) = -1 \end{cases}$ *and* $\begin{cases} b_1 - b_2 = c_1 - c_2 = 1 \\ b_1 b_2 = c_1 c_2 = 1, \end{cases}$ *in relationships (3.2), (3.4), and (3.5) above, we obtain the following theorem:*

Theorem 4.1. *For $n \in \mathbb{N}$, the new generating function for the product of Tribonacii numbers and square Gaussian Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} T_n GF_n^2 z^n = \frac{N_{T_n GF_n^2}}{D},$$

with

$$\begin{aligned} N_{T_n GF_n^2} &= -2iz - (2+14i)\, z^3 - (-4+8i) z^4 + (-6+16i)\, z^5 \\ &\quad - (11-18i)\, z^6 + (9+26i)\, z^7 - (-10-2i)\, z^8 + (5+4i)\, z^9. \end{aligned}$$

and

$$D = 1 - z - 13z^2 - 24z^3 - 13z^4 - 23z^5 - 12z^6 - 37z^7 + 47z^8 - 22z^9 + 5z^{10} + z^{11} + z^{12}.$$

Proof. Since [7] $GF_n = iS_n(B) + (1-i)S_{n-1}(B)$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} T_n GF_n^2 z^n &= \sum_{n=0}^{\infty} T_n (iS_n(B) + (1-i)S_{n-1}(B)) \times (iS_n(C) + (1-i)S_{n-1}(C)) \\ &= -\sum_{n=0}^{\infty} T_n S_n(B) S_n(C) z^n + i(1-i) \sum_{n=0}^{\infty} T_n S_n(B) S_{n-1}(C) z^n \\ &\quad + i(1-i) \sum_{n=0}^{\infty} T_n S_{n-1}(B) S_n(C) z^n + (1-i)^2 \sum_{n=0}^{\infty} T_n S_{n-1}(B) S_{n-1}(C) z^n. \end{aligned}$$

Accordingin to relationships (3.2), (3.4) and (3.5), and after reducing to a common denominator, we obtain the result. □

Theorem 4.2. *For $n \in \mathbb{N}$, the new generating function for the product of Tribonacii numbers and square Gaussian Lucas numbers is given by*

$$\sum_{n=0}^{\infty} T_n GL_n^2 z^n = \frac{N_{T_n GL_n^2}}{D},$$

with

$$\begin{aligned} N_{T_n GL_n^2} &= (-8-6i)z + (-12-4i)\, z^2 + (-14-8i)\, z^3 - (32+4i)\, z^4 + (70+20i) z^5 \\ &\quad + (-57+6i) z^6 + (125+130i) z^7 - (26+62i) z^8 + (231+8i) z^9 \end{aligned}$$

Proof. From [7], we have $GL_n = (2-i)S_n(B) + (3i-1)S_{n-1}(B)$. Hence

$$\begin{aligned}\sum_{n=0}^{\infty} T_n GL_n^2 z^n &= \sum_{n=0}^{\infty} T_n((2-i)S_n(B) + (3i-1)S_{n-1}(B)) \times ((2-i)S_n(C) + (3i-1)S_{n-1}(C)) \\ &= (2-i)^2 \sum_{n=0}^{\infty} T_n S_n(B) S_n(C) z^n + (2-i)(3i-1) \sum_{n=0}^{\infty} T_n S_n(B) S_{n-1}(C) z^n \\ &\quad + (3i-1)(2-i) \sum_{n=0}^{\infty} T_n S_{n-1}(B) S_n(C) z^n + (3i-1)^2 \sum_{n=0}^{\infty} T_n S_{n-1}(B) S_{n-1}(C) z^n.\end{aligned}$$

According to relationships (3.2), (3.4), (3.5), and after reducing to a common denominator, we obtain the result. □

Second step. *Using the substitutions* $\begin{cases} S_1(-A) = -1 \\ S_2(-A) = -1 \\ S_3(-A) = -1 \end{cases}$ *and* $\begin{cases} b_1 - b_2 = c_1 - c_2 = 2 \\ b_1 b_2 = c_1 c_2 = 1 \end{cases}$ *in relationships (3.2), (3.4), (3.5) above, we obtain the following theorem:*

Theorem 4.3. *For $n \in \mathbb{N}$, the new generating function for the product of Tribonacii numbers and square Gaussian Pell numbers is given by*

$$\sum_{n=0}^{\infty} T_n GP_n^2 z^n = \frac{N_{T_n GP_n^2}}{D_1},$$

with

$$\begin{aligned}N_{T_n GP_n^2} &= -(3+4i)z - (11+8i)z^2 - (62+40i)z^3 + (63+120i)z^4 + (52+120i)z^5 \\ &\quad -(77+232i)z^6 + (302+264i)z^7 - (177+96i)z^8 + (99+60i)z^9.\end{aligned}$$

and

$$D_1 = 1 - 4z - 46z^2 - 252z^3 - 313z^4 + 248z^5 - 108z^6 - 520z^7 + 455z^8 - 244z^9 + 26z^{10} + 4z^{11} + z^{12}$$

Proof. From [7], we have $GP_n = iS_n(B) + (1-2i)S_{n-1}(B)$. Hence

$$\begin{aligned}\sum_{n=0}^{\infty} T_n GP_n^2 z^n &= \sum_{n=0}^{\infty} T_n(iS_n(B) + (1-2i)S_{n-1}(B))(iS_n(C) + (1-2i)S_{n-1}(C)) \\ &= -\sum_{n=0}^{\infty} T_n S_n(B) z^n + i(1-2i) \sum_{n=0}^{\infty} T_n S_n(B) S_{n-1}(C) z^n \\ &\quad + (1-2i)i \sum_{n=0}^{\infty} T_n S_{n-1}(B) S_n(C) z^n + (1-2i)^2 \sum_{n=0}^{\infty} T_n S_{n-1}(B) S_{n-1}(C) z^n.\end{aligned}$$

According to relationships (3.2), (3.4), (3.5) and after reducing to a common denominator, we obtain the result. □

Third step. *Using the substitutions* $\begin{cases} S_1(-A) = -1 \\ S_2(-A) = -1 \\ S_3(-A) = -1 \end{cases}$ *and* $\begin{cases} b_1 - b_2 = c_1 - c_2 = 1 \\ b_1 b_2 = c_1 c_2 = 2 \end{cases}$ *in relationships (3.2), (3.4), (3.5) above, we arrive at the following theorem:*

Theorem 4.4. *For $n \in \mathbb{N}$, the new generating function for the product of Tribonacii numbers and square Gaussian Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} T_n GJ_n^2 z^n = \frac{N_{T_n GJ_n^2}}{D_2},$$

with

$$N_{T_nGJ_n^2} = -(\frac{3}{4}-i)z+\left(\frac{-7}{4}+i\right)z^2-\left(\frac{75}{4}+13i\right)z^3+(-28+16i)z^4+(-109+100i)z^5$$
$$+(28-176i)z^6+(-116+496i)z^7+(-672+128i)z^8+(148+48i)z^9.$$

and

$$D_2 = 1-z-37z^2-77z^3+4z^4+292z^5-192z^6-1712z^7$$
$$+5824z^8-4416z^9+3328z^{10}+1024z^{11}+4096z^{12}.$$

Proof. From [7], we have $GJ_n = \frac{i}{2}S_n(B)+(1-\frac{i}{2})S_{n-1}(B)$. Hence

$$\sum_{n=0}^{\infty} T_nGJ_n^2z^n = \sum_{n=0}^{\infty} T_n(\frac{i}{2}S_n(B)+(1-\frac{i}{2})S_{n-1}(B))\times(\frac{i}{2}S_n(B)+(1-\frac{i}{2})S_{n-1}(B))$$
$$= -\frac{1}{2}\sum_{n=0}^{\infty} T_nS_n(B)S_n(C)z^n+\frac{i}{2}(1-\frac{i}{2})\sum_{n=0}^{\infty} T_nS_n(B)S_{n-1}(C)z^n$$
$$+\frac{i}{2}(1-\frac{i}{2})\sum_{n=0}^{\infty} T_nS_{n-1}(B)S_n(C)z^n+(1-\frac{i}{2})^2\sum_{n=0}^{\infty} T_nS_{n-1}(B)S_{n-1}(C)z^n.$$

According to relationships (3.2), (3.4), (3.5), and after reducing to a common denominator, we obtain the result. □

4.2 Generating Functions for Products of Padovan numbers and square Gaussian numbers.

In this part, we derive new generating functions for the products of Padovan numbers with some well-know square Gaussian numbers. This case consists of three related parts. Replacing b_2 by $(-b_2)$ and c_2 by $(-c_2)$ in relationships (3.2), (3.3), (3.4), (3.5), (3.7), and (3.9) above, we have

First step. *By imposing the following restrictions* $\begin{cases} S_1(-A)=0 \\ S_2(-A)=-1 \\ S_3(-A)=-1 \end{cases}$ *and* $\begin{cases} b_1-b_2=c_1-c_2=1 \\ b_1b_2=c_1c_2=2 \end{cases}$ *in relationships (3.2), (3.3), (3.4), (3.5), (3.7), and (3.9), we obtain the following theorem.*

Theorem 4.5. *For $n \in \mathbb{N}$, the new generating function for the product of Padovan and square Gaussian Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} P_nGF_n^2z^n = \frac{N_{P_nGF_n^2}}{d},$$

with

$$N_{P_nGF_n^2} = 2iz^2+(-1+6i)z^3+(-2+18i)z^4-(5+2i)z^5+7z^6+(7+22i)z^7$$
$$-(12+12i)z^8+(3-2i)z^9+(3+4i)z^{10},$$

and,

$$d = 1-9z^2-16z^3+16z^4+23z^5-43z^6$$
$$-16z^7+34z^8-15z^9-4z^{10}+z^{11}+z^{12}.$$

Proof. From [7] , we have $GF_n = iS_n(B) + (1-i)S_{n-1}(B)$. Which implies that

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n GF_n^2 z^n &= \sum_{n=0}^{\infty}(S_n(A) + S_{n-1}(A)) \times (iS_n(B) + (1-i)S_{n-1}(B)) \\
&\quad \times (iS_n(C) + (1-i)S_{n-1}(C)) \\
&= -\sum_{n=0}^{\infty} S_n(A)S_n(B)S_n(C)z^n - \sum_{n=0}^{\infty} S_{n-1}(A)S_n(B)S_n(C)z^n \\
&\quad +(2+2i)\sum_{n=0}^{\infty} S_n(A)S_n(B)S_{n-1}(C)z^n + (2+2i)\sum_{n=0}^{\infty} S_{n-1}(A)S_n(B)S_{n-1}(C)z^n \\
&\quad -2i\sum_{n=0}^{\infty} S_n(A)S_{n-1}(B)S_{n-1}(C)z^n - 2i\sum_{n=0}^{\infty} S_{n-1}(A)S_{n-1}(B)S_{n-1}(C)z^n.
\end{aligned}$$

According to relationships (3.2), (3.3), (3.4), (3.5), (3.7), and (3.9) and after reducing to a common denominator, we obtain the result. □

Theorem 4.6. *For $n \in \mathbb{N}$, the new generating function for the product of Padovan and square Gaussian Lucas numbers is given by*

$$\sum_{n=0}^{\infty} P_n GL_n^2 z^n = \frac{N_{P_n GL_n^2}}{d}$$

with

$$\begin{aligned}
N_{P_n GL_n^2} &= (8+6i)z^2 + (27+14i)z^3 + (78+46i)z^4 + (7-26i)z^5 + (-21+28i)z^6 \\
&\quad +(67+94i)z^7 - (12+84i)z^8 + (-17+6i)z^9 + (7+24i)z^{10}.
\end{aligned}$$

Proof. From [7], we have $GL_n = (2-i)S_n(B) + (3i-1)S_{n-1}(B)$. Thus,

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n GL_n^2 z^n &= \sum_{n=0}^{\infty}(S_n(A) + S_{n-1}(A)) \times ((2-i)S_n(B) + (3i-1)S_{n-1}(B)) \\
&\quad \times ((2-i)S_n(C) + (3i-1)S_{n-1}(C)) \\
&= (3-4i)\sum_{n=0}^{\infty} S_n(A)S_n(B)S_n(C)z^n + (3-4i)\sum_{n=0}^{\infty} S_{n-1}(A)S_n(B)S_n(C)z^n \\
&\quad +(2+14i)\sum_{n=0}^{\infty} S_n(A)S_n(B)S_{n-1}(C)z^n + (2+14i)\sum_{n=0}^{\infty} S_{n-1}(A)S_n(B)S_{n-1}(C)z^n \\
&\quad -(8+6i)\sum_{n=0}^{\infty} S_n(A)S_{n-1}(B)S_{n-1}(C)z^n - (8+6i)\sum_{n=0}^{\infty} S_{n-1}(A)S_{n-1}(B)S_{n-1}(C)z^n.
\end{aligned}$$

According to relationships (3.2), (3.3), (3.4), (3.5), (3.7), and (3.9) and after reducing to a common denominator, we obtain the result.

□

Second step. *By imposing the following restrictions* $\begin{cases} S_1(-A) = 0 \\ S_2(-A) = -1 \\ S_3(-A) = -1 \end{cases}$ *and* $\begin{cases} b_1 - b_2 = c_1 - c_2 = 2 \\ b_1 b_2 = c_1 c_2 = 1 \end{cases}$ *in relationships (3.2), (3.3), (3.4), (3.5), (3.7), (3.9) above, we obtain the following theorem.*

Theorem 4.7. *For $n \in \mathbb{N}$ the new generating function for the product of Padovan and square Gaussian Pell numbers is given by*

$$\sum_{n=0}^{\infty} P_n GP_n^2 z^n = \frac{N_{P_n GP_n^2}}{d_1}$$

$$\begin{aligned} N_{P_nGP_n^2} &= (12+16i)z^2+(71+96i)z^3+(103+144i)z^4-(86+76i)z^5 \\ &+34z^6+(163+176i)z^7-(105+60i)z^8 \\ &+(21-4i)z^9+(21+20i)z^{10}, \end{aligned}$$

and

$$d_1 = \begin{array}{c} 1-36z^2-196z^3+70z^4+332z^5-430z^6 \\ -196z^7+373z^8-192z^9-10z^{10}+4z^{11}+z^{12}. \end{array}$$

Proof. From [7], we have $GP_n = iS_n(B)+(1-2i)S_{n-1}(B)$. Therefore

$$\begin{aligned} \sum_{n=0}^{\infty} P_nGP_n^2z^n &= \sum_{n=0}^{\infty}(S_n(A)+S_{n-1}(A))\times(iS_n(B)+(1-2i)S_{n-1}(B)) \\ &\quad \times(iS_n(C)+(1-2i)S_{n-1}(C)) \\ &= -\sum_{n=0}^{\infty}S_n(A)S_n(B)S_n(C)z^n-\sum_{n=0}^{\infty}S_{n-1}(A)S_n(B)S_n(C)z^n \\ &\quad +(4+2i)\sum_{n=0}^{\infty}S_n(A)S_n(B)S_{n-1}(C)z^n+(4+2i)\sum_{n=0}^{\infty}S_{n-1}(A)S_n(B)S_{n-1}(C)z^n \\ &\quad -(3+4i)\sum_{n=0}^{\infty}S_n(A)S_{n-1}(B)S_{n-1}(C)z^n \\ &\quad -(3+4i)\sum_{n=0}^{\infty}S_{n-1}(A)S_{n-1}(B)S_{n-1}(C)z^n. \end{aligned}$$

According to relationships (3.2), (3.3), (3.4), (3.5), (3.7), (3.9) above and after reducing to a common denominator, we obtain the result. □

Thrid step. *Using the substitutions* $\begin{cases} S_1(-A)=0 \\ S_2(-A)=-1 \\ S_3(-A)=-1 \end{cases}$ *and* $\begin{cases} b_1-b_2=c_1-c_2=1 \\ b_1b_2=c_1c_2=2 \end{cases}$ *in relationships (3.2), (3.3), (3.4), (3.5), (3.7), (3.9) above, we have*

Theorem 4.8. *For $n \in \mathbb{N}$ the new generating function for the product of Padovan and square Gaussian Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} P_nGJ_n^2z^n = \frac{N_{P_nGJ_n^2}}{d_2},$$

with

$$\begin{aligned} N_{P_nGJ_n^2} &= (-4+4i)z^2+(\frac{-25}{2}+10i)z^3+(\frac{-97}{2}+70i)z^4-(18+16i)z^5 \\ &+(-22+88i)z^6+(20+720i)z^7-(48+832i)z^8 \\ &+(320-128i)z^9+(320+768i)z^{10}, \end{aligned}$$

and

$$\begin{aligned} d_2 &= 1-25z^2-49z^3+168z^4+264z^5-1312z^6-784z^7 \\ &+409z^8-2880z^9-3072z^{10}+1024z^{11}+4096z^{12}. \end{aligned}$$

Proof. From [7], we have $GJ_n = \frac{i}{2}S_n(B) + (1-\frac{i}{2})S_{n-1}(B)$. Thus

$$\begin{aligned}
\sum_{n=0}^{\infty} P_n GJ_n^2 z^n &= \sum_{n=0}^{\infty}(S_n(A)+S_{n-1}(A)) \times (\frac{i}{2}S_n(B)+(1-\frac{i}{2})S_{n-1}(B)) \\
&\quad \times(\frac{i}{2}S_n(C)+(1-\frac{i}{2})S_{n-1}(C)) \\
&= -\frac{1}{4}\sum_{n=0}^{\infty}S_n(A)S_n(B)S_n(C)z^n - \frac{1}{4}\sum_{n=0}^{\infty}S_{n-1}(A)S_n(B)S_n(C)z^n \\
&\quad +(\frac{1}{2}+i)\sum_{n=0}^{\infty}S_n(A)S_n(B)S_{n-1}(C)z^n + (\frac{1}{2}+i)\sum_{n=0}^{\infty}S_{n-1}(A)S_n(B)S_{n-1}(C)z^n \\
&\quad +(\frac{3}{4}-i)\sum_{n=0}^{\infty}S_n(A)S_{n-1}(B)S_{n-1}(C)z^n + (\frac{3}{4}-i)\sum_{n=0}^{\infty}S_{n-1}(A)S_{n-1}(B)S_{n-1}(C)z^n.
\end{aligned}$$

According to relationships (3.2), (3.3), (3.4), (3.5), (3.7), (3.9)above and after reducing to a common denominator, we obtain the result. □

5 Conclusion

In this research, a new theorem and its corollaries for determining generating functions were proposed. The suggested theorem and corollaries are founded on Symmetric functions. The results obtained correspond with those found in prior works.

References

[1] *A. Boussayoud, A. Abderrezzak, Complete Homogeneous Symmetric Functions and Hadamard Product,Ars Combinatoria.* ***144*** *(2019), 81–90.*

[2] *A. Boussayoud, A. Abderrezzak, M. Kerada, Some applications of symmetric functions,Integers.* ***15(1)*** *(2015), 1–7.*

[3] *A. Boussayoud, M. Boulyer, M. Kerada, A simple and accurate method for determination of some generalized sequence of numbers,Int. J. Pure Appl. Math.* ***108*** *(2016), 503–511.*

[4] *A. Boussayoud, On some identities and generating functions for Pell-Lucas numbers, Online. J. Anal. Comb.* ***12(1)*** *(2017), 1–10.*

[5] *A. Boussayoud, S. Boughaba, M. Kerada, Generating Functions k-Fibonacci and k-Jacobsthal numbers at negative indices, Electron. J. Math. Anal. Appl.* ***6*** *(2018), 195–202.*

[6] *A. Boussayoud, M. Chelgham, S. Boughaba, On some identities and generating functions for Mersenne numbers and polynomials, Turkish Journal of Analysis and Number Theory.* ***6(3)*** *(2018), 93–97.*

[7] *S. Boughaba, A. Boussayoud, S. Araci, M. Kerada, Construction of Generating Functions of Gaussian Fibonacci Numbers and Polynomials, Universal Journal of Mathematics and Applications, (Submitted).*

[8] *Kh. Boubellouta, A. Boussayoud, S. Araci, M. Kerada, Some theorems on generating function and their applications Advanced Studies in Contemporary Mathematics.* ***30(3)*** *(2020), 320–324.*

[9] *M. Chelgham A. Boussayoud, Construction of symmetric functions of generalized Tribonacci numbers,J. Sci. Technol. Arts.* ***50*** *(2020), 65–74*

[10] *A. Daoui, A. Boussayoud, Kh. Boubellouta, Complex bivariate Fibonacci and Lucas polynomials and new generating functions with some special numbers, Palestine Journal of Mathematics* ***13(3)*** *(2024), 593–610.*

[11] *M. Feinberg, Fibonacci-Tribonacci Fibonacci Quart* ***1*** *(1963), 70–74.*

[12] *A. F. Horadam, Basic properties of a certain generalized sequence of numbers, The Fibonacci Quarterly* ***3*** *(1965), 161–176.*

[13] *A. F. Horadam, J. M. Mahon, Pell and Pell-Lucas Polynomials, The Fibonacci Quarterly* ***23(1)*** *(1985), 7–20.*

[14] *E. Kilic, Tribonacci sequences with certain indices and their sums, Ars Combinatoria* ***86*** *(2008), 13–22.*

[15] *M. Merca, A generalization of the symmetry between complete and elementary symmetric functions, Indian Journal of Pure and Applied Mathematics* ***45****(2014) ,75–89.*

[16] *N. Saba, A. Boussayoud, M. Ferkioui, S. Boughaba, Symmetric functions of binary products of Gaussian Jacobsthal Lucas polynomials and Chebyshev polynomials, Palestine Journal of Mathematics* ***10(2)***, *(2021), 452–464.*

[17] *N. Saba, A. Boussayoud, A new class of ordinary generating functions of binary products of Mersenne Lucas numbers with several numbers, Palestine Journal of Mathematics* ***12(2)*** *(2023), 450–463.*

[18] *A. G. Shannon, P. G. Anderson, A. F. Horadam, Properties of Cordonnier, Perrin and Van der Laan numbers, International Journal of Mathematical Education in Science and Technology* ***37(7)****(2006), 825–831.*

[19] *S. Sompong, N. Wora-Ngon, A. Piranan, N. Wongkaentow, Some Matrices with Padovan Q-matrix Property, AIP Conference Proceedings.* ***1905*** *(2017), 030035,* https://doi.org/10.1063/1.5012181.

[20] *Y. Soykan, Yuksel. On generalized Padovan numbers, (2021).*

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