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Controlled continuous g-frames and their duals in Hilbert spaces

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Abstract. *Frame theory has recently undergone a major transformation, establishing itself as a vital tool in various applications. This paper explores the concept of controlled continuous g-frames in Hilbert spaces, extending the standard definition of continuous g-frames. We introduce controlled continuous dual g-frames, analyze their key properties, and characterize all such duals for a given controlled continuous g-frame. These findings extend and generalize several existing results in the field.*

1 Introduction

Gabor [10] introduced a method using a family of elementary functions for reconstructing functions (signals) in 1946. *The idea of frames originated in the* 1952 *paper by Duffin and Schaeffer [7] to address some deep questions in non-harmonic Fourier series. After some decades, Daubechies, Grossmann, and Meyer [6] formally defined the concept of a frame in the abstract Hilbert spaces in* 1986. *After their work, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames. A continuous frame is a concept of generalization of frames, proposed by Kaiser [11] and independently by Ali, Antoine, and Gazeau [3], to a family indexed by some locally compact space endowed with a Radon measure. These frames are called frames associated with measurable spaces by Gabrado and Han in [9]. Askari-Hemmat, Dehghan, and Radjabalipour in [4] called them generalized frames.*

A continuous g-frame is an extension of g-frames and continuous frames, which was first introduced by Abdollahpour and Faroughi in [1]. In recent years, functional analysts has shown considerable interest in the study of continuous g-frames in Hilbert spaces. For a comprehensive overview, readers are referred to [8, 12, 13, 14]. In frame theory, it is important to find controlled continuous dual g-frames that minimize data transmission and reconstruction processes and reconstruct vectors (or signals) in terms of the frame elements.

We begin with a few preliminaries that are needed in the sequel. Let H, L be separable Hilbert spaces and (Ω, μ) be a positive measure space. Let $\{H_w\}_{w\in\Omega}$ be a family of closed subspaces of L. Throughout we consider the index set as Ω, and we denote simply $\{H_w\}_w$ for $\{H_w\}_{w\in\Omega}$. We denote the set of all bounded linear operators from H into H_w by $L(H, H_w)$ and the set of all bounded linear operators on H with bounded inverse by $GL(H)$. The set of all positive operators in $GL(H)$ is denoted by $GL^+(H)$, and I_H represents the identity operator on H. Note that if $P, Q \in GL(H)$, then P^*, P^{-1}, and PQ are also in $GL(H)$. For $P, Q \in GL^+(H)$, the concept of (P, Q)-controlled continuous g-frames for H with respect to $\{H_\omega\}_\omega$ has been introduced in [2]. All integrals in the theory of continuous frames are weak integrals (Pettis integrals) and not strong integrals (Bochner integrals). Operators and integrals are interchangeable.

We consider the space

$$\ell^2(\{H_w\}_w) = \left\{\{f_w\}_w : f_w \in H_w, w \in \Omega, \int_\Omega \|f_w\|^2 \, d\mu(w) < \infty\right\}$$

with the inner product given by

$$\langle \{f_w\}_w, \{g_w\}_w\rangle = \int_\Omega \langle f_w, g_w\rangle \, d\mu(w).$$

It is clear that $\ell^2(\{H_w\}_w)$ is a Hilbert space.

Lemma 1.1. *Let $T : H \to H$ be a linear operator. Then the following statements are equivalent:*

(1) There exist constants $0 < A \le B < \infty$, such that $AI_H \le T \le BI_H$.

(2) T is positive and there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \|T^{\frac{1}{2}} f\|^2 \le B\|f\|^2, \quad \forall f \in H.$$

(3) $T \in GL^+(H)$.

Definition 1.2. [5] Let (Ω, μ) be a measure space with a positive measure μ and $P \in GL(H)$. A P-controlled continuous frame is a map $F : \Omega \to H$ such that there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \int_\Omega \langle f, F(w)\rangle\langle PF(w), f\rangle \, d\mu(w) \le B\|f\|^2, \quad \forall f \in H.$$

Definition 1.3. [1] For each $w \in \Omega$, let $\Lambda_w \in L(H, H_w)$. We say that $\Lambda = \{\Lambda_w\}_w$ is a continuous g-frame for H with respect to $\{H_w\}_w$ if

(1) for each $f \in H$, $\{\Lambda_w f\}_w$ is strongly measurable.

(2) there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w) \le B\|f\|^2, \quad \forall f \in H. \tag{1.1}$$

The numbers A, B are called lower and upper frame bounds for the continuous g-frame, respectively. If only the right-hand inequality of (1.1) is satisfied, we call $\{\Lambda_w\}_w$ is a continuous g-Bessel family for H with respect to $\{H_w\}_w$ with bound B. If $A = B = \lambda$, we call $\{\Lambda_w\}_w$ the λ-tight continuous g-frame. Moreover, if $\lambda = 1$, $\{\Lambda_w\}_w$ is called the Parseval continuous g-frame.

For a given continuous g-frame $\Lambda = \{\Lambda_w\}_w$ for H with respect to $\{H_w\}_w$, there exists a unique positive and invertible operator (called the frame operator) $S_\Lambda : H \to H$ such that for each $f, g \in H$:

$$\langle S_\Lambda f, g\rangle = \int_\Omega \langle f, \Lambda_w^* \Lambda_w g\rangle \, d\mu(w)$$

and $AI_H \le S_\Lambda \le BI_H$.

Definition 1.4. [2] Let $P, Q \in GL^+(H)$ and $\Lambda_w \in L(H, H_w)$. We say that $\{\Lambda_w\}_w$ is a (P, Q)-controlled continuous g-frame for H with respect to $\{H_w\}_w$ if there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \int_\Omega \langle \Lambda_w P f, \Lambda_w Q f\rangle \, d\mu(w) \le B\|f\|^2, \quad \forall f \in H. \tag{1.2}$$

The numbers A and B are called controlled continuous g-frame bounds. If the right hand inequality of (1.2) holds for all $f \in H$, then $\{\Lambda_w\}_w$ is a called a (P, Q)-controlled continuous g-Bessel family with bound B.

If $Q = I_H$, we call $\{\Lambda_w\}_w$ is a P-controlled continuous g-frame for H with respect to $\{H_w\}_w$. If $Q = P$, we call $\{\Lambda_w\}_w$ is a (P, P)-controlled continuous g-frame for H with respect to $\{H_w\}$.

For a (P,Q)-controlled continuous g-Bessel family $\{\Lambda_w\}_w$ with bound B, the operator $T_{P\Lambda Q}$: $\ell^2(\{H_w\}_w) \to H$ given by

$$T_{P\Lambda Q}\{f_w\}_w = \int_\Omega (PQ)^{\frac{1}{2}}\Lambda_w^* f_w \, d\mu(w), \quad \forall \{f_w\}_w \in \ell^2(\{H_w\}_w)$$

is well-defined and its adjoint is given by

$$T_{P\Lambda Q}^* : H \to \ell^2(\{H_w\}_w), \quad T_{P\Lambda Q}^* f = \{\Lambda_w (QP)^{\frac{1}{2}} f\}_w, \quad \forall f \in H.$$

$T_{P\Lambda Q}$ is called the synthesis operator and $T_{P\Lambda Q}^*$ is called the analysis operator of $\{\Lambda_w\}_w$. For a (P,Q)-controlled continuous g-frame $\{\Lambda_w\}_w$ with bounds A and B, the operator

$$S_{P\Lambda Q} : H \to H, \quad S_{P\Lambda Q} f = \int_\Omega Q\Lambda_w^*\Lambda_w P f \, d\mu(w), \quad \forall f \in H.$$

is called the frame operator of $\{\Lambda_w\}_w$. It is a positive invertible operator and $S_{P\Lambda Q} = QS_\Lambda P$.

Example 1.5. Let P and Q be any positive definite matrices of order 2. For each $w \in [0, 2\pi]$, we define $\Lambda_w = \begin{pmatrix} \cos w \\ \sin w \end{pmatrix}$. Then $\{P\Lambda_w Q\}$ is a (P,Q)-controlled continuous g-frame for $\mathbb{R}^2$.

2 Controlled continuous g-frames in Hilbert spaces

In this section, we present certain conditions under which continuous g-frames become (P,Q)-controlled continuous g-frames. Given any operators $P, Q \in GL^+(H)$, if we have a (P,Q)-controlled continuous g-frame $\{\Lambda_w\}_w$ for H with respect to $\{H_w\}_w$, we shall get frame bounds of the continuous g-frame $\{\Lambda_w\}_w$ in terms of controlled continuous g-frame bounds and operator norms of $(PQ)^{\frac{1}{2}}$ and $(PQ)^{-\frac{1}{2}}$. It is very useful in estimating bounds of some continuous g-frame provided it is a (P,Q)-controlled continuous g-frame for some operators $P, Q \in GL^+(H)$.

Theorem 2.1. *Let $P, Q \in GL^+(H)$. Then $\{\Lambda_w\}_{w\in\Omega}$ is a (P,Q)-controlled continuous g-frame for H with respect to $\{H_w\}_w$ if and only if $\{\Lambda_w\}_w$ is a continuous g-frame for H with respect to $\{H_w\}_w$.*

Proof. Assume that $\{\Lambda_w\}_w$ is a (P,Q)-controlled continuous g-frame for H with respect to $\{H_w\}_w$ with bounds A and B. Then for each $f \in H$, we have

$$\begin{aligned}
A\|f\|^2 &= A\|(PQ)^{\frac{1}{2}}(PQ)^{-\frac{1}{2}}f\|^2 \\
&\le A\|(PQ)^{\frac{1}{2}}\|^2\|(PQ)^{-\frac{1}{2}}f\|^2 \\
&\le \|(PQ)^{\frac{1}{2}}\|^2 \int_\Omega \langle \Lambda_w P(PQ)^{-\frac{1}{2}}f, \Lambda_w Q(PQ)^{-\frac{1}{2}}f\rangle \, d\mu(w) \\
&= \|(PQ)^{\frac{1}{2}}\|^2 \Big\langle \int_\Omega Q\Lambda_w^*\Lambda_w P(PQ)^{-\frac{1}{2}}f \, d\mu(w), (PQ)^{-\frac{1}{2}}f \Big\rangle \\
&= \|(PQ)^{\frac{1}{2}}\|^2 \Big\langle QS_\Lambda P(PQ)^{-\frac{1}{2}}f, (PQ)^{-\frac{1}{2}}f \Big\rangle \\
&= \|(PQ)^{\frac{1}{2}}\|^2 \langle S_\Lambda P^{\frac{1}{2}}Q^{-\frac{1}{2}}f, Q^{\frac{1}{2}}P^{-\frac{1}{2}}f\rangle \\
&= \|(PQ)^{\frac{1}{2}}\|^2 \langle Q^{\frac{1}{2}}P^{-\frac{1}{2}}S_\Lambda P^{\frac{1}{2}}Q^{-\frac{1}{2}}f, f\rangle \\
&= \|(PQ)^{\frac{1}{2}}\|^2 \langle S_\Lambda f, f\rangle,
\end{aligned}$$

hence,

$$\frac{A}{\|(PQ)^{\frac{1}{2}}\|^2}\|f\|^2 \le \int_\Omega \|\Lambda_w f\|^2 \, d\mu(w).$$

On the other hand for each $f \in H$, we have

$$\begin{aligned}\int_{\Omega} \|\Lambda_w f\|^2 \, d\mu(w) &= \langle S_\Lambda f, f\rangle \\ &= \langle (PQ)^{-\frac{1}{2}}(PQ)^{\frac{1}{2}} S_\Lambda f, f\rangle \\ &= \langle (PQ)^{\frac{1}{2}} S_\Lambda f, (PQ)^{-\frac{1}{2}} f\rangle \\ &= \langle S_\Lambda (PQ)(PQ)^{-\frac{1}{2}} f, (PQ)^{-\frac{1}{2}} f\rangle \\ &= \langle Q S_\Lambda P (PQ)^{-\frac{1}{2}} f, (PQ)^{-\frac{1}{2}} f\rangle \\ &= \langle S_{P\Lambda Q} (PQ)^{-\frac{1}{2}} f, (PQ)^{-\frac{1}{2}} f\rangle \\ &\leq B\|(PQ)^{-\frac{1}{2}}\|^2 \|f\|^2.\end{aligned}$$

Finally, we conclude that $\{\Lambda_w\}_w$ is a continuous g-frame for H with respect to $\{H_w\}_w$ with bounds $\frac{A}{\|(PQ)^{\frac{1}{2}}\|^2}$ and $B\|(PQ)^{-\frac{1}{2}}\|^2$.

Conversely, suppose that $\{\Lambda_w\}_w$ is a continuous g-frame for H with respect to $\{H_w\}_w$ with bounds A and B. Then,

$$A\langle f, f\rangle \leq \langle S_\Lambda f, f\rangle \leq B\langle f, f\rangle, \quad \forall f \in H.$$

Since $P, Q \in GL^+(H)$, by Lemma 1.1, there exist constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ such that

$$\alpha_1 I_H \leq P \leq \beta_1 I_H, \quad \alpha_2 I_H \leq Q \leq \beta_2 I_H.$$

So,

$$\alpha_1\alpha_2 A\|f\|^2 \leq \int_{\Omega} \langle \Lambda_w P f, \Lambda_w Q f\rangle \, d\mu(w) \leq \beta_1\beta_2 B\|f\|^2, \quad \forall f \in H.$$

Therefore $\{\Lambda_w\}_w$ is a (P, Q)-controlled continuous g-frame for H with respect to $\{H_w\}_w$. □

Corollary 2.2. *Let $\{\Lambda_w\}_{w\in\Omega}$ be a sequence in $L(H, H_w)$. Then the following statements are equivalent :*

(a) $\{\Lambda_w\}_{w\in\Omega}$ is a continuous g-frame.

(b) For every $P, Q \in GL^+(H)$, $\{\Lambda_w\}_{w\in\Omega}$ is a (P, Q)-controlled continuous g-frame.

(c) There exist $R, S \in GL^+(H)$ such that $\{\Lambda_w\}_{w\in\Omega}$ is a (R, S)-controlled continuous g-frame.

Definition 2.3. Let (Ω, μ) be a positive measure space. For each $w \in \Omega$, let Ω_w be a measurable subset of Ω. We say that the collection $\{e_{w,v}\}_{v\in\Omega_w}$ is a continuous orthonormal basis of H_w if

(1) $f = \int_{\Omega_w} \langle f, e_{w,v}\rangle e_{w,v} \, d\mu(v)$, for all $f \in H_w$;

(2) $\langle e_{w,v}, e_{w,v}\rangle = 1, \forall v \in \Omega_w$;

(3) $\langle e_{w,v_1}, e_{w,v_2}\rangle = 0, \forall v_1, v_2 \in \Omega_w$ and $v_1 \neq v_2$.

Theorem 2.4. *Let $P, Q \in GL^+(H)$. Then $\{\Lambda_w\}_w$ is a (P, Q)-controlled continuous g-frame for H with respect to $\{H_w\}_w$ if and only if $\{u_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ is a (P, Q)-controlled continuous g-frame for H, where $\{u_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ is the sequence induced by $\{\Lambda_w\}_w$ with respect to $\{e_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ (i.e., $u_{w,v} = \Lambda_w^* e_{w,v}$).*

Proof. For each $w \in \Omega$, let $\{e_{w,v}\}_{v\in\Omega_w}$ be an continuous orthonormal basis for H_w. Then we have

$$\Lambda_w P f = \int_{\Omega_w} \langle \Lambda_w P f, e_{w,v}\rangle e_{w,v} \, d\mu(v) = \int_{\Omega_w} \langle f, P\Lambda_w^* e_{w,v}\rangle e_{w,v} \, d\mu(v)$$

and

$$\Lambda_w Q f = \int_{\Omega_w} \langle \Lambda_w Q f, e_{w,v}\rangle e_{w,v} d\mu(v) = \int_{\Omega_w} \langle f, Q\Lambda_w^* e_{w,v}\rangle e_{w,v} \, d\mu(v).$$

It is easy to check that

$$\langle \Lambda_w Pf, \Lambda_w Qf \rangle = \int_{\Omega_w} \langle f, Pu_{w,v} \rangle \langle Qu_{w,v}, f \rangle \, d\mu(v)$$

$$\int_{\Omega} \langle \Lambda_w Pf, \Lambda_w Qf \rangle \, d\mu(w) = \int_{\Omega} \int_{\Omega_w} \langle f, Pu_{w,v} \rangle \langle Qu_{w,v}, f \rangle \, d\mu(v) \, d\mu(w).$$

Since

$$A\|f\|^2 \leq \int_{\Omega} \langle \Lambda_w Pf, \Lambda_w Qf \rangle \, d\mu(w) \leq B\|f\|^2, \quad \forall f \in H,$$

we get that

$$A\|f\|^2 \leq \int_{\Omega} \int_{\Omega_w} \langle f, Pu_{w,v} \rangle \langle Qu_{w,v}, f \rangle \, d\mu(v) \, d\mu(w) \leq B\|f\|^2, \quad \forall f \in H.$$

The proof is completed. □

Proposition 2.5. *Let $P, Q \in GL^+(H)$. Then $\{\Lambda_w\}_w$ is a (P,Q)-controlled continuous g-frame for H with respect to $\{H_w\}_w$ if and only if $\{Pu_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ is a QP^{-1}-controlled continuous g-frame for H, where $\{u_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ is the sequence induced by $\{\Lambda_w\}_w$ with respect to $\{e_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ (i.e., $u_{w,v} = \Lambda_w^* e_{w,v}$).*

Proof. From the proof of Theorem 2.4, we have

$$\int_{\Omega} \langle \Lambda_w Pf, \Lambda_w Qf \rangle \, d\mu(w) = \int_{\Omega} \int_{\Omega_w} \langle f, Pu_{w,v} \rangle \langle Qu_{w,v}, f \rangle \, d\mu(v) \, d\mu(w).$$

Then

$$A\|f\|^2 \leq \int_{\Omega} \langle \Lambda_w Pf, \Lambda_w Qf \rangle \, d\mu(w) \leq B\|f\|^2, \quad \forall f \in H,$$

is equivalent to

$$A\|f\|^2 \leq \int_{\Omega} \int_{\Omega_w} \langle f, Pu_{w,v} \rangle \langle QP^{-1} Pu_{w,v}, f \rangle d\mu(v) \, d\mu(w) \leq B\|f\|^2, \quad \forall f \in H.$$

Hence the proof is completed. □

Combining all results presented in the section, we get the following result which gives several interesting characterizations of (P,Q)-controlled continuous g-frames for H with respect to $\{H_w\}_w$.

Corollary 2.6. *Let $P, Q \in GL^+(H)$. Then the following are equivalent:*

(i) $\{\Lambda_w\}_w$ is a (P,Q)-controlled continuous g-frame for H with respect to $\{H_w\}_w$.

(ii) $\{\Lambda_w\}_w$ is a $((PQ)^{\frac{1}{2}}, (PQ)^{\frac{1}{2}})$-controlled continuous g-frame for H with respect to $\{H_w\}_w$.

(iii) $\{\Lambda_w\}_w$ is a QP-controlled continuous g-frame for H with respect to $\{H_w\}_w$.

(iv) $\{u_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ is a (P,Q)-controlled continuous g-frame for H, where $\{u_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ is the sequence induced by $\{\Lambda_w\}_w$ with respect to $\{e_{w,v}\}_{w\in\Omega, v\in\Omega_w}$.

(v) $\{Pu_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ is a QP^{-1}-controlled continuous g-frame for H, where $\{u_{w,v}\}_{w\in\Omega, v\in\Omega_w}$ is the sequence induced by $\{\Lambda_w\}_w$ with respect to $\{e_{w,v}\}_{w\in\Omega, v\in\Omega_w}$.

(vi) $\{\Lambda_w\}_w$ is a continuous g-frame for H with respect to$\{H_w\}_w$.

Moreover, $\{\Lambda_w\}_w$ is a continuous g-frame for H with respect to $\{H_w\}_w$ with bounds $\frac{A}{\|(PQ)^{\frac{1}{2}}\|^2}$ and $B\|(PQ)^{-\frac{1}{2}}\|^2$.

3 Controlled continuous dual g-frames in Hilbert spaces

In this section, we define controlled continuous dual g-frames and characterize them by operator theory.

Definition 3.1. Let $P, Q \in GL^+(H)$ and let $\{\Lambda_w\}_w, \{\Gamma_w\}_w$ be (P, P)-controlled continuous and (Q, Q)-controlled continuous g-Bessel families for H with respect to $\{H_w\}_w$, respectively. We say that $\{\Gamma_w\}_w$ is a (P, Q)-controlled continuous dual g-frame of $\{\Lambda_w\}_w$ if

$$f = \int_\Omega P\Lambda_w^*\Gamma_w Qf \, d\mu(w), \quad \forall f \in H.$$

In particular, if $Q = I_H$, $\{\Gamma_w\}_w$ is called a P-controlled continuous dual g-frame of $\{\Lambda_w\}_w$.

Definition 3.2. Let $P, Q \in GL^+(H)$ and let $\{\Lambda_w\}_w, \{\Gamma_w\}_w$ be (P, P)-controlled continuous and (Q, Q)-controlled continuous g-Bessel families for H with respect to $\{H_w\}_w$ respectively. For the pair $\{\Lambda_w\}_w$ and $\{\Gamma_w\}_w$, we define a (P, Q)-controlled continuous dual g-frame operator $S_{P\Lambda\Gamma Q}$ by

$$S_{P\Lambda\Gamma Q}f = \int_\Omega P\Lambda_w^*\Gamma_w Qf \, d\mu(w), \quad \forall f \in H.$$

It is easy to check that $S_{P\Lambda\Gamma Q}$ is a well-defined bounded operator, and

$$S_{P\Lambda\Gamma Q} = T_{P\Lambda P}T^*_{Q\Gamma Q} = PT_\Lambda T^*_\Gamma Q = PS_{\Lambda\Gamma}Q,$$

where $S_{\Lambda\Gamma}f = \int_\Omega \Lambda_w^*\Gamma_w f \, d\mu(w)$. Note that $\{\Gamma_w\}_w$ is a (P, Q)-controlled continuous dual g-frame of $\{\Lambda_w\}_w$ if and only if $S_{P\Lambda\Gamma Q} = I_H$.

Theorem 3.3. *Let $P, Q \in GL^+(H)$ and let $\{\Lambda_w\}_w, \{\Gamma_w\}_w$ be (P, P)-controlled continuous and (Q, Q)-controlled continuous g-Bessel families with bounds B_Λ and B_Γ respectively. If $S_{P\Lambda\Gamma Q}$ is bounded below, then $\{\Lambda_w\}_w$ and $\{\Gamma_w\}_w$ are (P, P)-controlled continuous and (Q, Q)-controlled continuous g-frames respectively.*

Proof. Suppose that there exists a constant $\lambda > 0$ such that

$$\|S_{P\Lambda\Gamma Q}f\| \geq \lambda\|f\|, \quad \forall f \in H.$$

Hence

$$\begin{aligned}
\lambda\|f\| &\leq \|S_{P\Lambda\Gamma Q}f\| \\
&= \sup_{\|g\|=1} \left|\langle \int_\Omega P\Lambda_w^*\Gamma_w Qf \, d\mu(w), g\rangle\right| \\
&= \sup_{\|g\|=1} \left| \int_\Omega \langle \Gamma_w Qf, \Lambda_w Pg\rangle \, d\mu(w)\right| \\
&\leq \sup_{\|g\|=1} \left(\int_\Omega \|\Gamma_w Qf\|^2 \, d\mu(w))\right)^{\frac{1}{2}} \left(\int_\Omega \|\Lambda_w Pg\|^2 \, d\mu(w)\right)^{\frac{1}{2}} \\
&\leq \sqrt{B_\Lambda}\left(\int_\Omega \|\Gamma_w Qf\|^2 \, d\mu(w)\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus

$$\frac{\lambda^2}{B_\Lambda}\|f\|^2 \leq \int_\Omega \|\Gamma_w Qf\|^2 \, d\mu(w), \quad \forall f \in H.$$

On the other hand, Since

$$S^*_{P\Lambda\Gamma Q} = (PS_{\Lambda\Gamma}Q)^* = QS_{\Gamma\Lambda}P = S_{Q\Gamma\Lambda P},$$

then $S_{Q\Gamma\Lambda P}$ is also bounded below. Similarly, we can prove that $\{\Lambda_w\}_w$ is a (P, P)-controlled continuous g-frame. Hence the proof is completed. □

Theorem 3.4. *Let $P, Q \in GL^+(H)$, $\{\Lambda_w\}_w$ and let $\{\Gamma_w\}_w$ be (P,P)-controlled continuous and (Q,Q)-controlled continuous g-Bessel families for H with respect to $\{H_w\}_w$ respectively. Then the following conditions are equivalent:*

(1) $f = \int_\Omega P\Lambda_w^*\Gamma_w Qf \, d\mu(w)$, $\forall f \in H$.

(2) $f = \int_\Omega Q\Gamma_w^*\Lambda_w Pf \, d\mu(w)$, $\forall f \in H$.

(3) $\langle f, g\rangle = \int_\Omega \langle \Lambda_w Pf, \Gamma_w Qg\rangle \, d\mu(w) = \int_\Omega \langle \Gamma_w Qf, \Lambda_w Pg\rangle \, d\mu(w)$, $\forall f, g \in H$.

(4) $\|f\|^2 = \int_\Omega \langle \Lambda_w Pf, \Gamma_w Qf\rangle \, d\mu(w) = \int_\Omega \langle \Gamma_w Qf, \Lambda_w Pf\rangle \, d\mu(w)$, $\forall f \in H$.

In case the equivalent conditions are satisfied, $\{\Lambda_w\}_w$ and $\{\Gamma_w\}_w$ are (P,P)-controlled continuous and (Q,Q)-controlled continuous g-frames respectively.

Proof. (1) $\Longleftrightarrow$ (2). Let $T_{P\Lambda P}$ be the synthesis operator of the (P,P)-controlled continuous g-Bessel family $\{\Lambda_w\}_w$ and $T_{Q\Gamma Q}$ be the synthesis operator of the (Q,Q)-controlled continuous g-Bessel family $\{\Gamma_w\}_w$. Condition (1) means that $T_{P\Lambda P}T^*_{Q\Gamma Q} = I_H$, and it is equivalent to $T_{Q\Gamma Q}T^*_{P\Lambda P}$, which is identical to the statement (2). Conversely, (2) $\Longrightarrow$ (1) similarly.

(2) $\Longleftrightarrow$ (3). It is clear that (2) $\Longrightarrow$ (3). For any $f, g \in H$, $\langle f, g\rangle = \int_\Omega \langle \Lambda_w Pf, \Gamma_w Qg\rangle \, d\mu(w)$ shows that

$$\Big\langle f - \int_\Omega Q\Gamma_w^*\Lambda_w Pf \, d\mu(w), g\Big\rangle = 0, \quad \forall g \in H.$$

Hence (3) $\Longrightarrow$ (2) is proved.

(3) $\Longleftrightarrow$ (4). (3) $\Longrightarrow$ (4) is obvious. To prove (4) $\Longrightarrow$ (3), we apply condition (4) and get

$$\begin{aligned}\|f+g\|^2 &= \int_\Omega \langle \Lambda_w P(f+g), \Gamma_w Q(f+g)\rangle \, d\mu(w) \\ &= \int_\Omega \langle \Lambda_w Pf, \Gamma_w Qf\rangle \, d\mu(w) + \int_\Omega \langle \Lambda_w Pf, \Gamma_w Qg\rangle \, d\mu(w) \\ &\quad + \int_\Omega \langle \Lambda_w Pg, \Gamma_w Qf\rangle \, d\mu(w) + \int_\Omega \langle \Lambda_w Pg, \Gamma_w Qg\rangle \, d\mu(w).\end{aligned}$$

Similarly,

$$\begin{aligned}\|f-g\|^2 &= \int_\Omega \langle \Lambda_w Pf, \Gamma_w Qf\rangle \, d\mu(w) - \int_\Omega \langle \Lambda_w Pf, \Gamma_w Qg\rangle \, d\mu(w) \\ &\quad - \int_\Omega \langle \Lambda_w Pg, \Gamma_w Qf\rangle \, d\mu(w) + \int_\Omega \langle \Lambda_w Pg, \Gamma_w Qg\rangle \, d\mu(w).\end{aligned}$$

$$\begin{aligned}\|f+ig\|^2 &= \int_\Omega \langle \Lambda_w Pf, \Gamma_w Qf\rangle \, d\mu(w) - i\int_\Omega \langle \Lambda_w Pf, \Gamma_w Qg\rangle \, d\mu(w) \\ &\quad + i\int_\Omega \langle \Lambda_w Pg, \Gamma_w Qf\rangle \, d\mu(w) + \int_\Omega \langle \Lambda_w Pg, \Gamma_w Qg\rangle \, d\mu(w).\end{aligned}$$

$$\begin{aligned}\|f-ig\|^2 &= \int_\Omega \langle \Lambda_w Pf, \Gamma_w Qf\rangle \, d\mu(w) + i\int_\Omega \langle \Lambda_w Pf, \Gamma_w Qg\rangle \, d\mu(w) \\ &\quad - i\int_\Omega \langle \Lambda_w Pg, \Gamma_w Qf\rangle \, d\mu(w) + \int_\Omega \langle \Lambda_w Pg, \Gamma_w Qg\rangle \, d\mu(w).\end{aligned}$$

By polarization identity,

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{4}\Big(\|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2\Big) \\ &= \int_\Omega \langle \Lambda_w Pf, \Gamma_w Qg\rangle \, d\mu(w).\end{aligned}$$

In case the equivalent conditions are satisfied, $S_{Q\Gamma\Lambda P} = I_H$ implies $\|S_{Q\Gamma\Lambda P}\| = 1$, hence by Theorem 3.3, $\{\Lambda_w\}_w$ and $\{\Gamma_w\}_w$ are (P,P)-controlled continuous and (Q,Q)-controlled continuous g-frames respectively. □

Theorem 3.5. *Let $P, Q \in GL^+(H)$. A sequence $\{\Lambda_w\}_w$ is a (P,Q)-controlled continuous g-Bessel family for H with respect to $\{H_w\}_w$ with bound B if and only if the operator $T_{P\Lambda Q} : \ell^2(\{H_w\}_w) \to H$ given by*

$$T_{P\Lambda Q}(\{f_w\}_w) = \int_\Omega (PQ)^{\frac{1}{2}} \Lambda_w^* f_w \, d\mu(w)$$

is well-defined and bounded with $\|T_{P\Lambda Q}\| \leq \sqrt{B}$.

Proof. The necessary condition follows from the definition of (P,Q)-controlled continuous g-Bessel sequence. We only need to prove the sufficient condition. Suppose that $T_{P\Lambda Q}$ is well-defined and bound operator with $\|T_{P\Lambda Q}\| \leq \sqrt{B}$. For any $f \in H$, we have

$$\begin{aligned}
\int_\Omega \langle \Lambda_w P f, \Lambda_w Q f\rangle \, d\mu(w) &= \int_\Omega \langle Q\Lambda_w^* \Lambda_w P f, f\rangle \, d\mu(w) \\
&= \langle Q S_\Lambda P f, f\rangle \\
&= \langle (QP)^{\frac{1}{2}} S_\Lambda (QP)^{\frac{1}{2}} f, f\rangle \\
&= \Big\langle \int_\Omega (QP)^{\frac{1}{2}} \Lambda_w^* \Lambda_w (QP)^{\frac{1}{2}} f \, d\mu(w), f \Big\rangle \\
&\leq \|T_{P\Lambda Q}\| \Big(\int_\Omega \|\Lambda_w (QP)^{\frac{1}{2}} f\|^2 \, d\mu(w)\Big)^{\frac{1}{2}} \|f\| \\
&= \|T_{P\Lambda Q}\| \Big(\int_\Omega \langle \Lambda_w P f, \Lambda_w Q f\rangle \, d\mu(w)\Big)^{\frac{1}{2}} \|f\|.
\end{aligned}$$

Hence we get

$$\int_\Omega \langle \Lambda_w P f, \Lambda_w Q f\rangle \, d\mu(w) \leq \|T_{P\Lambda Q}\|^2 \|f\|^2 \leq B\|f\|^2.$$

This shows that $\{\Lambda_w\}_w$ is a (P,Q)-controlled continuous g-Bessel family for H with respect to $\{H_w\}_w$ with bound B. □

Theorem 3.6. *Let $P, Q \in GL^+(H)$ and $\{\Lambda_w\}_w$ be a (P,P)-controlled g-frame for H with respect to $\{H_w\}_w$ with the synthesis operator $T_{P\Lambda P}$. Then a (Q,Q)-controlled continuous g-frame $\{\Gamma_w\}_w$ is a (P,Q)-controlled continuous dual g-frame of $\{\Lambda_w\}_w$ if and only if*

$$Q\Gamma_w^* e_{w,v} = U(e_{w,v}\delta_w), \quad w \in \Omega, v \in \Omega_w$$

where $U : \ell^2(\{H_w\}_w) \to H$ is a bounded left-inverse of $T_{P\Lambda P}^*$.

Proof. If $\{g_w\}_w \in \ell^2(\{H_w\}_w)$, then

$$\{g_w\}_w = \int_\Omega g_w \delta_w \, d\mu(w) = \int_\Omega \int_{\Omega_w} \langle g_w, e_{w,v}\rangle e_{w,v}\delta_w \; d\mu(v) \, d\mu(w).$$

Roughly speaking, $\{e_{w,v}\delta_w\}_{w\in\Omega, v\in\Omega_w}$ is a continuous orthonormal basis of $\ell^2(\{H_w\}_w)$. Suppose that there exists a bounded left-inverse $U : \ell^2(\{H_w\}_w) \to H$ of $T_{P\Lambda P}^*$ such that

$$Q\Gamma_w^* e_{w,v} = U(e_{w,v}\delta_w), \quad w \in \Omega, v \in \Omega_w.$$

For any $f \in H$, we have

$$\begin{aligned} f &= UT^*_{P\Lambda P} f \\ &= U\Big(\int_\Omega \int_{\Omega_w} \langle \Lambda_w Pf, e_{w,v}\rangle e_{w,v}\delta_w \, d\mu(v)\, d\mu(w)\Big) \\ &= \int_\Omega \int_{\Omega_w} \langle f, P\Lambda^*_w e_{w,v}\rangle U(e_{w,v}\delta_w)\, d\mu(v)\, d\mu(w) \\ &= \int_\Omega \int_{\Omega_w} \langle f, Pu_{w,v}\rangle Q\Gamma^*_w e_{w,v}\, d\mu(v)\, d\mu(w) \\ &= \int_\Omega Q\Gamma^*_w \Big(\int_{\Omega_w} \langle Pf, u_{w,v}\rangle e_{w,v}\, d\mu(v)\Big)\, d\mu(w) \\ &= \int_\Omega Q\Gamma^*_w \Lambda_w Pf\, d\mu(w), \end{aligned}$$

where $u_{w,v} = \Lambda^*_w e_{w,v}$. By the definition of controlled continuous dual g-frame, $\{\Gamma_w\}_w$ is a (P,Q)-controlled continuous dual g-frame of $\{\Lambda_w\}_w$.

On the other hand, suppose that a (Q,Q)-controlled continuous g-frame $\{\Gamma_w\}_w$ is a (P,Q)-controlled continuous dual g-frame of $\{\Lambda_w\}_w$. For any $f \in H$, we have

$$f = \int_\Omega P\Lambda^*_w \Gamma^*_w Qf\, d\mu(w) = \int_\Omega Q\Gamma^*_w \Lambda_w Pf\, d\mu(w),$$

that is, $T_{Q\Lambda Q}T^*_{P\Lambda P} = I_H$. Let $U = T_{Q\Gamma Q}$. Then $U : \ell^2(\{H_w\}_w) \to H$ is a bounded left-inverse of $T^*_{P\Lambda P}$. A calculation as above shows that

$$\int_\Omega \int_{\Omega_w} \langle f, Pu_{w,v}\rangle Q\Gamma^*_w e_{w,v}\, d\mu(v)\, d\mu(w) = f = \int_\Omega \int_{\Omega_w} \langle f, Pu_{w,v}\rangle U(e_{w,v}\delta_w)\, d\mu(v)\, d\mu(w), \forall f \in H.$$

Combining this with the fact $\{e_{w,v}\}_{v\in\Omega_w}$ is a continuous orthonormal basis of H_w, we have

$$Q\Gamma^*_w e_{w,v} = U(e_{w,v}\delta_w), \quad w \in \Omega, v \in \Omega_w.$$

□

Theorem 3.7. *Let $P \in GL^+(H)$ and $\{\Lambda_w\}_w$ be a (P,P)-controlled continuous g-frame for H with respect to $\{H_w\}_w$ with the synthesis operator and frame operator $T_{P\Lambda P}$ and $S_{P\Lambda P}$, respectively. Then $\{\Gamma_w\}_w$ is a P-controlled continuous dual g-frame of $\{\Lambda_w\}_w$ if and only if*

$$\Gamma_w f = (Tf)_w + \Lambda_w S^{-1}_{P\Lambda P} Pf, w \in \Omega, f \in H,$$

where $T : H \to \ell^2(\{H_w\}_w)$ is a bounded linear operator satisfying $T_{P\Lambda P}T = 0$.

Proof. If $T : H \to \ell^2(\{H_w\}_w)$ is a bounded linear operator satisfying $T_{P\Lambda P}T = 0$, then $\{\Gamma_w\}_w$ is a g-Bessel family for H with respect to $\{H_w\}_w$. In fact, any $f \in H$ we have

$$\begin{aligned} \int_\Omega \|\Gamma_w f\|^2\, d\mu(w) &= \int_\Omega \|(Tf)_w + \Lambda_w S^{-1}_{P\Lambda P} Pf\|^2\, d\mu(w) \\ &\le 2\Big(\int_\Omega \|\Lambda_w S^{-1}_{P\Lambda P} Pf\|^2\, d\mu(w) + \|Tf\|^2\Big) \\ &\le 2(B\|S^{-1}_{P\Lambda P} P\|^2 + \|T\|^2)\|f\|^2, \end{aligned}$$

where B is the upper bound of $\{\Lambda_w\}_{w\in\Omega}$. Furthermore,

$$\begin{aligned} \int_\Omega P\Lambda^*_w \Gamma_w f\, d\mu(w) &= \int_\Omega P\Lambda^*_w ((Tf)_w + \Lambda_w S^{-1}_{P\Lambda P} Pf)\, d\mu(w) \\ &= T_{P\Lambda P}Tf + \int_\Omega P\Lambda^*_w \Lambda_w S^{-1}_{P\Lambda P} Pf\, d\mu(w) = f. \end{aligned}$$

Thus, $\{\Gamma_w\}_{w\in\Omega}$ is a P-controlled continuous dual g-frame of $\{\Lambda_w\}_{w\in\Omega}$.

Now, we prove the converse. Assume that $\{\Gamma_w\}_{w\in\Omega}$ is a P-controlled continuous dual g-frame of $\{\Lambda_w\}_{w\in\Omega}$. Define the operator T as follows:

$$T : H \to \ell^2(\{H_w\}_{w\in\Omega}), \quad f \to (Tf)_w, \quad f \in H$$

satisfying

$$\Gamma_w f = (Tf)_w + \Lambda_w S_{P\Lambda P}^{-1} P f, w \in \Omega.$$

For any $f \in H$, we have

$$\begin{aligned}
\|Tf\|^2 &= \int_\Omega \|\Gamma_w f - \Lambda_w S_{P\Lambda P}^{-1} P f\|^2 \, d\mu(w) \\
&\le \int_\Omega \|\Gamma_w f\|^2 \, d\mu(w) + \int_\Omega \|\Lambda_w S_{P\Lambda P}^{-1} P f\|^2 \, d\mu(w) \\
&\quad + 2\Big(\int_\Omega \|\Gamma_w f\|^2 \, d\mu(w)\Big)^{\frac{1}{2}} \Big(\int_\Omega \|\Lambda_w S_{P\Lambda P}^{-1} P f\|^2 \, d\mu(w)\Big)^{\frac{1}{2}} \\
&\le (B_1 + A^{-1} + 2\sqrt{B_1 A^{-1}})\|f\|^2,
\end{aligned}$$

where B_1 is the frame upper bound of $\{\Gamma_w\}_{w\in\Omega}$ and A is the frame lower bound of $\{\Lambda_w\}_{w\in\Omega}$. Thus, T is a linear bounded operator. Moreover, for any $f, g \in H$, we have

$$\begin{aligned}
\langle T_{P\Lambda P} T f, g\rangle &= \int_\Omega \langle P\Lambda_w^* T f, g\rangle \, d\mu(w) \\
&= \int_\Omega \langle P\Lambda_w^* (\Gamma_w f - \Lambda_w S_{P\Lambda P}^{-1} P f), g\rangle \, d\mu(w) \\
&= \int_\Omega \langle P\Lambda_w^* \Gamma_w f, g\rangle \, d\mu(w) - \int_\Omega \langle P\Lambda_w^* \Lambda_w S_{P\Lambda P}^{-1} P f, g\rangle \, d\mu(w) \\
&= \langle f, g\rangle - \langle f, g\rangle = 0.
\end{aligned}$$

That is, $T_{P\Lambda P} T = 0$. The proof is completed. □

4 Conclusion

In summary, this paper advances the study of Hilbert space frame theory by extending standard continuous g-frames into the realm of controlled continuous g-frames. By providing a comprehensive characterization of their dual frames, the research optimizes vector reconstruction and data transmission efficiency. These results offer a significant generalization of existing literature, providing more versatile and robust mathematical tools for functional analysis.

References

[1] *M. R. Abdollahpour, M. H. Faroughi, Continuous g-frames in Hilbert spaces, Southeast Asian Bull. Math. 32 (2008), 1-19.*

[2] *Y. Alizadeh, M. Abdollahpour, Controlled continuous g-frames and their multipliers in Hilbert spaces, Sahand Commun. Math. Anal., 15(1) (2019), 37-48.*

[3] *S. T. Ali, J. P. Antoine, J. P. Gazeau,* Continuous frames in Hilbert spaces*, Ann. Phys.* **222** *(1993), 1-37.*

[4] *A. Askari-Hemmat, M. A. Dehghan, M. Radjabalipour,* Generalized frames and their redundancy*, Proc. Am. Math. Soc.* **129** *(2001), 1143-1147.*

[5] *P. Balazs, D. Bayer, A. Rahimi, Multipliers for continuous frames in Hilbert spaces, J. Phys. A Math. Theor. 45, 2240023 (20p) (2012).*

[6] *I. Daubechies, A. Grossmann, Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys.,* **27** *(1986), 1271-1283.*

[7] *R. J. Duffin, A. C. Schaeffer,* A class of nonharmonic fourier series*, Trans. Am. Math. Soc.* **72** *(1952), 341-366.*

[8] *R. El Jazzar, M. Rossafi, M. Mouniane, K-g-frames in Hilbert modules over locally-C^*-algebras, Palest. J. Math. 14(4) (2025), 44-56.*

[9] *J. P. Gabardo, D. Han,* Frames associated with measurable space, *Adv. Comput. Math.* **18** *(2003), no. 3, 127-147.*

[10] *D. Gabor,* Theory of communications, *J. Elect. Eng.* **93** *(1946), 429-457.*

[11] *G. Kaiser,* A friendly guide to wavelets, *Birkhäuser, Boston, 1994.*

[12] *A. Karara, M. Rossafi, M. Klilou, S. Kabbaj, Construction of continuous K-g-Frames in Hilbert C^*-Modules, Palest. J. Math. 13(4) (2024), 198-209.*

[13] *K. Mahesh Krishna, P. Sam Johnson, Dilation theorem for p-approximate Schauder frames for separable Banach spaces, Palest. J. Math. 11 (2) (2022) 384-394.*

[14] *S. Ramesan, K. T. Ravindran, Scalability and K-frames, Palest. J. Math. 12 (1) (2023) 493-500.*

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