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Arithmetic Properties of Congruences for (a, b, m)-Copartitions

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Abstract We consider $cp_{a,b,m}(n)$, the number of (a, b, m)-copartitions of n. We find infinitely many congruences modulo 2 and 3 for some particular value of a, b and m applying theta function identities.

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. If $p(n)$ denote the number of partition of n, then the generating function of n is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where, as is customary, for any complex number a and $|q| < 1$,

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{n-1}).$$

Ramanujan's general theta function is given by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \tag{1.1}$$

$$= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \qquad |ab| < 1. \tag{1.2}$$

Identity (1.2) is the Jacobi's triple product identity in Ramanujan's notation [2, Ch. 16, Entry 19]. It follows from (1.1) and (1.2) that [2, Ch. 16, Entry 22],

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1. \tag{1.3}$$

Andrews studied interesting properties of $EO^*(n)$, including that its generating function is simply, $\frac{1}{2}(\nu(q) + \nu(-q))$, where

$$\nu(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}}$$

is Watson's third order mock theta function. Chern[5] provided a combinatorial proof of the generating function for $EO^*(n)$ and studied several further properties of $EO^*(n)$ in [6].

Burson and Eichhorn in [3] generalized $EO^*(n)$ by introducing new partition-theoretic objects called copartitions, which reveal an inherent symmetry in partitions counted by $EO^*(n)$ that was not previously obvious. Copartitions are counted by the function $cp_{a,b,m}(n)$, where $cp_{1,1,2}(n) = EO^*(2n)$ [3, Theorem 3.5].

Definition 1.1. An (a, b, m)-copartition is a triple of partition (γ, ρ, σ), where each of the parts of γ is at least a and congruent to $a \pmod{m}$, each of the parts of γ is at least b and congruent to $b \pmod{m}$, and ρ is a partition with σ parts, each of which has size equal to m times the number of parts of γ.

Example 1.2. The $(1, 3, 4)$-copartitions of size 12 are

$$(\{9, 1^3\}, \phi, \phi), (\{5^2, 1^2\}, \phi, \phi), (\{5, 1^7\}, \phi, \phi), (\{1^{12}\}, \phi, \phi)$$

$$(\{5\}, \{4\}, \{3\}), (\{1\}, \{4\}, \{7\}) and\ (\phi, \phi, \{3^4\})$$

$\ni The$(1,3,4)$-Copartitionsof$ 12$aretherefore countedby$ $\text{cp}_{1,3,4}(12) = 7$.
$\ni Nowthegeneratingfunctionforthecopartitionscanbegivenasfollows:$

Theorem 1.3. *[3, Theorem 3.6] Define $cp_{a,b,m}(w, s, n)$ to be the number of (a, b, m)-copartitions of size n that have w ground parts and s sky parts. Then,*

$$cp_{a,b,m}(x, y, q) = \sum_{n=0}^{\infty}\sum_{w=0}^{\infty}\sum_{s=0}^{\infty} cp_{a,b,m}(w, s, n)x^s y^w q^n = \frac{(xyq^{a+b}; q^m)_\infty}{(xq^b; q^m)_\infty (yq^a; q^m)_\infty} \tag{1.4}$$

$\ni BursonandEichhorn$[4]$alsostudiedtheparityof$ $\text{cp}_{3,1,4}(n)$ and $cp_{5,1,6}(n)$.

Corollary 1.4. *[4, Corollary 4.5] $\ni Foranyprimep$>3,$p \equiv 3 \pmod 4$, let $24\delta \equiv 1 \pmod{p^2}$. Then*

$$cp_{3,1,4}(p^2k + pt - 5\delta) \equiv 0 \pmod 2 \tag{1.5}$$

$\ni fort$=1, 2, $\cdots p - 1$ and every non-negative integer k.

Corollary 1.5. *[4, Corollary 4.10] For any prime $p > 2$, $p \equiv 2 \pmod 3$, let $6\delta \equiv 1 \pmod{p^2}$, we have*

$$cp_{5,1,6}(p^2k + pt - \delta) \equiv 0 \pmod 2 \tag{1.6}$$

for $t = 1, 2, \cdots, p - 1$ and every non-negative integer k.

In this paper, we establish more infinite families of congruences of $cp_{3,1,4}(n)$ and $cp_{5,1,6}(n)$ modulo 2 and 3.

2 Preliminary Results

In this section, we list few dissection formulas which are useful in proving our main results. As is customary, we use

$$f_k = f(-q^k) = \prod_{n=1}^{\infty}(1 - q^{nk}) = (q^k; q^k)_\infty, \quad k \geq 1.$$

$\ni Thefollowinglemmagivesa$p$-dissectionof$f(-q):

Lemma 2.1. *[7, Theorem 2.2] For any prime $p \geq 5$,*

$$f(-q) = \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{3p^2-(6k+1)p}{2}}, -q^{\frac{3p^2+(6k+1)p}{2}}) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}), \quad (2.1)$$

where $\pm$ depends on the conditions that $(\pm p-1)/6$ should be an integer. Moreover, note that $(3k^2+k)/2 \not\equiv (p^2-1)/24 \pmod{p}$ as k runs through the range of the summation.

Lemma 2.2. *For positive integers k and m, we have*

$$f_{2k}^m \equiv f_k^{2m} \pmod{2}.$$
$$f_{3k}^m \equiv f_k^{3m} \pmod{3}.$$

3 Main Results

In this section, we state and prove our main results.

Theorem 3.1. *For any prime $p \geq 5$ and $\alpha, n \geq 0$, we have*

$$\sum_{n=0}^{\infty} cp_{3,1,4}\left(p^{2\alpha}n + 5\frac{p^{2\alpha}-1}{24}\right) q^n \equiv f^5(-q) \pmod{2}. \quad (3.1)$$

Proof. Setting $x = y = 1$, $a = 3$, $b = 1$ and $m = 4$ in (1.4), we have

$$\sum_{n=0}^{\infty} cp_{3,1,4}(n)q^n = \frac{(q^4;q^4)_\infty}{(q;q^4)_\infty(q^3;q^4)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q^2)_\infty} = \frac{f_4 f_2}{f_1}. \quad (3.2)$$

From Lemma 2.2, we have

$$f_4 \equiv f_1^4 \pmod{2} \quad (3.3)$$

$\ni and f_2 \equiv f_1^2 \pmod{2}(3.4)$
$\ni Invoking (3.3) and (3.4) in (3.2), we obtain \sum_{n=0}^{\infty} cp_{3,1,4}(n)q^n \equiv f_1^5 \pmod{2}.(3.5)$ Applying Lemma 2.1, we obtain
$\ni \sum_{n=0}^{\infty} cp_{3,1,4}\left(p^{2\alpha}n + 5\frac{p^{2\alpha}-1}{24}\right) q^n$

$$\equiv \left(\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{3p^2-(6k+1)p}{2}}, -q^{\frac{3p^2+(6k+1)p}{2}}) + (-1)^{\pm\frac{p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}) \right)^5 \pmod{2}. \quad (3.6)$$

Extracting the term containing $q^{pn+5\frac{p^2-1}{24}}$ from both sides of (3.6) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} cp_{3,1,4}\left(p^{2\alpha+1}n + 5\frac{p^{2\alpha+1}-1}{24}\right) q^n \equiv f^5(-q^p) \pmod{2}. \quad (3.7)$$

Again extracting the term containing q^{pn} from both sides of (3.7) and replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} cp_{3,1,4}\left(p^{2\alpha+2}n + 5\frac{p^{2\alpha+2}-1}{24}\right) q^n \equiv f^5(-q) \pmod{2}, \quad (3.8)$$

which is the $\alpha+1$ term of (3.1). □

Corollary 3.2. *For $p \geq 5$, $\alpha \geq 1$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} cp_{3,1,4}\left(p^{2\alpha}n + \frac{(24j+5p)p^{2\alpha-1}-5}{24}\right)q^n \equiv 0 \pmod{2}, \tag{3.9}$$

where $j = 1, 2, \cdots p-1$.

Proof. Comparing the coefficients of q^{pn+j}, $1 \leq j \leq p-1$ in (3.7), we easily obtain (3.9). □

Corollary 3.3. *For $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, we have*

$$cp_{3,1,4}\left(p^{2\alpha+1}n + \frac{(24j+5)p^{2\alpha}-5}{24}\right) \equiv 0 \pmod{2}, \tag{3.10}$$

for $j = 1, 2, \cdots p-1$ and $\left(\frac{24j+5}{p}\right) = -1$.

Proof. According to Lemma (2.1) and Theorem 3.1, for any integer j with $0 \leq j \leq p-1$, if $j \not\equiv (3k^2+k)/2 \pmod{p}$ for $|k| \leq (p-1)/2$, then we have

$$cp_{3,1,4}\left(p^{2\alpha}(pn+j) + 5\frac{p^{2\alpha}-1}{24}\right) \equiv 0 \pmod{2},$$

which gives (3.10). □

Theorem 3.4. *For the primes $p_1, p_2, p_3 \cdots p_l$, $l \geq 1$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} cp_{3,1,4}\left(\prod_{s=1}^{l} p_s^2 + 5\left(\frac{\prod_{s=1}^{l} p_s^2 - 1}{24}\right)\right)q^n \equiv f^5(-q) \pmod{2}. \tag{3.11}$$

Proof. Proof can be completed by induction on l where the initial case is (3.1). Assume that (3.11) is true for l. Then based on Lemma 2.1 for prime p_{l+1}, we have

$$\sum_{n=0}^{\infty} cp_{3,1,4}\left(\prod_{s=1}^{l} p_s^2\left(p_{l+1}^2 n + 5\frac{p_{l+1}^2-1}{24}\right) + 5\left(\frac{\prod_{s=1}^{l} p_s^2 - 1}{24}\right)\right)q^n$$
$$= \sum_{n=0}^{\infty} cp_{3,1,4}\left(\prod_{s=1}^{l+1} p_s^2 n + 5\left(\frac{\prod_{s=1}^{l+1} p_s^2 - 1}{24}\right)\right)q^n \equiv f^5(-q) \pmod{2},$$

which is the case of $l+1$. □

Theorem 3.5. *For any prime $p \geq 5$ and α, $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} cp_{5,1,6}\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{6}\right) \equiv f_1^2 f_2 \pmod{3}. \tag{3.12}$$

Proof. Setting $x = y = 1$, $a = 5$, $b = 1$ and $m = 6$ in (1.4), we have

$$\sum_{n=0}^{\infty} cp_{5,1,6}(n)q^n = \frac{(q^6;q^6)_\infty}{(q;q^6)_\infty(q^5;q^6)_\infty} = \frac{(q^6;q^6)_\infty(q^3;q^6)_\infty}{(q;q^2)_\infty} = \frac{f_2 f_3}{f_1}. \tag{3.13}$$

$\ni ApplyingLemma2.2in(3.13), wehave \sum_{n=0}^{\infty} cp_{5,1,6}(n)q^n \equiv f_1^2 f_2 \pmod{3}$.(3.14) Applying Lemma 2.1 to (3.14), we have

$$\ni \sum_{n=0}^{\infty} cp_{5,1,6}\left(p^{2\alpha}n+\frac{p^{2\alpha}-1}{6}\right)q^n \equiv$$

$$\left(\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{3p^2-(6k+1)p}{2}}, -q^{\frac{3p^2+(6k+1)p}{2}}) + (-1)^{\pm\frac{p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2})\right)^2$$

$$\left(\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{3k^2+k} f(-q^{3p^2-(6k+1)p}, -q^{3p^2+(6k+1)p}) + (-1)^{\pm\frac{p-1}{6}} q^{\frac{p^2-1}{12}} f(-q^{2p^2})\right) \pmod 3. \tag{3.15}$$

Extracting the term containing $q^{pn+\frac{p^2-1}{6}}$ from both sides of (3.15) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} cp_{5,1,6}\left(p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{6}\right)q^n \equiv f_p^2 f_{2p} \pmod 3. \tag{3.16}$$

Again extracting the term containing q^{pn} from both sides of (3.16) and replacing q^p by q, we arrive at

$$\sum_{n=0}^{\infty} cp_{5,1,6}\left(p^{2\alpha+2}n+\frac{p^{2\alpha+2}-1}{6}\right)q^n \equiv f_1^2 f_2 \pmod 3. \tag{3.17}$$

which is the $\alpha+1$ term of (3.12). □

Corollary 3.6. *For $p \geq 5$, $\alpha \geq 1$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} cp_{5,1,6}\left(p^{2\alpha}n+\frac{(6j+p)p^{2\alpha-1}-1}{6}\right) \equiv 0 \pmod 3. \tag{3.18}$$

where $j = 1, 2, \cdots p-1$.

Proof. Comparing the coefficients of q^{pn+j}, $1 \leq j \leq p-1$ in (3.12), we easily obtain (3.18). □

Corollary 3.7. *For $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, we have*

$$cp_{5,1,6}\left(p^{2\alpha+1}n+\frac{(6j+1)p^{2\alpha}-1}{6}\right) \equiv 0 \pmod 3, \tag{3.19}$$

for $j = 1, 2, \cdots p-1$ and $\left(\frac{6j+1}{p}\right) = -1$.

Proof. According to Lemma 2.1 and Theorem 3.5, for any integer j with $0 \leq j \leq p-1$, if $j \not\equiv (3k^2+k)/2 \pmod p$ for $|k| \leq (p-1)/2$, then we have

$$cp_{5,1,6}\left(p^{2\alpha}(pn+j)+\frac{p^{2\alpha}-1}{6}\right) \equiv 0 \pmod 3,$$

which gives (3.19). □

Theorem 3.8. *For the primes $p_1, p_2, p_3 \cdots p_l$, $l \geq 1$ and $n \geq 0$, we have*

$$\sum_{n=0}^{\infty} cp_{5,1,6}\left(\prod_{s=1}^{l} p_s^2 + \left(\frac{\prod_{s=1}^{l} p_s^2 - 1}{6}\right)\right)q^n \equiv f_1^2 f_2 \pmod 3. \tag{3.20}$$

Proof. Proof can be completed by induction on l where the initial case is (3.12). Assume that (3.20) is true for l. Then based on Lemma 2.1 for prime p_{l+1}, we have

$$\sum_{n=0}^{\infty} cp_{5,1,6}\left(\prod_{s=1}^{l} p_s^2\left(p_{l+1}^2 n + \frac{p_{l+1}^2-1}{6}\right) + \left(\frac{\prod_{s=1}^{l} p_s^2 - 1}{6}\right)\right) q^n$$
$$= \sum_{n=0}^{\infty} cp_{5,1,6}\left(\prod_{s=1}^{l+1} p_s^2 n + \left(\frac{\prod_{s=1}^{l+1} p_s^2 - 1}{6}\right)\right) q^n \equiv f_1^2 f_2 \pmod 3,$$

which is the case of $l+1$. □

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