Palestine Journal of Mathematics

Vol 15(1)(2026) , 233–241

Two Equivalent Contractions and Common Fixed Point in Complete Fuzzy Metric Spaces

Dev Raj Joshi, Sanjay Mishra and Piyush Kumar Tripathi

Communicated by: Hichem Ben-El-Mechaiekh

MSC 2020 Classifications: Primary 47H10; Secondary 54H25.

Keywords and phrases: H-contraction, ϕ–contraction, and alternating distance function.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract In this research work, we prove a class of new common fixed point theorems in complete fuzzy metric spaces (CFMS). To prove these theorems, we use contractions existing in the literature, setting some new conditions for two mappings. Our result also reveals that the property of subset, commutativity, and compatibility are not necessary for the existence of a unique common fixed point of two self–mappings in fuzzy metric spaces.

1 Introduction

In 1975, Kramosil and Michalek [9] advanced the concept of fuzzy metric spaces, which can be seen as a generalization of statistical and probabilistic metric spaces. Following the work of Kramosil and Michalek, it became possible to obtain various fixed-point results in fuzzy metric spaces by employing different types of contractions for self mappings. Later, Grabiec [3] introduced the concept of completeness in fuzzy metric spaces, also known as G-complete fuzzy metric spaces, and extended the Banach contraction theorem to G-complete fuzzy metric spaces. Based on Grabiec's work, Fang [1] later established new fixed-point theorems for contractive type mappings in G-complete fuzzy metric spaces. Shortly thereafter, Mishra et al. [10] derived several general fixed-point theorems for asymptotically commutative maps in the same space, generalizing several fixed-point theorems in metric, Menger, fuzzy, and uniform spaces. In addition to the works relying on the G-complete fuzzy metric spaces, George and Veeramani [2] modified the definition of the Cauchy sequence introduced by Grabiek [3] due to the fact that the real set $\mathbb{R}$ is not complete with Grabiec's completeness definition. Furthermore, they modified the notion of a fuzzy metric space introduced by Kramosil and Michalek [9] in the context of the intuition of fuzziness. They then defined Hausdorff and first countable topology. Subsequently, the notion of a complete fuzzy metric space introduced by George and Veeramani became another new characterization of completeness in fuzzy metric spaces. Several fixed-point theorems have also been formulated based on this metric space. Gregory and Sapena [4] presented a fixed-point theorem for complete fuzzy metric spaces in the sense of George and Veeramani.

Many researchers have established common fixed-point results to generalize fixed-point theory in fuzzy metric spaces. Sun et al. [18] proved a common fixed point theorem for four self-mappings. Shahzad et al. [15] established some common fixed point theorems in fuzzy b-metric spaces. Rehman et al. [13] demonstrated common fixed point theorems for a pair of self-mappings in fuzzy cone metric spaces. For more results about the common fixed point theorems in fuzzy metric spaces we can see [12], [5], [6], [21], [22] and [11]. After a brief analysis of the construction of common fixed point in complete fuzzy metric spaces, we develop common fixed point theorems for two self-mappings in fuzzy metric spaces using the contractions introduced by Shen et al. [16] and Wardowski [20]. The first two theorems apply to compatible mappings, while the third and fourth theorems establish the existence of a unique common fixed point without requiring commutativity or compatibility.

2 Preliminaries

Definition 2.1 ([14]). A binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ is called t-norm or triangular norm if it satisfied following properties:

(i) $*$ is commutative and associative;

(ii) $1*a=a*1=a$ for all $a\in[0,1]$;

(iii) $a*b\le c*d$ if $a\le c$ and $b\le d$, for all $a,b,c,d\in[0,1]$.

George and Veeramani introduced the following revised definition of fuzzy metric space.

Definition 2.2 ([2]). Let X be a set, the binary operation $*\colon [0,1]\times[0,1]\to[0,1]$ be a continuous t-norm, and $\mathbb{R}^+$ is the real set $[0,\infty)$. A mapping $M\colon X\times X\times[0,\infty)\to[0,1]$ is fuzzy distance function. The fuzzy metric on X is the ordered triplet $(X,M,*)$ satisfies the following conditions for all $x,y,z\in X$ and $s,t\in\mathbb{R}^+$:

(i) $M(x,y,t)>0$;

(ii) $M(x,y,t)=1$ if and only if $x=y$, for all $t\in\mathbb{R}^+$;

(iii) $M(x,y,t)=M(y,x,t)$;

(iv) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$;

(v) The function $M(x,y,\ldots)\colon [0,\infty)\to[0,1]$ is left continuous.

The metric spaces satisfying above five properties called GV-fuzzy metric space.

The concept of a common fixed point for a self-mapping was first introduced by Jungk [8] in 1974, and he proved the following theorem.

Theorem 2.3. *Let f be a continuous mapping of a complete metric space (X,d) into itself. Then f has a fixed point in X if and only if there exists $k\in(0,1)$ and a mapping $g\colon X\to X$ which commutes with f and satisfies $g(X)\subset f(X)$ and*

$$d(g(x),g(y))\le kd(f(x),f(y)), \tag{2.1}$$

for all $x,y\in X$. Indeed, f and g have a unique common fixed point if (2.1) holds.

In 1986, Jungk introduced compatible mappings and proved the general fixed point for self-mappings by using the concept of compatible mappings in place of commutative mappings. He defined compatible mappings as follows.

Definition 2.4 ([7]). Self mappings f and g on a metric space (X,d) are compatible if and only if

$$\lim_{n\to\infty} d(gf(x_n),fg(x_n))=0,$$

where (x_n) is a sequence in X such that

$$\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} g(x_n)=t \quad \text{for some } t\in X.$$

The following fuzzy version of the compatible mappings was introduced by Mishra et al. in [10].

Definition 2.5 ([10]). Two self-mappings f and g on a fuzzy metric space $(X,M,*)$ are compatible if

$$\lim_{n\to\infty} M(fg(x_n),gf(x_n),t)=1,$$

where (x_n) is a sequence in X such that

$$\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} g(x_n)=u \quad \text{for some } u\in X.$$

3 Different Types of Contraction Model in Fuzzy Metric Spaces

Building on Grabiec's work on fuzzy contractions, several models have been proposed for general fixed points in fuzzy metric spaces, analogous to Banach contractions. Vasuki [19] introduced the following contraction.

Theorem 3.1 ([19]). *Let f and g be R-weakly commuting mappings on a complete fuzzy metric spaces $(X, M, *)$ such that*

$$M(g(x), g(y), t) \geq r(M(f(x), f(y), t)),$$

where $r\colon [0, 1] \to [0, 1]$ is continuous function with $r(p) > p$, for all $p \in (0, 1)$. If one of the map f or g is continuous, then f and g have unique common fixed point.

Shingh and Jain [17] proved the following common fixed point theorem.

Theorem 3.2 ([17]). *Let f and g be semi-compatible mappings on a complete fuzzy metric spaces $(X, M, *)$ such that $M(f(x), g(y), t) \geq rM(x, y, t)$, where $r\colon [0, 1] \to [0, 1]$ is continuous function with $r(p) > p$, for all $p \in (0, 1)$. If one of the map f or g is continuous, then f and g have unique common fixed point.*

Shen et al. defined the altering distance function and further proved an interesting result in [16] as follows.

Definition 3.3 ([16]). A altering distance function $\phi\colon [0, 1] \to [0, 1]$ which satisfies the following properties:

(i) ϕ is strictly decreasing and left continuous,

(ii) $\lim_{\lambda \to 1^-} \phi(\lambda) = 0$, i.e., when $\phi(\lambda) = 0$ if and only if $\lambda = 1$.

Theorem 3.4 ([16]). *Let $(X, M, *)$ be an M-complete fuzzy metric space and f a self-map of X and suppose that $\phi\colon [0, 1] \to [0, 1]$ satisfies the above properties (1) and (2) of Definition (3.3). Furthermore, let τ be a function from $(0, \infty)$ into $(0, 1)$. If for any $t > 0$, f satisfies the following condition:*

$$\phi(M(fx, fy, t)) \leq \tau(t)\phi(M(x, y, t).$$

Wardowski defined family of mappings and H-contractive in [20] as follows.

Definition 3.5 ([20]). A family of mappings $\mu\colon (0, 1] \to [0, \infty)$ satisfying following two conditions:

(i) μ is strictly decreasing and left continuous,

(ii) μ is onto.

Here conditions (i) and (ii) imply that $\mu(1) = 0$. Such family of mappings is denoted by H.

Definition 3.6 ([20]). Let $(X, M, *)$ be a fuzzy metric space. A mapping $f\colon X \to X$ is said to be H-counteractive with respect to $\mu \in H$, if there exists $K \in (0, 1)$ such that

$$\mu[M(f(x), f(y), t)] \leq K\mu[M(x, y, t)] \quad \text{for all} \quad x, y \in X, \text{ and } t > 0.$$

4 Main Results

First, using the contractions introduced by Shen et al. [16] and Wardowski [20], we prove two common fixed point theorems for two self-mappings, using the quasi-compatibility property of mappings.

The fixed-point theorem established by Shen et al. [16] using ϕ-contraction and Wardowski's H-contraction [20] applies to a single mapping. However, we have extended these contractions to prove a common fixed-point theorem for two self-mappings in fuzzy metric spaces. Both theorems follow the same pattern for obtaining a common fixed point, as illustrated below.

Theorem 4.1. *Let $(X, M, *)$ be a complete fuzzy metric space and f and g are compatible self-mappings of X, with $g(X) \subset f(X)$ satisfying the condition:*

$$\phi[M(g(x), g(y), t)] \leq K(t)\, \phi[M(f(x), f(y), t)] \quad \text{for all } x, y \in X \text{ and } t > 0, \tag{4.1}$$

where function $\phi\colon [0,1] \to [0,1]$ is

(i) strictly decreasing and left continuous,

(ii) $\phi(\alpha) = 0 \iff \alpha = 1$ and $K\colon (0,\infty) \to (0,1)$ is any function.

If the self-map f is continuous, then f and g each have a unique fixed point in X.

Proof. Let $x_0 \in X$. Since $g(X) \subset f(X)$, for $x_0 \in X$, there exists $x_1 \in X$ such that $g(x_0) = f(x_1)$). In general, $y_n = g(x_n) = f(x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. If $y_n = y_{n+1}$, then $g(x_n) = g(x_{n+1})$ implying $(x_{n+1}) = g(x_{n+1})$. Thus, x_{n+1} is coincident point. So, let $y_n \neq y_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Now, by given condition

$$\begin{aligned} \phi[M(g(x_n), g(x_{n+1}), t)] &\leq K(t) \cdot \phi[M(f(x_n), f(x_{n+1}), t)] \\ \Rightarrow \quad \phi[M(f(x_{n+1}), f(x_{n+2}), t)] &\leq K(t) \cdot \phi[M(f(x_n), f(x_{n+1}), t)] \\ &< \phi[M(f(x_n), f(x_{n+1}), t)] \end{aligned}$$

Since ϕ is strictly decreasing, we must have,

$$M(f(x_{n+1}), f(x_{n+2}), t) > M(f(x_n), f(x_{n+1}), t).$$

This show that $(M(f(x_n), f(x_{n+1}), t))$ is an increasing sequence in $[0,1]$.

Let $\lim_{n\to\infty} M(f(x_n), f(x_{n+1}), t) = l < 1$. Now, by the given contraction

$$\phi[M(f(x_{n+1}), f(x_{n+2}), t)] \leq K(t) \cdot \phi[M(f(x_n), f(x_{n+1}), t)].$$

For every t, supposing that $n \to \infty$, since ϕ is left continuous, we have $\phi(l) \leq K(t) \cdot \phi(l) < \phi(l)$, showing that $\phi(l) = 0$. By the second property of ϕ, $\phi(l) = 0$ implies $l = 1$. Hence

$$\lim_{n\to\infty} M(f(x_n), f(x_{n+1}), t) = 1 \tag{4.2}$$

Next, we show that the sequence $(f(x_n))$ is a Cauchy sequence. Then, for $0 < \epsilon < 1$ and $n_0 \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for $n > n_0$, $p \geq 1$ and $t > 0$,

$$\begin{aligned} M(f(x_n), f(x_{n+p}), t) \geq M(f(x_n), f(x_{n+1}), \tfrac{t}{p} * M(f(x_{n+1}), f(x_{n+2}), \\ \tfrac{t}{p} * \ldots * M(f(x_{n+p-1}), f(x_{n+p}), \tfrac{t}{p}). \end{aligned}$$

Taking $n \to \infty$ from (4.2)

$$M(f(x_n), f(x_{n+p}), t) \geq 1 * 1 * \ldots 1 = 1, \quad (\text{since } 1 * 1 = 1)$$

Thus, $\lim_{n\to\infty} M(f(x_n), f(x_{n+p}), t) = 1$. Hence, the sequence $(f(x_n))$ is a Cauchy sequence in the complete fuzzy metric space $(X, M, *)$. Therefore, $(f(x_n))$ converges to a point $z \in X$, thus $\lim_{n\to\infty} f(x_n) = z$. Since $y_n = g(x_n) = f(x_{n+1})$, if $\lim_{n\to\infty} f(x_n) = z$, then $\lim_{n\to\infty} g(x_n) = z$. As f is continuous, $\lim_{n\to\infty} f(x_n) = f(z)$ and $\lim_{n\to\infty} fg(x_n) = f(z)$; the compatibility of the two mappings implies $\lim_{n\to\infty} M(fg(x_n), gf(x_n)) = 1$. This implies $M(f(z), gf(x_n), t) = 1$, thus $\lim_{n\to\infty} gf(x_n) = f(z)$.

Next, we will show that z is a common fixed point of f and g. By the given contraction (4.1)

$$\phi[M(gf(x_n), g(x_n), t)] \leq K(t)\phi[M(ff(x_n), f(x_n))] \quad \text{for all} \quad t > 0, n \in \mathbb{N}.$$

When $n \to \infty$, then $\phi[M(f(z), z, t)] \leq K(t)\phi[M(f(z), z, t)]$. Thus,

$$[1 - K(t)]\phi[M(f(z), z, t)] \leq 0 \tag{4.3}$$

Since, $K(t) \in (0,1)$, shows $\phi[M(f(z),z,t)] = 0$, i.e., $M(f(z),z,t) = 1$ (by condition (ii) of ϕ), hence $f(z) = z$. Again, by (4.1)

$$\phi[M(g(z),g(x_n),t)] \leq K(t)\phi[M(f(z),f(x_n),t)].$$

Taking $n \to \infty$, we have

$$\phi[M(g(z),z,t)] \leq K(t)\phi[M(f(z),z,t)].$$

This shows $\phi[M(g(z),z,t)] \leq 0$, since $f(z) = z$. Thus, $\phi[M(g(z),z,t)] = 0$, i.e., $M(g(z),z,t) = 1$. Hence, $g(z) = z$.

To show the uniqueness of the common fixed point, let $u \in X$ be another common fixed point other than z, then $u = f(u) = g(u)$. Now, by (4.1),

$$\phi[M(g(z),g(u),t)] \leq K(t)\phi[M(f(z),f(u),t)].$$

Thus, $\phi[M(z,u,t)] \leq K(t)\phi[M(z,u,t)]$, i.e., $\phi[M(z,u,t)] \leq 0$ (by (4.3)). This shows $\phi[M(z,u,t)] = 0$, i.e., $M(z,u,t) = 1$. Thus, $z = u$. Therefore, f and g have a unique fixed point. □

The following theorem is a direct application of Wardowski's contraction in complete metric spaces. A common fixed point for two self-mappings exists if they satisfy Wardowski's contraction condition [20].

Theorem 4.2. *Let $(X, M, *)$ be a complete fuzzy metric space and f, g are compatible self-mappings on X, with $g(X) \subset f(X)$ satisfying the condition*

$$\mu[M(g(x),g(y),t)] \leq K(t)\mu[M(f(x),f(y),t)] \quad \text{for all } x,y \in X \text{ and } t > 0, \tag{4.4}$$

where function $\mu\colon (0,1] \to [0,\infty)$ is

(i) strictly decreasing and left continuous,

(ii) $\mu(\alpha) = 0 \iff \alpha = 1$, and $K\colon (0,\infty) \to (0,1)$ is any function.

If self-map f is continuous, then f and g have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Since $g(X) \subset f(X)$, for $x_0 \in X$, there exists $x_1 \in X$ such that $g(x_0) = f(x_1)$. In general, $y_n = g(x_n) = f(x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$. If $y_n = y_{n+1}$, then $g(x_n) = g(x_{n+1})$ implying that $f(x_{n+1}) = g(x_{n+1})$. Thus, x_{n+1} is a coincident point. So, let $y_n \neq y_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Now, by given condition

$$\begin{aligned} & \mu[M(g(x_n),g(x_{n+1}),t)] \leq K(t) \cdot \mu[M(f(x_n),f(x_{n+1}),t)] \\ \Rightarrow \quad & \mu[M(f(x_{n+1}),f(x_{n+2}),t)] \leq K(t) \cdot \mu[M(f(x_n),f(x_{n+1}),t)] \\ & \qquad < \mu[M(f(x_n),f(x_{n+1}),t)] \end{aligned}$$

Since μ is strictly decreasing, we must have,

$$M(f(x_{n+1}),f(x_{n+2}),t) > M(f(x_n),f(x_{n+1}),t).$$

This implies that $(M(f(x_n),f(x_{n+1}),t))$ is an increasing sequence in $[0,1]$.

Let $\lim_{n\to\infty} M(f(x_n),f(x_{n+1}),t) = l < 1$. Now, by given contraction

$$\mu[M(f(x_{n+1}),f(x_{n+2}),t)] \leq K(t) \cdot \mu[M(f(x_n),f(x_{n+1}),t)].$$

For every t, by supposing that $n \to \infty$, since μ is left continuous, we have, $\mu(l) \leq K(t) \cdot \mu(l) < 1\mu(l)$ which shows that $\mu(l) = 0$. By the second property of μ, therefore, $l = 1$. Hence

$$\lim_{n\to\infty} M(f(x_n),f(x_{n+1}),t) = 1 \tag{4.5}$$

Next, we show that the sequence $(f(x_n))$ is a Cauchy sequence. For $0 < \epsilon < 1$ and $n_0 \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that for $n > n_0, p \geq 1$, and $t > 0$,

$$M(f(x_n), f(x_{n+p}), t) \geq M(f(x_n), f(x_{n+1}), \tfrac{t}{p}) * M(f(x_{n+1}), f(x_{n+2}), \tfrac{t}{p}) * \ldots * M(f(x_{n+p-1}), f(x_{n+p}), \tfrac{t}{p}).$$

Taking $n \to \infty$ from (4.5),

$$M(f(x_n), f(x_{n+p}), t) \geq 1 * 1 * \ldots * 1 = 1.$$

Thus, $\lim_{n\to\infty} M(f(x_n), f(x_{n+p}), t) = 1$. Hence, the sequence $(f(x_n))$ is a Cauchy sequence in the complete fuzzy metric space $(X, M, *)$. Therefore, $(f(x_n))$ converges to a point $z \in X$. Consequently, $\lim_{n\to\infty} f(x_n) = z$. Since $y_n = g(x_n) = f(x_{n+1})$, if $\lim_{n\to\infty} f(x_n) = z$, then $\lim_{n\to\infty} g(x_n) = z$. As f is continuous, thus $\lim_{n\to\infty} ff(x_n) = f(z)$ and $\lim_{n\to\infty} fg(x_n) = f(z)$. The compatibility of the two functions shows $\lim_{n\to\infty} M(fg(x_n), gf(x_n), t) = 1$, i.e., $M(f(z), gf(x_n), t) = 1$, i.e., $\lim_{n\to\infty} gf(x_n) = f(z)$.

Next, we will show that z is a common fixed point of f and g. Given the contraction (4.4), we have

$$\mu[M(gf(x_n), g(x_n), t)] \leq K(t)\mu[M(ff(x_n), f(x_n)] \quad \text{for all} \quad t > 0, n \in \mathbb{N}.$$

When $n \to \infty$, then $\mu[M(f(z), z, t)] \leq K(t)\mu[M(f(z), z, t)]$ shows

$$[1 - K(t)]\mu[M(f(z), z, t)] \leq 0 \quad (\text{since } K(t) \in (0, 1)) \tag{4.6}$$

Thus, $\mu[M(f(z), z, t)] = 0$, i.e., $M(f(z), z, t) = 1$ (by condition (ii) of ϕ). Hence, $f(z) = z$. Again, by (4.4)

$$\mu[M(g(z), g(x_n), t)] \leq K(t)\mu[M(f(z), f(x_n), t)].$$

Taking $n \to \infty$, we have

$$\begin{aligned}
& \mu[M(g(z), z, t)] \leq K(t)\mu[M(f(z), z, t)] \\
\Rightarrow \quad & \mu[M(g(z), z, t)] \leq 0 \quad (\text{since } f(z) = z) \\
\Rightarrow \quad & \mu[M(g(z), z, t)] = 0 \\
\Rightarrow \quad & M(g(z), z, t) = 1 \\
\Rightarrow \quad & g(z) = z
\end{aligned}$$

To show the uniqueness of the common fixed point, let $u \in X$ be another common fixed point other than z, then $u = f(u) = g(u)$. Now,

$$\begin{aligned}
& \mu[M(g(z), g(u), t)] \leq K(t)\mu[M(f(z), f(u), t)] \\
\Rightarrow \quad & \mu[M(z, u, t)] \leq K(t)\mu[M(z, u, t)] \\
\Rightarrow \quad & [1 - K(t)])\mu[M(z, u, t)] \leq 0 \quad (\text{by } (4.6)) \\
\Rightarrow \quad & \mu[M(z, u, t)] = 0 \\
\Rightarrow \quad & M(z, u, t) = 1 \\
\Rightarrow \quad & z = u
\end{aligned}$$

Hence, f and g each have a unique fixed point. □

Corollary 4.3. *If the map f is identity map on X, then the above two theorems become the theorem of Shen et al. [16] and Wardowski [20], respectively.*

Example 4.4. Let $X = [0, 1]$ and self-mappings

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{if } 0 \leq x \leq \frac{1}{8}; \\ \frac{1}{6} & \text{if } \frac{1}{8} < x \leq 1. \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{8} & \text{if } 0 \leq x \leq \frac{1}{8}; \\ \frac{1}{6} & \text{if } \frac{1}{8} < x \leq 1. \end{cases}$$

on X. Thus, $g(X) \subset f(X)$. For the sequence $(x_n = \frac{1}{8} - \frac{1}{8n})$, the mappings $f(x)$ and $g(x)$ are compatible. The altering distance function $\phi\colon [0,1] \to [0,1]$, defined by $\phi(r) = 1 - \sqrt{r}$. The function $K\colon (0,\infty) \to (0,1)$ given by $K(t) = \frac{t}{t+1}$ for all $t > 0$. The fuzzy metric on X is $M(x,y,t) = e^{\frac{-2d(x,y)}{t}}$, where $d(x,y) = |x-y|$. Here, $\phi(r)$ satisfies the properties (i) and (ii) of Definition (3.3). Clearly, it satisfies ϕ contraction, and hence by Theorem (4.1), there is a unique common fixed point at $x = \frac{1}{8}$. Similarly, if we define $\mu\colon [0,1] \to [0,\infty)$ by $\mu(r) = \frac{1-r}{r}$, then by Theorem (4.2), there is a unique common fixed point at $x = \frac{1}{8}$.

In Theorems (4.1) and (4.2), we applied Jungck contraction-type conditions, requiring the range of one mapping to be a subset of the range of the other. However, a common fixed point for any two mappings f and g exists under the following contraction, where the subset condition and compatibility are not necessarily required.

Theorem 4.5. *Let f and g be two self-mappings on the fuzzy metric space $(X, M, *)$ satisfying the conditions:*

$$\phi(M(f(x), g(y), t)) \leq K(t)\phi(M(x,y,t)) \tag{4.7}$$

$$\mu(M(f(x), g(y), t)) \leq K(t)\mu(M(x,y,t)) \tag{4.8}$$

for all $x, y \in X$ and $t > 0$. Where ϕ is the altering distance functions as defined in Definition (3.3), and $K\colon [0,\infty) \to [0,1]$ is any function. If one of f or g is continuous in X, then f and g have a unique common fixed point in X.

Proof. Let f be continuous in X. For any arbitrary point $x_0 \in X$, let $f(x_0) = x_1$ and $g(x_1) = x_2$. In general, we can construct sequences $x_{2n+1} = f(x_{2n})$ and $x_{2n+2} = g(x_{2n+1})$ for $n = 0, 1, \ldots$. Now, taking $x = x_{2n}$ and $y = x_{2n+1}$ in (4.7). We prove the theorem by using the contraction (4.7).

$$\begin{aligned} &\phi(M(f(x_{2n}), g(x_{2n+1}), t)) \leq K(t)\phi(M(x_{2n}, x_{2n+1}, t)) < \phi(M(x_{2n}, x_{2n+1}, t)) \\ \Rightarrow\quad &\phi(M(x_{2n+1}, x_{2n+2}, t)) < \phi(M(x_{2n}, x_{2n+1}, t)). \end{aligned}$$

Since ϕ is strictly decreasing, we have,

$$M(x_{2n+1}, x_{2n+2}, t) > M(x_{2n}, x_{2n+1}, t) \quad \text{for all} \quad n \in \mathbb{N} \cup \{0\}.$$

This shows that the sequence $(M(x_{2n}, x_{2n+1}))$ is increasing in $[0,1]$. Thus, $(M(x_n, x_{n+1}))$ is increasing in $[0,1]$. Let $\lim_{n\to\infty} M(x_n, x_{n+1}) = l \leq 1$. If possible, suppose $l < 1$. Now, using the contraction (4.7) we have,

$$\begin{aligned} &\phi(M(f(x_n), g(x_{n+1}), t)) \leq K(t)\phi(M(x_n, x_{n+1}, t)) \\ \Rightarrow\quad &\phi(M(x_{n+1}, x_{n+2}), t)) \leq K(t)\phi(M(x_n, x_{n+1}, t)). \end{aligned}$$

Taking $n \to \infty$, we have,

$$\phi(l) \leq K(t)\phi(l) < \phi(l) \Rightarrow \phi(l) = 0.$$

Hence, by second property of ϕ, $l = 1$. Thus,

$$\lim_{n\to\infty} M(x_n, x_{n+1}) = 1 \tag{4.9}$$

Next, we show that (x_n) is a Cauchy sequence. Take any positive integer p and $t > 0$, then we have,

$$M(x_n, x_{n+p}, t) \geq M(x_n, x_{n+1}, \tfrac{t}{p}) * M(x_{n+1}, x_{n+2}, \tfrac{t}{p}) * \ldots * M(x_{n+p-1}, x_{n+p}, \tfrac{t}{p}).$$

It is p times. When $n \to \infty$, then we have

$$\begin{aligned} &\lim_{n\to\infty} M(x_n, x_{n+p}, t) \geq 1 * 1 * \cdots * 1 \quad (\text{since} 1 * 1 = 1) \\ \Rightarrow\quad &\lim_{n\to\infty} M(x_n, x_{n+p}, t) \geq 1 \\ \Rightarrow\quad &\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1 \end{aligned}$$

Hence, (x_n) is a Cauchy sequence. Since (X, M, t) is a complete metric space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. As f is continuous, $\lim_{n\to\infty} f(x_n) = f(u)$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = u$. Therefore, $f(u) = u$. Again, taking $x = x_n$ and $y = u$, then from (4.7)

$$\begin{aligned} & \phi(M(f(x_n), g(u), t)) \leq K(t)\phi(M(x_n, u, t)) < \phi(M(x_n, u, t)) \\ \Rightarrow\ & \phi(M(f(x_n), g(u), t)) < \phi(M(x_n, u, t)). \end{aligned}$$

Since ϕ is strictly decreasing, we have, $M(f(x_n), g(u), t) > M(x_n, u, t)$. As the fuzzy metric is continuous in X, we have,

$$\begin{aligned} & \lim_{n\to\infty} M(f(x_n), g(u), t) \geq \lim_{n\to\infty} M(x_n, u, t) \\ \Rightarrow\ & M(u, g(u), t) \geq M(u, u, t) \quad (\text{since } \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = u) \\ \Rightarrow\ & M(u, g(u), t) \geq 1 \\ \Rightarrow\ & M(u, g(u), t) = 1 \\ \Rightarrow\ & g(u) = u \end{aligned}$$

Hence, u is a common fixed point of f and g.

To show uniqueness, let there be another common fixed point $v \in X$, such that $f(v) = g(v) = v$. Now, by (4.7)

$$\begin{aligned} & \phi(M(f(u), g(v), t)) \leq K(t)\phi(M(u, v, t)) \\ \Rightarrow\ & \phi(M(u, v, t)) \leq K(t)\phi(M(u, v, t)) \\ \Rightarrow\ & (1 - K(t))\phi(M(u, v, t)) \leq 0 \\ \Rightarrow\ & \phi(M(u, v, t)) = 0 \quad (\text{since } 1 - K(t) \geq 0) \\ \Rightarrow\ & M(u, v, t)) = 1 \\ \Rightarrow\ & u = v \end{aligned}$$

Hence, u is the unique common fixed point. □

In the following theorem, a similar process of proof is used as above, even-though the mappings satisfy μ contraction.

Theorem 4.6. *Let f and g be two self-mappings on the fuzzy metric space $(X, M, *)$ satisfying the conditions:*

$$\mu(M(f(x), g(y), t)) \leq K(t)\mu(M(x, y, t)), \tag{4.10}$$

for all $x, y \in X$ and $t > 0$. Where μ is the altering distance functions as defined in Definition (3.5), and $K\colon [0, \infty) \to [0, 1]$ is any function. If one of f or g is continuous in X, then f and g have a unique common fixed point in X.

5 Conclusion

From Theorems (4.1), (4.2), (4.5), and (4.6) we conclude that the contraction introduced by Shen et al. [16] and Wardowski [20] are applicable to the common fixed point theorem. Furthermore, from Theorems (4.5) and (4.6), it is evident that the subset and compatibility conditions are not necessary for the existence of a common fixed point for two self-mappings. An open question remains as to whether these contractions are applicable to generalized fuzzy metric spaces and intuitionistic fuzzy metric spaces.

References

[1] J. Fang, *On fixed point theorems in fuzzy metric spaces*, Fuzzy Sets Syst. **46** (1992), 107–113. https://doi.org/10.1016/0165-0114(92)90271-5

[2] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets Syst. **64** (1994), 395–399. https://doi.org/10.1016/0165-0114(94)90162-7

[3] M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets Syst. **27** (1988), 385–389.

[4] V. Gregori, A. Sapena, *On fixed-point theorems in fuzzy metric spaces*, Fuzzy sets and systems. **125** (2002), no. 2, 245–252. https://doi.org/10.1016/S0165-0114(00)00088-9

[5] M. Jeyaraman, M. Suganthi, and W. Shatanawi, *Common fixed point theorems in intuitionistic generalized fuzzy cone metric spaces*, Mathematics. **8** (2020), no. 8, 1212. https://www.mdpi.com/2227-7390/8/8/1212

[6] M. Jeyaraman and S. Sowndrarajan, *Some common fixed point theorems for (ϕ, ψ)-weak contractions in intuitionistic generalized fuzzy cone metric spaces*, Malaya Journal of Matematik. **S** (2019), no. 1, 154–159. https://doi.org/10.26637/MJM0S01/0036

[7] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci. **9** (1986), 771–779. https://doi.org/10.1155/S0161171286000935

[8] G. Jungck, *Commuting mappings and fixed points*, Am. Math. Mon. **83** (1976), no. 4, 261–263. https://doi.org/10.2307/2318216

[9] I. Kramosil and J. Michalek, *Fuzzy metrics and statistical metric spaces*, Kybernetika. **11** (1975), no. 5, 336–344. http://dml.cz/dmlcz/125556

[10] S. N. Mishra, N. Sharma, and S. L. Singh, *Common fixed points of maps on fuzzy metric spaces*, Int. J. Math. Math. Sci. **17** (1994), 253–258.

[11] D. Rakic, T. Dosenovic, Z. D. Mitrovic, M. de la Sen, and S. Radenović, *Some fixed point theorems of ciric type in fuzzy metric spaces*, Mathematics. **8** (2020), no. 7, 297. https://doi.org/10.3390/math8020297

[12] K. P. R. Rao, I. Altun, and S. H. Bindu, *Common coupled fixed-point theorems in generalized fuzzy metric spaces*, Advances in Fuzzy Systems. (2011), Article ID 986748. https://doi.org/10.1155/2011/986748

[13] S. U. Rehman, Y. Li, S. Jabeen, and T. Mahmood, *Common fixed point theorems for a pair of self-mappings in fuzzy cone metric spaces*, Abstract and Applied Analysis, (2019), Article ID 2841606. https://doi.org/10.1155/2019/2841606

[14] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10** (1960), no. 1, 313–334.

[15] A. Shahzad, A. Shoaib, and Q. Mahmood, *Common fixed point theorems for fuzzy mappings in b-metric space*, Ital. J. Pure Appl. Math.. **38** (2017), 419–427.

[16] Y. Shen, D. Qiu, and W. Chen, *Fixed point theorems in fuzzy metric spaces*, Appl. Math. Lett. **25** (2012), no. 2, 138–141. https://doi.org/10.1016/j.aml.2011.08.002

[17] B. Singh and S. Jain, *Semi-compatibility, compatibility and fixed point theorems in fuzzy metric space*, J. Chungcheong Math. Soc. **18** (2005), no. 1, 1–22.

[18] G. Sun and K. Yang, *Generalized fuzzy metric spaces with properties*, Research Journal of Applied Sciences, Engineering and Technology. **2**, no. 7, (2010), 673–678.

[19] R. Vasuki, *Common fixed points for R-weakly commuting maps in fuzzy metric spaces*, Indian J. Pure Appl. Math. **30** (1999), no. 4, 419–424.

[20] D. Wardowski, *Fuzzy contractive mappings and fixed points in fuzzy metric spaces*, Fuzzy Sets Syst. **222** (2013), 108–114. https://doi.org/10.1016/j.fss.2013.01.012

[21] M.Jeyaraman, M.Sornavalli, R.Muthuraj and S. Sowndrarajan, *Common fixed point theorems for weakly compatible mappings in intuitionistic generalized fuzzy metric spaces*, Palest. J. Math. **9** (2020), no. 1, 476-484.

[22] V. Sharma, *Common fixed point theorem for compatible mappings of type(A-1) in fuzzy metric space*, Palest. J. Math. **8** (2019), no. 1, 312–317.

Author information

Dev Raj Joshi, Amity School of Applied Sciences, Amity University, Uttar Pradesh, Lucknow Campus, India.
E-mail: `dev.joshi@s.amity.edu`

Sanjay Mishra, Amity School of Applied Sciences, Amity University, Uttar Pradesh, Lucknow Campus, India.
E-mail: `drsanjaymishra1@gmail.com`

Piyush Kumar Tripathi, Amity School of Applied Sciences, Amity University, Uttar Pradesh, Lucknow Campus, India.
E-mail: `pktripathi@lko.amity.edu`

Received: 2025-02-27
Accepted: 2025-06-24