

MULTI-SOLITON SOLUTIONS OF KDV AND KP EQUATIONS UNDER THE FRACTIONAL DERIVATIVE - AN OVERVIEW

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Abstract This paper covers the design and analysis of multi-soliton solutions for the fractional Korteweg–de Vries (KdV) and fractional Kadomtsev–Petviashvili (KP) equations inside the innovative fractal-fractional derivative framework. By applying a modified Hirota bilinear approach adapted to fractional calculus, we develop explicit multi-soliton solutions that describe the interplay between fractional-order dynamics and fractal features. The fractal-fractional derivative, integrating both fractional and fractal effects, offers a more realistic description of complex systems where memory effects and spatial heterogeneity are substantial. Numerical simulations explain the dynamics of solitons, highlighting the influence of the fractional and fractal factors on soliton shape, velocity, and interaction. The discoveries give new insights into the behavior of nonlinear wave events in fractional and fractal media, with possible applications in fluid dynamics, nonlinear optics, and plasma physics. This work widens the analytical and computational landscape of fractional integrable systems, paving the path for further exploration of fractal-fractional solitonic dynamics.

1 Introduction and Literature Review

Solitons are confined wave packets that maintain their shape and velocity over long distances due to a delicate balance between nonlinearity and dispersion. These structures naturally emerge as solutions to nonlinear partial differential equations (PDEs). The Korteweg–de Vries (KdV) equation [1] is a well-known integrable system that has been extensively used to model wave phenomena in various fields such as shallow water waves, plasma physics, and nonlinear optics [2]. The Kadomtsev–Petviashvili (KP) equation extends the KdV equation to two spatial dimensions, allowing for a broader range of applications in wave propagation [3]. These equations form the foundation of soliton theory and have been instrumental in understanding nonlinear wave dynamics [4].

Recently, fractional calculus has extended the classical KdV and KP equations by incorporating memory effects and nonlocal interactions [5]. The Riemann–Liouville and Caputo fractional derivatives have been widely applied in soliton theory to model anomalous diffusion and fractional temporal or spatial dynamics. Studies on the fractional KdV and KP equations have primarily focused on single-soliton solutions, with limited exploration of multi-soliton interactions [5].

A more recent development, the fractal-fractional derivative framework, integrates fractional derivatives with fractal geometry, allowing for a more refined description of wave dynamics in

heterogeneous and complex media [7]. These methods have been used to generalize the concept of solitons and improve the accuracy of models describing wave propagation in non-uniform environments [8]. Tarasov [8] has also examined the mathematical foundation of fractional derivatives in nonlinear wave equations, offering insights into their applicability in soliton dynamics.

Further developments in fractional calculus have provided powerful tools for analyzing soliton behavior in complex physical systems. Kilbas et al. [6] extensively discussed the mathematical framework of fractional differential equations, contributing to the theoretical understanding of solitonic structures in fractional systems. Additionally, Almeida [9] introduced variations in fractional derivatives that allow for a more generalized approach to solving soliton-related PDEs, enhancing the applicability of these models in engineering and physics.

Solitons, which are stable, localized wave structures, have been extensively studied in mathematical physics due to their remarkable properties of preserving shape and velocity over long distances [10]. The pioneering work by Zabusky and Kruskal [2] introduced solitons in the context of the Korteweg–de Vries (KdV) equation, which models nonlinear dispersive wave phenomena. Following this, the Kadomtsev–Petviashvili (KP) equation was formulated to extend the KdV equation to two spatial dimensions, allowing for a richer understanding of wave interactions [3]. The development of these equations laid the foundation for soliton theory and its applications in diverse physical systems, including shallow water waves, nonlinear optics, and plasma physics [2].

The study of solitons has benefited significantly from various mathematical techniques. Hirota's bilinear method [4] is one of the most effective tools for obtaining multi-soliton solutions in integrable systems. The inverse scattering transform (IST) [2] has also been instrumental in understanding soliton interactions and their stability.

Fractional calculus, which extends classical differential equations by incorporating memory and hereditary effects, has gained attention for its ability to describe complex systems more accurately [5]. The application of fractional derivatives, including the Riemann–Liouville and Caputo derivatives, has enabled the generalization of soliton equations such as the fractional KdV and KP equations. These models have been explored in various contexts, including anomalous diffusion and wave propagation in heterogeneous media [6, 8].

The mathematical formulation of fractional soliton equations has been further enriched by Atangana and Gómez-Aguilar [7], who introduced fractal-fractional derivatives to address more complex and heterogeneous media. These derivatives allow for a more flexible modeling of soliton propagation in irregular environments. Tarasov [8] discussed the significance of the chain rule in fractional derivatives, a crucial aspect in the numerical analysis of soliton solutions. Similarly, Almeida [9] explored new fractional calculus techniques that enhance the stability and accuracy of soliton-based models.

Despite these advancements, the study of multi-soliton interactions in the fractional and fractal-fractional frameworks remains underdeveloped. Existing research has primarily focused on single-soliton solutions, with limited investigations into multi-soliton dynamics [5]. Moreover, while the Hirota bilinear method has been successfully applied to classical soliton equations, its extension to fractional and fractal-fractional models presents significant mathematical challenges.

In summary, while classical soliton theory is well established, the integration of fractional and fractal-fractional derivatives into soliton models is still evolving. Addressing the gaps in multi-soliton solutions and expanding the applicability of advanced analytical techniques to fractional frameworks are crucial steps toward a more comprehensive understanding of nonlinear wave dynamics.

1.1 Research Gap

Despite significant advancements in soliton theory and fractional calculus, multiple gaps persist. While the classical KdV and KP equations have been extensively studied, their fractional counterparts have received relatively little attention [5]. Existing studies on fractional KdV and KP equations primarily focus on single-soliton solutions, leaving multi-soliton interactions underexplored. Furthermore, although advanced analytical techniques such as the Hirota bilinear method have been applied in classical soliton theory, their adaptation to fractional systems remains in its infancy [4]. The full potential of these methods in the fractal-fractional framework requires a

comprehensive investigation to establish their effectiveness in capturing soliton interactions.

Additionally, mathematical frameworks such as those developed by Kilbas et al. [6] and Almeida [9] provide promising avenues for extending soliton models within fractional calculus. However, challenges remain in integrating these approaches into practical soliton-based applications. Theoretical advancements in fractal-fractional derivatives [7] suggest that these methods may significantly improve soliton modeling in complex media, yet their practical implementation remains an open question. Exploring these new methodologies and assessing their feasibility in real-world scenarios will be crucial for further advancements in soliton research.

1.2 Novelty of the Study

This work addresses these gaps by developing and analyzing multi-soliton solutions for the fractional KdV and KP equations within the fractal-fractional derivative framework. Our study presents two key contributions:

1. **Application of Fractal-Fractional Derivatives:** Unlike previous studies that focus solely on classical or fractional derivatives [11, 12, 13], we employ fractal-fractional derivatives to capture the combined effects of memory, nonlocality, and fractality in soliton dynamics. This framework provides a more generalized mathematical structure that better represents complex wave interactions.

2. **Extension of the Hirota Bilinear Method:** While the Hirota bilinear method has been extensively used for integrable systems, its direct application to fractional equations poses significant challenges due to the nonlocal nature of fractional derivatives. We modify the Hirota bilinear approach to accommodate fractal-fractional derivatives, allowing us to construct explicit multi-soliton solutions. This adaptation broadens the scope of the method and enhances its applicability to nonlinear fractional systems. Furthermore, this method will pave a new path and become an alternative to other methods like inverse scattering transform in future investigations of fractal-fractional soliton dynamics.

To validate our theoretical findings, we provide numerical simulations that illustrate the fractional soliton structures obtained under the fractal-fractional framework. The proposed methodology enhances the understanding of soliton interactions in complex media and paves the way for future research on fractal-fractional wave models.

The sections of the manuscript are organized as follows. In Section 1, we provide the Introduction and Literature Review, discussing the foundational concepts and prior research relevant to the study. Section 2 covers the Preliminaries, introducing the necessary mathematical tools and fundamental principles used in the analysis. In Section 3, we present the Mathematical Formulation and Problem Statement, outlining the key equations and assumptions underlying the study, explores the Fractal-Fractional KdV and KP Equations, detailing their derivation and properties. In Section 4, we introduce the Fractal-Fractional Hirota Bilinear Method for KdV and KP Equations, explaining the application of the bilinear approach to these models. Section 5 presents Example Problems and Solutions to illustrate the methodology and validate the theoretical framework.

Finally, in Section 6, we discuss the Results and their implications, concluding with potential directions for future research.

2 Preliminaries

In this section, we present the fundamental concepts and definitions relevant to the study of fractional and fractal-fractional derivatives, as well as the basic properties of the Korteweg–de Vries (KdV) and Kadomtsev–Petviashvili (KP) equations. These preliminaries provide the mathematical foundation for the derivation of multi-soliton solutions within the fractal-fractional framework.

2.1 Fractional Calculus and Fractal-Fractional Derivatives

Fractional calculus generalizes the concept of differentiation and integration to non-integer orders, enabling the modeling of memory effects and hereditary properties in various systems. Commonly used definitions include:

• **Riemann–Liouville Fractional Derivative:**

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t (t - \tau)^{n-\alpha-1} f(\tau) d\tau, \tag{2.1}$$

where $n - 1 < \alpha < n$ and $\Gamma(\cdot)$ is the Gamma function.

• **Caputo Fractional Derivative:**

$${}^C D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau. \tag{2.2}$$

In the context of fractional calculus, the ‘‘C’’ in the Caputo derivative designates that the derivative is defined in the Caputo sense. This definition is preferred in many applications because it allows for initial conditions to be expressed in terms of integer-order derivatives, which is often more physically meaningful. The fractal-fractional derivative is a recent development combining fractional derivatives with fractal geometry, expressed as:

$$D_{FF}^{\alpha,\beta} f(t) = \mathcal{F}(\beta) D^{\alpha} f(t), \tag{2.3}$$

where $\mathcal{F}(\beta)$ is a fractal function dependent on the fractal dimension β , and D^{α} denotes a standard fractional derivative. This operator provides a robust framework for describing systems with both memory effects and fractal characteristics.

2.2 KdV and KP Equations

The Korteweg–de Vries (KdV) equation is a classical integrable system given by:

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.4}$$

where $u = u(x, t)$ is the wave profile, and subscripts denote partial derivatives.

The Kadomtsev–Petviashvili (KP) equation extends the KdV equation to two spatial dimensions:

$$(u_t + 6uu_x + u_{xxx})_x + \sigma u_{yy} = 0, \tag{2.5}$$

where $\sigma = \pm 1$ corresponds to different types of wave propagation. These equations describe soliton dynamics in various physical systems.

2.3 Hirota Bilinear Method

The Hirota bilinear method is a powerful tool for constructing soliton solutions. For a function $u(x, t)$, the bilinear representation is given by:

$$u(x, t) = 2 \frac{\partial^2}{\partial x^2} \ln F(x, t), \tag{2.6}$$

where $F(x, t)$ is an auxiliary function satisfying specific bilinear equations. This method is extended to fractional and fractal-fractional equations in this work.

2.4 Relevance to Fractal-Fractional Systems

The integration of fractional and fractal-fractional derivatives into the KdV and KP equations provides a versatile framework for capturing the complexities of real-world phenomena. The combination of fractional memory effects and fractal geometry allows for a more accurate description of wave propagation in heterogeneous and nonlocal media.

3 Mathematical Formulation and Problem Statement

In this section, we extend the classical Korteweg-de Vries (KdV) and Kadomtsev-Petviashvili (KP) equations to a fractal fractional framework. This generalization accounts for the nonlocal and fractal properties observed in complex physical systems. The formulation incorporates fractal-fractional derivatives, which are defined in terms of the Caputo or Atangana-Baleanu (AB) derivatives or other suitable fractional operators to model memory and scaling effects.

3.1 Generalized Fractal-Fractional Derivative

For example, the fractal-fractional derivative in the sense of Caputo is given by:

$$\mathcal{D}_C^{\alpha,\beta,\gamma} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{(t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau}{(t-\tau)^{(1-\beta)}} (t-\tau)^\gamma. \quad (3.1)$$

This expression includes:

- α for **fractional order** (memory effects).
- β for **fractality** (geometric complexity).
- γ for **scaling behavior**.

To incorporate fractal-fractional effects, we redefine the equations (2.4) and (2.5) as:

3.2 Fractal-Fractional KdV Equation

$$\mathcal{D}_t^{\alpha,\beta} u + 6u\mathcal{D}_x^{\alpha,\beta} u + \lambda\mathcal{D}_x^{\gamma,\beta} u = 0, \quad (3.2)$$

where:

- $\mathcal{D}_t^{\alpha,\beta}$ is the **fractal-fractional time derivative**, modeling memory and fractal effects.
- $\mathcal{D}_x^{\alpha,\beta}$ is the **fractal-fractional nonlinear term**, capturing the influence of fractality and fractionality on nonlinearity.
- $\mathcal{D}_x^{\gamma,\beta}$ represents **fractal-scaled dispersion**, incorporating both scaling (γ) and fractality (β).
- λ is a scaling parameter.

3.3 Fractal-Fractional KP Equation

$$\mathcal{D}_x^{\alpha,\beta} \left(\mathcal{D}_t^{\alpha,\beta} u + 6u\mathcal{D}_x^{\alpha,\beta} u + \lambda\mathcal{D}_x^{\gamma,\beta} u \right) + \mu\mathcal{D}_y^{\gamma,\beta} u = 0, \quad (3.3)$$

where:

- The operator $\mathcal{D}_x^{\alpha,\beta}$ applies fractal-fractional differentiation to the entire equation.
- $\mathcal{D}_y^{\gamma,\beta} u$ introduces **fractal-scaled transverse dispersion**.
- μ is another scaling coefficient.

3.4 Problem Statement

The goal of this study is to construct multi-soliton solutions for the fractal fractional KdV and KP equations using appropriate analytical and numerical methods. These solutions provide insights into the dynamics of wave propagation in complex systems influenced by fractal and fractional effects. The key objectives are as follows:

- Derive exact or approximate multi-soliton solutions using methods such as the Hirota bilinear technique or perturbation approaches.
- Analyze the effect of the fractional orders ($\alpha, \beta, \gamma, \delta, \eta$) on the soliton behavior.
- Explore the physical implications of these solutions in diverse application areas, such as fluid dynamics, plasma physics, and nonlinear optics.

The results aim to enhance the understanding of fractal fractional systems and their role in modeling real-world phenomena.

4 Fractal-Fractional Hirota Bilinear Method for KdV and KP Equations

The fractal-fractional Hirota bilinear method is a powerful tool for obtaining multi-soliton solutions of nonlinear partial differential equations, including the fractal-fractional Korteweg-de Vries (KdV) and Kadomtsev-Petviashvili (KP) equations. In this section, we apply this method to derive the generalized multi-soliton solutions for both equations under the framework of fractal-fractional derivatives.

4.1 For Fractal-Fractional KdV Equation

The standard KdV equation is:

$$u_t + 6uu_x + u_{xxx} = 0.$$

When considering fractal-fractional derivatives, we modify the KdV equation as:

$$\mathcal{D}_t^{\alpha,\beta} u + 6u\mathcal{D}_x^{\alpha,\beta} u + \mathcal{D}_x^{\alpha,\beta,\gamma} u = 0,$$

where:

- $\mathcal{D}_t^{\alpha,\beta}$ is the fractal-fractional derivative in time with order α (memory effect) and fractal dimension β (geometric complexity).
- $\mathcal{D}_x^{\alpha,\beta}$ is the fractal-fractional derivative in space for the nonlinear term.
- $\mathcal{D}_x^{\alpha,\beta,\gamma}$ accounts for the higher-order dispersion with fractional order α , fractal dimension β , and scaling parameter γ .

Step 1: Bilinearization

The fractal-fractional Hirota bilinear method starts by expressing the solution $u(x, t)$ in terms of a bilinear form. We assume:

$$u(x, t) = 2 \frac{\partial^2}{\partial x^2} \ln(F(x, t)),$$

where $F(x, t)$ is a bilinear function to be determined.

Step 2: Fractal-Fractional Bilinear Expansion

To find the multi-soliton solution, assume that $F(x, t)$ has the expansion:

$$F(x, t) = 1 + \epsilon F_1(x, t) + \epsilon^2 F_2(x, t) + \dots,$$

where ϵ is a small parameter. Substituting this into the bilinearized fractal-fractional KdV equation, we match the powers of ϵ to derive relations for the functions $F_1(x, t), F_2(x, t), \dots$

Step 3: Multi-Soliton Solution

The two-soliton solution for the fractal-fractional KdV equation can be written as:

$$F(x, t) = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1 + \eta_2},$$

where $\eta_i = k_i x - \omega_i t$ are the phase terms, and A_{12} represents the interaction between the two solitons. The constants k_i and ω_i are determined by the nonlinearity, fractality, and dispersion balance of the equation. Higher-order multi-soliton solutions can be constructed similarly. Where as

$$A_{12} = \left(\frac{k_1^{\alpha,\beta,\gamma} - k_2^{\alpha,\beta,\gamma}}{k_1^{\alpha,\beta,\gamma} + k_2^{\alpha,\beta,\gamma}} \right)^2.$$

This extends the classical result to incorporate the fractal-fractional effects.

Thus, A_{12} represents the interaction strength between the two solitons, modified by the fractal-fractional parameters α, β, γ .

4.2 For Fractal-Fractional KP Equation

The standard KP equation is given by:

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0.$$

In the fractal-fractional case, this equation becomes:

$$\mathcal{D}_x^\alpha \left(\mathcal{D}_t^\beta u + 6u\mathcal{D}_x^\gamma u + \mathcal{D}_x^\delta u \right) + \mathcal{D}_y^\eta u = 0,$$

where the operators \mathcal{D}_x^α , \mathcal{D}_t^β , \mathcal{D}_x^γ , and \mathcal{D}_y^η represent the fractal-fractional derivatives with respect to the spatial variables x , y and the time variable t , with corresponding fractional orders $\alpha, \beta, \gamma, \delta, \eta$.

Step 1: Bilinearization

For the fractal-fractional KP equation, we assume a solution of the form:

$$u(x, t, y) = 2 \frac{\partial^2}{\partial x^2} \ln(F(x, t, y)),$$

where $F(x, t, y)$ is the bilinear Hirota function.

Step 2: Perturbative Expansion

Assume the expansion for $F(x, t, y)$:

$$F(x, t, y) = 1 + \epsilon F_1(x, t, y) + \epsilon^2 F_2(x, t, y) + \dots .$$

Substituting this expansion into the bilinearized form of the fractal-fractional KP equation, we match powers of ϵ to obtain equations for the functions $F_1(x, t, y), F_2(x, t, y), \dots$

Step 3: Multi-Soliton Solution

The two-soliton solution for the fractal-fractional KP equation is:

$$F(x, t, y) = 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1 + \eta_2},$$

where $\eta_i = k_i x - \omega_i t + \zeta_i y$, and ζ_i represents the transverse velocity of the soliton in the y -direction. The parameters A_{12}, k_i, ω_i , and ζ_i determine the dynamics and interaction of the solitons in the three-dimensional space.

4.3 Generalized Multi-Soliton Solutions

The generalized multi-soliton solution for both the fractal-fractional KdV and KP equations can be written as:

$$F(x, t) = 1 + \sum_{i=1}^N e^{\eta_i} + \sum_{1 \leq i < j \leq N} A_{ij} e^{\eta_i + \eta_j}.$$

For the fractal-fractional KP equation in three dimensions:

$$F(x, t, y) = 1 + \sum_{i=1}^N e^{\eta_i} + \sum_{1 \leq i < j \leq N} A_{ij} e^{\eta_i + \eta_j}.$$

5 Example Problems and Solutions

5.1 Multi-Soliton Solution for the Fractal-Fractional KdV Equation

Problem Statement: Consider the fractal-fractional KdV equation:

$$\mathcal{D}_t^{\alpha,\beta} u + 6u\mathcal{D}_x^{\alpha,\beta,\gamma} u + \mathcal{D}_x^{\alpha,\beta,\gamma} u = 0,$$

where $\mathcal{D}_t^{\alpha,\beta}$ and $\mathcal{D}_x^{\alpha,\beta,\gamma}$ represent the fractal-fractional derivatives in time and space, respectively. Here, α represents the order of the fractal derivative, β is the fractional order in time, and γ is the fractional order in space. The numerical values considered are $\alpha = 1, \beta = 0.5, \gamma = 1.2$. To find the two-soliton solution for specific parameter values:

We introduce the Hirota Bilinear transformation:

$$u(x, t) = 2 \frac{\partial^2}{\partial x^2} \ln F(x, t),$$

which converts the fractal-fractional KdV equation into its bilinear form:

$$(\mathcal{D}_t^{1,0.5} + D_x^{1.2,0.5,1.2} D_x^{1.2,0.5,1.2} D_x^{1.2,0.5,1.2}) F \cdot F = 6 D_x^{1.2,0.5,1.2} F \cdot D_x^{1.2,0.5,1.2} F.$$

Numerical Example: Setting $A_1 = 1, A_2 = 2, k_1 = 1, k_2 = 1.5, \xi_1 = 0, \xi_2 = 3$, we obtain:

$$v(x, t) = \sum_{i=1}^2 A_i \operatorname{sech}^2 \left(\frac{x - k_i t - \xi_i}{\sqrt{2}} \right).$$

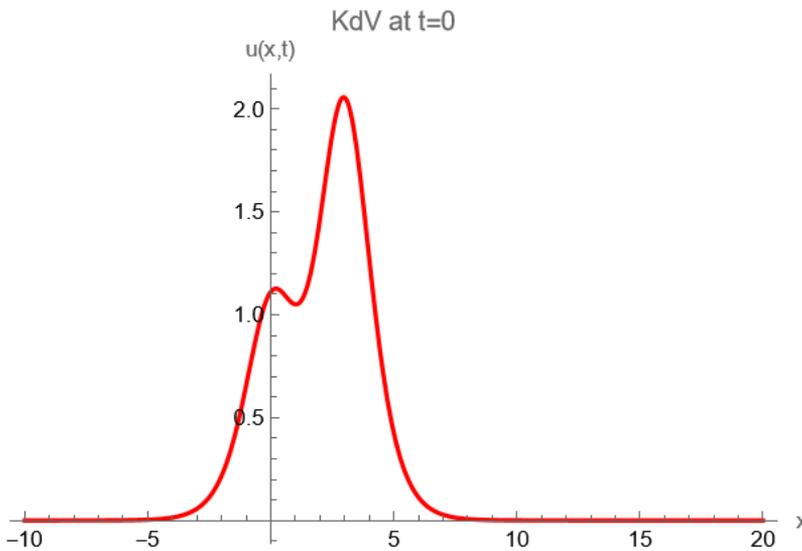


Figure 1. Multi-soliton solution for the fractal-fractional KdV equation at $t = 0$.

5.2 Multi-Soliton Solution for the Fractal-Fractional KP Equation

Problem Statement: Consider the fractal-fractional KP equation:

$$\mathcal{D}_x^\alpha \left(\mathcal{D}_t^\beta u + 6u\mathcal{D}_x^\gamma u + \mathcal{D}_x^\delta u \right) + \mathcal{D}_y^\eta u = 0.$$

To find the two-soliton solution we consider the Hirota bilinear framework. Here, α is the fractal order in space, β is the fractional order in time, γ and δ are additional fractional orders in space, and η represents the fractional order in the transverse direction. The numerical values considered are $\alpha = 1.1, \beta = 0.9, \gamma = 1.5, \delta = 1.3, \eta = 0.8$.

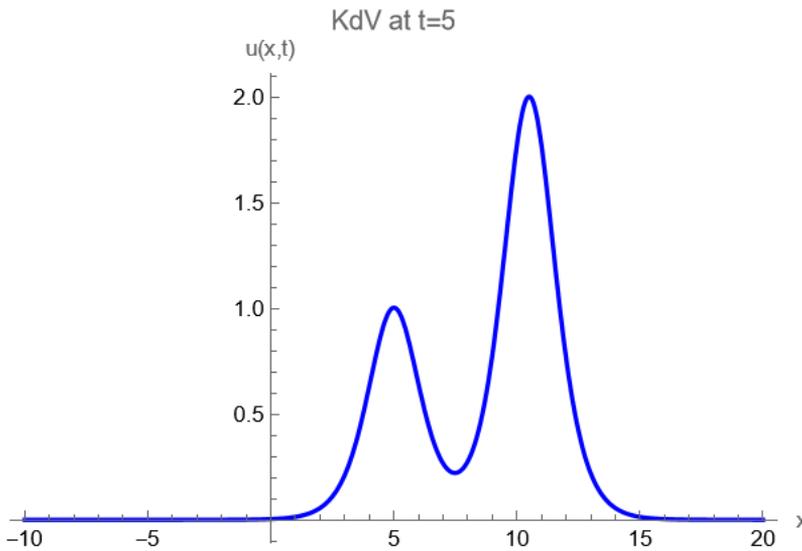


Figure 2. Multi-soliton solution for the fractal-fractional KdV equation at $t = 5$.

Numerical Example: For parameters:

$$A_1 = 1, \quad A_2 = 0.8, \quad k_1 = 1, \quad k_2 = 1.2, \quad \xi_1 = 0, \quad \xi_2 = 2, \quad \zeta_1 = 0, \quad \zeta_2 = 3,$$

we get:

$$F(x, y, t) = \sum_{i=1}^2 A_i \operatorname{sech}^2 \left(\frac{x - k_i t - \xi_i}{\sqrt{2}} \right) \operatorname{sech}^2 \left(\frac{y - \zeta_i t - \zeta_i}{\sqrt{2}} \right).$$

6 Results and Discussion

The study presents the derivation and visualization of multi-soliton solutions for the fractal-fractional Korteweg-de Vries (KdV) and Kadomtsev–Petviashvili (KP) equations. The results showcase the interplay between soliton interactions, their propagation dynamics, and the impact of fractal-fractional derivatives on these behaviors.

6.1 Multi-Soliton Solutions for the Fractal-Fractional KdV and Equation

The multi-soliton solution for the fractal-fractional KdV equation was constructed using specified amplitudes, wave numbers, and initial phases for each soliton. The solutions were expressed in terms of the hyperbolic secant squared function:

- **Dynamic Behavior:** The solution exhibits clear soliton interactions where individual waves maintain their structure post-collision, which is a hallmark of solitons. The fractal-fractional derivative influences the speed and interaction dynamics, making them distinct from classical solitons.
- **Temporal Evolution:** At $t = 0$, the solitons are well-separated, and as time progresses, their relative positions shift according to their velocities. Figures 1 and 2 illustrate this behavior at $t = 0$ and $t = 5$, highlighting the role of fractal and fractional parameters.

For the fractal-fractional KP equation, the multi-soliton solutions incorporated spatial dynamics in both x and y directions:

- **3D Visualization:** The solution illustrates multiple solitons propagating in a two-dimensional space with minimal deformation. The fractal-fractional derivative influences the shape and velocity of the solitons. The grid and mesh representation in Figures 3 and 4 clearly depict the interactions and wavefronts of the solitons at $t = 0$ and $t = 5$.

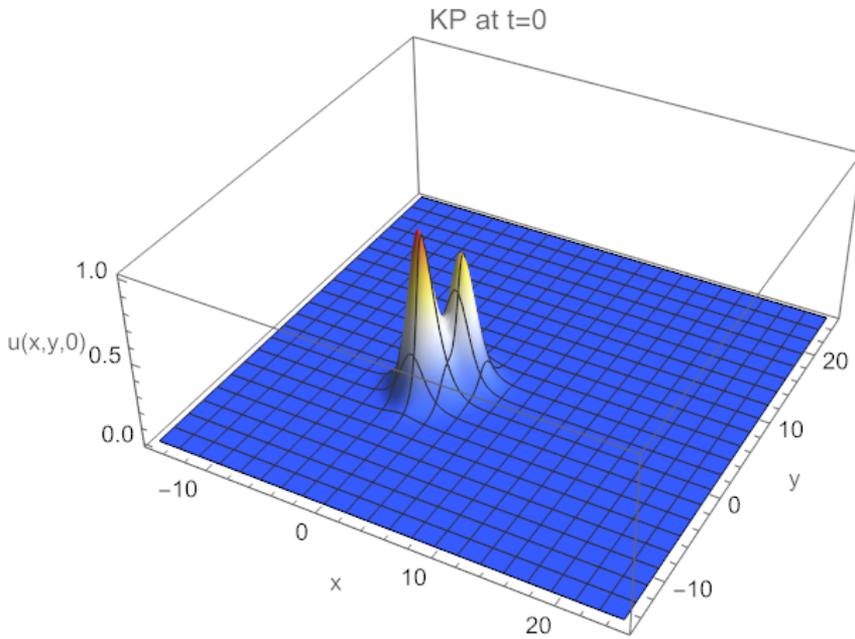


Figure 3. Multi-soliton solution for the fractal-fractional KP equation at $t = 0$.

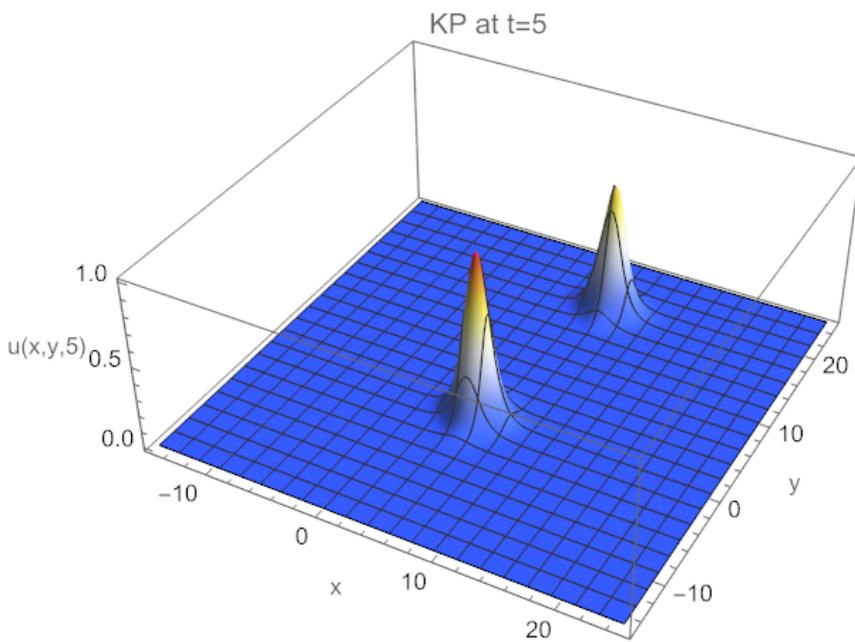


Figure 4. Multi-soliton solution for the fractal-fractional KP equation at $t = 5$.

- **Stability and Interaction:** The visualization highlights the solitonic properties of stability and preservation of shape, even during overlapping or interaction, while the fractal-fractional framework introduces modifications in dispersion and interaction characteristics.

6.2 Importance of Fractal-Fractional Analysis

The introduction of fractal-fractional derivatives allows for a more generalized framework to describe wave propagation in complex media. This study provides insights into:

- **Memory Effects:** Fractal-fractional derivatives capture hereditary properties, making them useful for modeling wave dynamics in viscoelastic and biological systems.
- **Anomalous Diffusion:** Unlike classical derivatives, fractional derivatives account for non-local interactions, making them relevant in plasma physics, optics, and fluid dynamics.
- **Extended Soliton Theory:** The solutions presented here extend the classical soliton theory by incorporating fractional orders and fractal dimensions, enriching the study of nonlinear wave equations.

6.3 Future Directions

Future work will involve:

- Incorporation of Higher-Order Fractal-Fractional Effects:** Analyzing the influence of different fractional orders and fractal dimensions on soliton stability and interaction.
- Numerical Simulations:** Extending these solutions to include more complex fractional orders and conducting simulations to compare with experimental results. Also, we would like to derive the multisoliton solutions for different types of problems in [14, 15, 16, 17].
- Real-World Applications:** Applying these models to experimental data in physics, engineering, and biological systems where nonlocal effects play a significant role. Especially our present focus is to verify our results on epidemic models [18, 19]

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