Palestine Journal of Mathematics

Vol 15(1)(2026) , 254–266

Fuzzy Soft PMS-algebras and Fuzzy Soft PMS-ideals of PMS-algebras

Nibret Melese Kassahun, Berhanu Assaye Alaba, Yohannes Gedamu Wondifraw and Zelalem Teshome Wale

Communicated by: Madeleine Al Tahan

MSC 2010 Classifications: Primary 06F35, 03G25; Secondary 03E72, 08A72.

Keywords and phrases: PMS-algebras, PMS-ideals, soft sets, fuzzy soft sets, fuzzy soft PMS-algebras, fuzzy soft PMS-ideals.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Corresponding Author: Nibret Melese Kassahun

Abstract: This paper defines fuzzy soft PMS-algebras and fuzzy soft PMS-ideals within PMS algebras and examines their fundamental properties. We establish the equivalence between fuzzy soft PMS-algebras and classical soft PMS-algebras expressed through level sets. Furthermore, we explore additional properties of fuzzy soft PMS-algebras and characterize soft set operations on fuzzy soft PMS-ideals. Finally, we studied the properties of fuzzy soft PMS-algebras and fuzzy soft PMS ideals in PMS-algebras under homomorphisms.

1 Introduction

Researchers are developing new mathematical approaches to handle uncertainties. Probability theory, interval-valued theory, fuzzy set theory, and intuitionistic fuzzy set theory are some of the mathematical theories introduced by researchers to overcome uncertainties. However, these theories need a set of parameters to prevent such difficulties. Molodtsov [1] proposed a new mathematical tool to deal with uncertainties by setting parameters. Soft sets are new mathematical tools used to handle uncertainties in decision-making problems in different fields of study. As a result, researchers are investigating a variety of applications. For instance, Maji et al. [2] further studied applications of soft sets in decision-making problems. Moreover, Roy et al. [3] apply soft sets to solve problems that exist in our real-life activities. Maji et al. [4] also proposed soft-set operations. However, in applications, soft set operations introduced by Maji et al. [4] have some limitations, as raised by Ali et al. [5]. So, Ali et al. [5] developed novel soft-set operations called restricted and extended intersections. Continuing the study of soft sets, researchers integrated the idea of soft sets into different algebraic structures. Hence, many scholars have studied the algebraic structure of soft sets. In 2007, Aktas and Cagman [6] studied the notion of soft groups and characterized soft operations. Jun et al. [7] also studied soft BCK algebras with their detailed properties. Recently, Kassahun et al. [8] introduced soft PMS algebras and investigated further properties under the homomorphism of mappings. Zadeh [9] proposed another mathematical tool to deal with uncertainties called fuzzy sets. As a result, many authors applied the fuzzy sets to different algebraic structures. Rosenfeld [10] studied fuzzy groups as the algebraic structure of fuzzy sets. In 2016, Sithar Selvam and Nagalakshmi [11] introduced fuzzy PMS algebras and characterized their properties. S. Shaqaqha [12] introduced and examined some properties of complex fuzzy ideals of Gamma Rings, applying fuzzy set theory to rings. Moreover, fuzzy sets are applied to various algebraic structures and have been generalized into numerous concepts. For instance, in 2022, Muhiuddin et al. [13] proposed linear Diophantine fuzzy set theory applied to BCK/BCI-algebras. Moreover, in 2023, Al-Tahan et al. [14] extended this concept to linear Diophantine fuzzy n-fold weak subalgebras of a BE-algebra. After the introduction of fuzzy sets, hesitant fuzzy sets were introduced by Torra (see [15, 16, 17, 18]) as a more general extension for modeling uncertainty, especially in decision-making. Hesitant fuzzy sets have since been applied in various algebraic structures. For instance, Girum et al. [19] introduced the concept of hesitant fuzzy T-ideals in TM-algebra and studied their properties in detail. By merging soft sets with fuzzy sets, the authors introduced a better mathematical tool to deal with uncertainties. Maji et al. [20] proposed the concept of fuzzy soft sets as a generalization of soft sets. Aygunoglu et al. [21] introduced the notion of fuzzy soft groups by merging soft sets with groups and studying some of their properties. In 2010, Jun et al. [22] applied the idea of fuzzy soft sets to the theory of BCK algebras as a generalization of fuzzy soft sets. Next, Yang [23] applied the ideas of fuzzy soft sets to semigroups and characterized their properties under homomorphisms.

Motivated by the above-reviewed articles, this paper introduces fuzzy soft PMS-subalgebras and fuzzy soft PMS-ideals within PMS-algebras and explores their properties. We demonstrate the equivalence between classical soft PMS-algebras (represented by level soft sets) and fuzzy soft PMS-algebras. Furthermore, we characterize fuzzy soft PMS-algebras and fuzzy soft PMS-ideals under homomorphisms.

2 Preliminaries

In this section, we define fuzzy sets, soft sets, fuzzy soft sets, and PMS-algebras, all of which are essential for our main findings.

2.1 Fuzzy Soft sets

Definition 2.1. [1] Let $P(U)$ be the power set of U, a pair $\langle\phi, A\rangle$ is called a soft set over U where ϕ is a mapping given by $\phi : A \longrightarrow P(U)$.

Definition 2.2. [5] The extended intersection of soft sets $\langle\phi, A\rangle$ and $\langle\sigma, B\rangle$ over a common universe U is the soft set $\langle\gamma, C\rangle$, where $C = A \cup B$ and for every $a \in C$, $\gamma[a] = \begin{cases} \phi[a], if\ a \in A - B \\ \sigma[a], if\ a \in B - A \\ \phi[a] \cap \sigma[a], if\ a \in A \cap B \end{cases}$

Definition 2.3. [8] Let $\langle\phi, A\rangle$ be a soft set over X, $\langle\phi, A\rangle$ is said to be a soft PMS-algebra over X if and only if $\phi(x)$ is a PMS-subalgebra of X for all $x \in A$.

Definition 2.4. [8] Let $\langle\phi, A\rangle$ be a soft set over X, $\langle\phi, A\rangle$ is said to be a soft PMS-ideal over X if and only if $\phi(x)$ is PMS-ideal of X for all $x \in A$.

Definition 2.5. [20] A pair $\langle\phi, A\rangle$ is called a fuzzy soft set over X, where $\phi : A \longrightarrow P(X)$ is a mapping, $P(X)$ being the set of all fuzzy subsets of X.

Definition 2.6. [6] Let $\langle\phi, A\rangle$ be a fuzzy soft set over X. For each $t \in [0, 1]$, the set $\langle\phi, A\rangle^t = \langle\phi^t, A\rangle$ is called a t -level set of $< \phi, \mathcal{A} >$, where $\phi^t[a] = \{x \in X : \phi[a](x) \geq t\}$ for each $a \in A$. Obviously, $\langle\phi^t, A\rangle$ is a soft set over X.

Definition 2.7. [20] Let $\langle\phi, A\rangle$ and $\langle\sigma, B\rangle$ be fuzzy soft sets over X with $A \cap B \neq \emptyset$. The intersection of them, denoted by $\langle\phi, A\rangle \widetilde{\cap} \langle\sigma, B\rangle = \langle\gamma, C\rangle$, is a fuzzy soft set $\langle\gamma, C\rangle$ over X, where $C = A \cap B$, and for each $a \in A$, $\gamma[a] = \phi[a] \cap \sigma[a]$.

Definition 2.8. [24] The union of two fuzzy soft sets $\langle\phi, A\rangle$ and $\langle\sigma, B\rangle$ over X, denoted by $\langle\phi, A\rangle \widetilde{\cup} \langle\sigma, B\rangle = \langle\gamma, C\rangle$, is a fuzzy soft set $\langle\gamma, C\rangle$ over X, where $C = A \cup B$, and for each $a \in A$ $\gamma[a] = \begin{cases} \phi[a], if\ a \in A - B \\ \sigma[a], if a \in B - A \\ \phi[a] \cup \sigma[a], if\ a \in A \cap B \end{cases}$

Definition 2.9. [20] If $\langle\phi, A\rangle$ and $\langle\sigma, B\rangle$ are fuzzy soft sets over a common universe X, then denoted $\langle\phi, A\rangle \widetilde{\wedge} \langle\sigma, B\rangle$ is defined by $\langle\phi, A\rangle \widetilde{\wedge} \langle\sigma, B\rangle = \langle\gamma, A \times B\rangle$, where $\gamma[a, b] = \phi[a] \cap \sigma[b]$ for all $(a, b) \in A \times B$.

2.2 PMS-algebras

In this section, we will address the concepts and basic properties of PMS-algebras that we need for the main results in the next sections. These concepts are taken from Selvam and Nagalakshmi, [25].

Definition 2.10. [25] A PMS-algebra is an algebra $(X, \star, 0)$ of type $(2, 0)$ with a constant "0" and a binary operation " $\star$ " satisfying the following axioms:

(i) $0 \star x = x$;

(ii) $(y \star x) \star (z \star x) = z \star y$ for all $x, y, z \in X$.

In X, we define a binary relation "$\leq$" by $x \leq y$ if and only if $x \star y = 0$.

Proposition 2.11. *[25] In a PMS-algebra $(X;\ \star, 0)$ the following properties hold for all $x, y, z \in X$.*

(i) $x \star x = 0$.

(ii) $(y \star x) \star x = y$.

(iii) $x \star (y \star x) = y \star 0$.

(iv) $(y \star x) \star z = (z \star x) \star y$.

(v) $(x \star y) \star 0 = y \star x$.

Definition 2.12. [25] Let S be a non empty subset of a PMS-algebra $(X, \star, 0)$. Then S is called a PMS-subalgebra of X if $x \star y \in S$ for all $x, y \in S$.

Definition 2.13. [25] Let X be a PMS-algebra and I be a subset of X. Then I is called a PMS-ideal of X if

(i) $0 \in I$;

(ii) $z \star y \in I$ and $z \star x \in I \Rightarrow y \star x \in I$ for all $x, y, z \in X$.

Definition 2.14. [25] Let X and Y be PMS-algebras. Then a mapping $f : X \longrightarrow Y$ is said to be a homomorphism of PMS-algebras if $f(x \star y) = f(x) \star f(y)$ for all $x, y \in X$. f is called an epimorphism if it is onto, and f is an endomorphism if it is a mapping from a PMS-algebra X to it self.

Note: [25] If f is a homomorphism of PMS-algebras, then $f(0) = 0$.

Definition 2.15. [9] Let X be a nonempty set. A fuzzy subset μ of X is the mapping μ from X into $[0, 1]$.

Definition 2.16. [11] Let X be a PMS-algebra and μ be a fuzzy subset of X. Then μ is a fuzzy PMS-subalgebra of X if $\mu(x \star y) \geq min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Definition 2.17. [11] Let X be a PMS-algebra. A fuzzy subset μ of X is called a fuzzy PMS-ideal of X if

(i) $\mu(0) \geq \mu(x)$;

(ii) $\mu(y \star x) \geq min\{\mu(z \star y), \mu(z \star x)\}$ for all $x, y, z \in X$.

Definition 2.18. [26] Let μ be a fuzzy subset of X; and for any $t \in [0, 1]$. Then a fuzzy set μ_t of X defined by $\mu^t(x) = min\{\mu(x), t\}$ for all $x \in X$ is called a t-fuzzy subset of X.

Definition 2.19. [9] Let $f : X \longrightarrow Y$ be a function; μ be a fuzzy subset of X, and θ be a fuzzy subset of Y. Then the image of μ under f, denoted by $f(\mu)$, is a fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} sup\{\mu(x) : x \in f^{-1}(Y)\}, if\ f^{-1}(Y) \neq \emptyset \\ 0, if\ f^{-1}(Y) = \emptyset \end{cases}$$

and the pre-image of θ under f, denoted by $f^{-1}(\theta)$, is a fuzzy subset of X defined by $f^{-1}(\theta)(x) = \theta(f(x))$ for all $x \in X$.

3 Fuzzy Soft PMS-algebras

In this section, we introduce the notion of fuzzy soft PMS-subalgebras and investigate some of their properties. Throughout this and the next section, X denotes a PMS-algebra, unless otherwise specified.

Definition 3.1. Let X be a PMS-algebra and $\langle \phi, A \rangle$ be a fuzzy soft set over X, where A is a set of parameters. Then $\langle \phi, A \rangle$ is a fuzzy soft PMS-algebra over X if and only if for each $\alpha \in A$ and $x, y \in X$, $\phi[\alpha](x \star y) \geq min\{\phi[\alpha](x), \phi[\alpha](y)\}$.

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a set with the following table.

Table 3.1 PMS-algebra

$\star$	0	1	2	3
0	0	1	2	3
1	2	0	1	2
2	1	2	0	1
3	3	1	2	0

Then $(X; \star, 0)$ is a PMS-algebra. Let $\langle \phi, A \rangle$ be a fuzzy soft set, where $A = X$. Let $\alpha \in A$. Define a fuzzy soft set $\phi : A \longrightarrow P(X)$ by the following table

Table 3.2 Fuzzy soft PMS-algebra

$\star$	0	1	2	3
0	1	0.8	0.8	0.5
1	1	0.7	0.4	0.3
2	1	0.6	0.2	0.1
3	1	0.5	0.1	0

Hence $\phi[0], \phi[1], \phi[2], \phi[3]$ are all fuzzy PMS-subalgebras of X. Thus $\langle \phi, A \rangle$ is a fuzzy soft PMS-algebra over X.

Proposition 3.3. *Let $\langle \phi, A \rangle$ be a fuzzy soft PMS-algebra over a PMS-algebra X. Then $\phi[\alpha](0) \geq \phi[\alpha](x), \forall x \in X$.*

Proof. Let $x \in X$ and $\alpha \in A$. Then
$\phi[\alpha](0) = \phi[\alpha](x \star x) \geq min\{\phi[\alpha](x), \phi[\alpha](x)\} = \phi[\alpha](x)$.
Hence $\phi[\alpha](0) \geq \phi[\alpha](x)$ for all $x \in X$ and for any $\alpha \in A$. □

Theorem 3.4. *Let $\langle \phi, A \rangle$ be a fuzzy soft PMS-algebra over X. If $x \star y \leq z$, then $\phi[\alpha](x) \geq min\{\phi[\alpha](y), \phi[\alpha](z)\}$ for all $x, y, z \in X$ and $\alpha \in A$.*

Proof. Let $x \star y \leq z$. Then $(x \star y) \star z = 0$.
Now

$$\begin{aligned} \phi[\alpha](x) &= \phi[\alpha](0 \star x) \\ &= \phi[\alpha](((x \star y) \star z) \star x) \\ &= \phi[\alpha](((z \star y) \star x) \star x) \\ &= \phi[\alpha]((x \star x) \star (z \star y)) \\ &= \phi[\alpha](z \star y) \\ &\geq min\{\phi[\alpha](y), \phi[\alpha](z)\}. \end{aligned}$$

□

Theorem 3.5. *Let $\langle \phi, A \rangle$ be a Fuzzy soft PMS-algebra over a PMS-algebra X. If B is a subset of A, then $\langle \phi|_B,\ B \rangle$ is a fuzzy soft PMS-algebra over X.*

Proof. Suppose $\langle\phi, A\rangle$ is a fuzzy soft PMS-algebra over X. Let $B \subseteq A$ and $x \in X$. Then $\phi|_B[\alpha](x) \neq \emptyset$. Since $\phi[\alpha](x)$ is a fuzzy PMS-subalgebra of X for all $\alpha \in A$ and $\phi|_B(x) \subseteq \phi[\alpha](x), \forall\alpha \in B$, then $\phi|_B(x)$ is a fuzzy PMS-subalgebra of X. Hence, $\langle\phi|_B, B\rangle$ is a fuzzy soft PMS-algebra over X. □

Theorem 3.6. *Let $\langle\phi, A\rangle$ be a fuzzy soft PMS-algebra over X. Then the set $X_{\phi[\alpha]} = \{x \in X : \phi[\alpha](x) = \phi[\alpha](0);\ \alpha \in A\}$ is a PMS-subalgebra of X.*

Proof. Suppose that $\langle\phi, A\rangle$ be a fuzzy soft PMS-algebra over X. Let $x, y \in X$ such that $x, y \in X_{\phi[\alpha]}$. Then $\phi[\alpha](x) = \phi[\alpha](0) = \phi[\alpha](y)$. Since $\phi[\alpha]$ is a fuzzy PMS subalgebra as $\langle\phi, A\rangle$ is a fuzzy soft PMS-algebra, we have $\phi[\alpha](x \star y) \geq min\{\phi[\alpha](x), \phi[\alpha](y)\} = \phi[\alpha](0)$. Again by Proposition 3.3, we have $\phi[\alpha](0) \geq \phi[\alpha](x \star y)$ implies that $\phi[\alpha](x \star y) = \phi[\alpha](0)$. Hence $x \star y \in X_{\phi[\alpha]}$. Thus $X_{\phi[\alpha]}$ is a PMS subalgebra of X.

□

Theorem 3.7. *Let $\langle\phi, A\rangle$ be a fuzzy soft set over a PMS-algebra X. Then (ϕ, A) is a fuzzy soft PMS-algebra of X if and only if $\langle\phi^t, A\rangle$ is a soft PMS-algebra over X for each $t \in [0, 1]$.*

Proof. Suppose $\langle\phi, A\rangle$ is a fuzzy soft PMS-algebra over X. For any $t \in [0, 1]$ and $\alpha \in A$, let $x, y \in \phi^t[\alpha]$. Then $\phi[\alpha](x) \geq t$ and $\phi[\alpha](y) \geq t$. Since $\phi[\alpha]$ is a fuzzy PMS-subalgebra, $\phi[\alpha](x \star y) \geq min\{\phi[\alpha](x), \phi[\alpha](y)\} \geq t$. It follows that $x \star y \in \phi^t[\alpha]$. Hence $\phi^t[\alpha]$ is a PMS-subalgebra of X. Thus, by Definition 2.3, $\langle\phi^t, A\rangle$ is a soft PMS-algebra over X for each $t \in [0, 1]$.
Conversely, assume that $\langle\phi^t, A\rangle$ is a soft PMS-algebra over X for each $t \in A$. For each $a \in A$ and $x, y \in X$, let $\phi[\alpha](x) = t_1$ and $\phi[\alpha](y) = t_2$. Then $x \in \phi^{t_1}[\alpha]$ and $y \in \phi^{t_2}[\alpha]$. Take $t = min\{t_1, t_2\}$. Then, $x, y \in \phi^t[\alpha]$. Since $\phi^t[\alpha]$ is a PMS-subalgebra of X, $x \star y \in \phi^t[\alpha]$. It follows that $\phi[\alpha](x \star y) \geq t = min\{\phi[\alpha](x), \phi[\alpha](y)\}$. So $\phi[\alpha]$ is a fuzzy PMS-subalgebra of X. Therefore $\langle\phi, A\rangle$ is a fuzzy soft PMS-algebra over X. □

Theorem 3.8. *Let X be a PMS-algebra, and let $\langle\phi, A\rangle$ and $\langle\sigma, B\rangle$ be fuzzy soft PMS-algebras over X. If $A \cap B \neq \emptyset$, then $\langle\phi, A\rangle \,\widetilde{\cap}\, \langle\sigma, B\rangle$ is a fuzzy soft PMS-algebra over X.*

Proof. Suppose $\langle\phi, A\rangle \,\widetilde{\cap}\, \langle\sigma, B\rangle = \langle\gamma, C\rangle$, where $C = A \cap B$. Let $\alpha \in C$. Then $\gamma[\alpha] = \phi[\alpha] \cap \sigma[\alpha]$. Let $x, y \in X$.

$$\begin{aligned}
\gamma[\alpha](x \star y) &= (\phi[\alpha] \cap \sigma[\alpha])(x \star y) \\
&= min\{\phi[\alpha](x \star y), \sigma[\alpha](x \star y)\} \\
&\geq min\{min\{\phi[\alpha](x), \phi[\alpha](y)\}, min\{\sigma[\alpha](x), \sigma[\alpha](y)\}\} \\
&= min\{min\{\phi[\alpha](x), \sigma[\alpha](x)\}, min\{\phi[\alpha](y), \sigma[\alpha](y)\}\} \\
&= min\{(\phi[\alpha] \cap \sigma[\alpha])(x), (\phi[\alpha] \cap \sigma[\alpha])(y)\} \\
&= min\{\gamma[\alpha](x), \gamma[\alpha](y)\}
\end{aligned}$$

Thus, $\gamma[\alpha]$ is a fuzzy PMS-subalgebra of X, and hence $\langle\gamma, C\rangle = \langle\phi, A\rangle \,\widetilde{\cap}\, \langle\sigma, B\rangle$ is a fuzzy soft PMS-algebra over X. □

Theorem 3.9. *Let X be a PMS-algebra; and let $\langle\phi, A\rangle$ and $\langle\sigma, B\rangle$ be fuzzy soft PMS-algebras over X. If $A \cap B = \emptyset$, then $\langle\phi, A\rangle \,\widetilde{\cup}\, \langle\sigma, B\rangle$ is a fuzzy soft PMS-algebra over X.*

Proof. Let $\langle\phi, A\rangle \,\widetilde{\cup}\, \langle\sigma, B\rangle = \langle\gamma, C\rangle$, where $C = A \cup B$. Since $A \cap B = \emptyset$, either $\alpha \in A\backslash B$ or $\alpha \in B\backslash A$ for all $\alpha \in C$. If $\alpha \in A\backslash B$, then $\gamma[\alpha] = \phi[\alpha]$ is a fuzzy PMS-subalgebra of X because $\langle\phi, A\rangle$ is a fuzzy soft PMS-algebra over X. If $\alpha \in B\backslash A$, then $\gamma[\alpha] = \sigma[\alpha]$ is a fuzzy PMS-subalgebra of X because (σ, B) is a fuzzy soft PMS-algebra over X. Hence $\langle\phi, A\rangle \,\widetilde{\cup}\, \langle\sigma, B\rangle = \langle\gamma, C\rangle$ is a fuzzy soft PMS-algebra over X. □

Theorem 3.10. *If $\langle\phi, A\rangle$ and $\langle\sigma, B\rangle$ are fuzzy soft PMS-algebras over a PMS-algebra X, then the extended intersection of $\langle\phi, A\rangle$ and $\langle\sigma, B\rangle$ is a fuzzy soft PMS-algebra over X.*

Proof. Let $\langle \phi, A\rangle \widetilde{\cap}_e \langle \sigma, B\rangle = \langle \gamma, C\rangle$ be the extended intersection of $\langle \phi, A\rangle$ and $\langle \sigma, B\rangle$. Then $C = A \cup B$. For any $\alpha \in C$, if $\alpha \in A - B$, then $\gamma[\alpha] = \phi[\alpha]$. Since $\phi[\alpha]$ is a fuzzy PMS-subalgebra, $\gamma[\alpha](x \star y) = \phi[\alpha](x \star y) \geq min\{\phi[\alpha](x), \phi[\alpha](y)\} = min\{\gamma[\alpha], \gamma[\alpha](y)\}$, and so $\gamma[\alpha](x \star y) \geq min\{\gamma[\alpha](x), \gamma[\alpha](y)\}$. Hence $\gamma[\alpha]$ is a fuzzy PMS-subalgebra. If $\alpha \in B - A$, then $\gamma[\alpha] = \sigma[\alpha]$. Since $\sigma[\alpha]$ is a fuzzy PMS-subalgebra, $\gamma[\alpha](x \star y) = \sigma[\alpha](x \star y) \geq min\{\sigma[\alpha](x), \sigma[\alpha](y)\} = min\{\gamma[\alpha], \gamma[\alpha](y)\}$, and so $\gamma[\alpha](x \star y) \geq min\{\gamma[\alpha](x), \gamma[\alpha](y)\}$. Hence $\gamma[\alpha]$ is a fuzzy PMS-subalgebra. If $A \cap B \neq \emptyset$, then

$$\begin{aligned}
\gamma[\alpha](x \star y) &= (\phi[\alpha] \cap \sigma[\alpha])(x \star y) \\
&= min\{\phi[\alpha](x \star y), \sigma[\alpha](x \star y)\} \\
&\geq min\{min\{\phi[\alpha](x), \phi[\alpha](y)\}, min\{\sigma[\alpha](x), \sigma[\alpha](y)\}\} \\
&= min\{min\{\phi[\alpha](x), \sigma[\alpha](x)\}, min\{\phi[\alpha](y), \sigma[\alpha](y)\}\} \\
&= min\{(\phi[\alpha] \cap \sigma[\alpha](x)), (\phi[\alpha] \cap \sigma[\alpha])(y)\} \\
&= min\{\gamma[\alpha](x), \gamma[\alpha](y)\}
\end{aligned}$$

Hence $\gamma[\alpha]$ is a fuzzy PMS-subalgebra for all $\alpha \in A \cap B$, and so $\langle \gamma, C\rangle$ is a fuzzy soft PMS-algebra. □

Theorem 3.11. *If $\langle \phi, A\rangle$ and $\langle \sigma, B\rangle$ are fuzzy soft PMS-algebras over a PMS-algebra X, then the fuzzy soft set $\langle \phi, A\rangle \widetilde{\barwedge} \langle \sigma, B\rangle$ is a fuzzy soft PMS-algebra over X.*

Proof. Let $(\alpha_1, \alpha_2) \in A \times B$. Then $\gamma[\alpha_1, \alpha_2] = \phi[\alpha_1] \cap \sigma[\alpha_2]$. Let $x, y \in X$.

$$\begin{aligned}
\gamma[\alpha_1, \alpha_2](x \star y) &= (\phi[\alpha_1] \cap \sigma[\alpha_2])(x \star y) \\
&= min\{\phi[\alpha_1](x \star y), \sigma[b](x \star y)\} \\
&\geq min\{min\{\phi[\alpha_1](x), \phi[\alpha_1](y)\}, min\{\sigma[\alpha_1](x), \sigma[\alpha_2](y)\}\} \\
&= min\{min\{\phi[\alpha_1](x), \sigma[\alpha_2](x)\}, min\{\phi[\alpha_1](y), \sigma[\alpha_2](y)\}\} \\
&= min\{\gamma[\alpha_1, \alpha_2](x), \gamma[\alpha_1, \alpha_2](y)\}.
\end{aligned}$$

Hence $\gamma[\alpha_1, \alpha_2]$ is a fuzzy PMS-subalgebra of X.
Therefore $\langle \phi, A\rangle \widetilde{\barwedge} \langle \sigma, B\rangle = \langle \gamma, C\rangle$ is a fuzzy soft PMS-algebra of X, where $C = A \cap B$. □

Theorem 3.12. *Let $\langle \phi, A\rangle$ be a fuzzy soft PMS-ideal of a PMS-algebra X such that $x \star y \leq z$. Then $\phi[\alpha](x) \geq min\{\phi[\alpha](y), \phi[\alpha](z)\}$ for all $x, y, z \in X$ and $\alpha \in A$.*

Proof. Suppose $\langle \phi, A\rangle$ be a fuzzy soft PMS-ideal. Then $\phi[\alpha]$ is a fuzzy PMS-ideal of X. Since $x \star y \geq z$, we have $(x \star y) \star z = 0$.

$$\begin{aligned}
\phi[\alpha](x) &= \phi[\alpha](0 \star x) \\
&= \phi[\alpha](((x \star y) \star z) \star x) \\
&= \phi[\alpha](((z \star y) \star x) \star x) \\
&= \phi[\alpha]((x \star x) \star (z \star y)) \\
&= \phi[\alpha](z \star y) \\
&\geq min\{\phi[\alpha](z), \phi[\alpha](y)\}
\end{aligned}$$

Hence $\phi[\alpha](x) \geq min\{\phi[\alpha](y), \phi[\alpha](z)\}$. □

Definition 3.13. Let $\langle \phi_1, A\rangle$ and $\langle \phi_2, A\rangle$ be fuzzy soft PMS-algebras over a PMS-algebra X. Then $\langle \phi_1, A\rangle$ is a fuzzy soft PMS-subalgebra of $\langle \phi_2, A\rangle$ if $\phi_1[\alpha]$ is a fuzzy PMS-subalgebra of $\phi_2[\alpha]$ for each $\alpha \in A$.

Theorem 3.14. *Let $\langle \phi_1, A\rangle$ and $\langle \phi_2, A\rangle$ be fuzzy soft PMS-algebras over X and $f : X \longrightarrow Y$ be an epimorphism. Then*
(i). $\langle f(\phi_1), A\rangle$ is a fuzzy soft PMS-algebra over Y with sup property.
(ii). If $\langle \phi_1, A\rangle$ is a fuzzy soft PMS-subalgebra of $\langle \phi_2, A\rangle$ with sup property, then $\langle f(\phi_1), A\rangle$ is a fuzzy soft PMS-subalgebra of $\langle f(\phi_2), A\rangle$.

Proof. (i). Let $\alpha \in A$ and let $x, y, z \in Y$ with $a \in f^{-1}(x), b \in f^{-1}(y)$ such that $\phi_1[\alpha](a) = sup\{\phi_1[\alpha](t) : t \in f^{-1}(x)\}, \phi_1[\alpha](b) = sup\{\phi_1[\alpha](t) : t \in f^{-1}(y)\}$.

$$\begin{aligned} f(\phi_1[\alpha])(x \star y) &= sup\{\phi_1[\alpha](t) : t \in f^{-1}(x \star y)\} \\ &= \phi_1[\alpha](a \star b) \\ &\geq min\{\phi_1[\alpha](a), \phi_1, \phi[\alpha](b)\} \\ &= min\{sup\{\phi_1[\alpha](t) : t \in f^{-1}(x), sup\{\phi_1[\alpha](t) : t \in f^{-1}(y)\}\} \\ &= min\{f(\phi_1[\alpha])(x), f(\phi_1[\alpha])(y)\} \end{aligned}$$

Hence $f(\phi_1[\alpha])$ is a fuzzy PMS-subalgebra of X, and so $\langle f(\phi_1), A\rangle$ is a fuzzy soft PMS-algebra over X.
(ii). To prove this, use a similar approach as in (i). □

Theorem 3.15. *Let $\langle \phi_1, A\rangle$ and $\langle \phi_2, A\rangle$ be fuzzy soft PMS-algebras over X and $f : X \longrightarrow Y$ be a homomorphism. Then*
(i). $\langle f^{-1}(\phi_1), A\rangle$ is a fuzzy soft PMS-algebra over Y.
(ii). If $\langle \phi_1, A\rangle$ is a fuzzy soft PMS-subalgebra of $\langle \phi_2, A\rangle$, then $\langle f^{-1}(\phi_1), A\rangle$ is a fuzzy soft PMS-subalgebra of $\langle f^{-1}(\phi_2), A\rangle$.

Proof. (i). Let $\alpha \in A$ and $x, y \in X$. Then

$$\begin{aligned} f^{-1}(\phi_1[\alpha])(x \star y) &= \phi_1[\alpha](f(x \star y)) \\ &= \phi_1[\alpha](f(x) \star f(y)) \\ &\geq min\{\phi_1[\alpha](f(x)), \phi_1[\alpha](f(y))\} \\ &= min\{f^{-1}(\phi_1[\alpha])(x), f^{-1}(\phi_1[\alpha](y))\} \end{aligned}$$

Hence $f^{-1}(\phi_1[\alpha])$ is a fuzzy PMS-subalgebra of X, and so $\langle f^{-1}(\phi_1[\alpha]), A\rangle$ is a fuzzy soft PMS-algebra over X.
(ii). To prove this, use a similar approach as in (i). □

Definition 3.16. Let $f : X \longrightarrow X$ be an endomorphism of a PMS-algebra X. Then for a fuzzy soft set $\langle \phi, A\rangle$ in X, we define a fuzzy soft set $\langle \phi, A\rangle^f = \langle \phi^f, A\rangle = \phi[\alpha](f(x))$ for all $x \in X$.

Theorem 3.17. *Let $f : X \longrightarrow Y$ be a homomorphism of PMS-algebras. If $\langle \phi, A\rangle$ is a fuzzy soft PMS-algebra over Y, then the fuzzy soft set $\langle \phi^f, A\rangle$ is a fuzzy soft PMS-algebra over X.*

Proof. Let $\alpha \in A$ and $x, y \in X$. Then

$$\begin{aligned} \phi^f[\alpha](x \star y) &= \phi[\alpha](f(x \star y)) \\ &= \phi[\alpha](f(x) \star f(y)) \\ &\geq min\{\phi[\alpha](f(x)), \phi[\alpha](f(y))\} \\ &= min\{\phi^f[\alpha](x), \phi^f[\alpha](y)\} \end{aligned}$$

Hence $\phi^f[\alpha]$ is a fuzzy PMS-subalgebra of X, and so $\langle \phi^f, A\rangle$ is a fuzzy soft PMS-algebra over X. □

However, the converse of Theorem 3.17 is true if f is an epimorphism of PMS-algebras as shown in Theorem 3.18

Theorem 3.18. *Let $f : X \longrightarrow Y$ be an epimorphism of PMS-algebras. If $\langle \phi^f, A\rangle$ is a fuzzy soft PMS-algebra over X, then $\langle \phi, A\rangle$ is a fuzzy soft PMS-algebra over Y.*

Proof. Let $\alpha \in A$ and let $a, b \in Y$. Then there exists $x, y \in X$ such that $f(x) = a$ and $f(y) = b$.

Now

$$\begin{aligned}\phi[\alpha](a \star b) &= \phi[\alpha](f(x) \star f(y)) \\ &= \phi[\alpha](f(x \star y)) \\ &= \phi^f[\alpha](x \star y) \\ &\geq min\{\phi^f[\alpha](x), \phi^f[\alpha](y)\} \\ &= min\{\phi[\alpha](f(x)), \phi[\alpha](f(y))\} \\ &= min\{\phi[\alpha](a), \phi[\alpha](b)\}\end{aligned}$$

Hence $\phi[\alpha]$ is a fuzzy PMS-subalgebra of Y, and so $\langle\phi, A\rangle$ is a fuzzy soft PMS-algebra over Y. □

4 Fuzzy Soft PMS-ideals

In this section, we introduce the notion of fuzzy soft PMS-ideals and investigate some of their properties. Throughout this and the next section, X denotes a PMS-algebra, unless otherwise specified.

Definition 4.1. Let X be a PMS-algebra and $\langle\phi, A\rangle$ be a fuzzy soft set over X, where A is a set of parameters. Then $\langle\phi, A\rangle$ is a fuzzy soft PMS-ideal over X if and only if for each $\alpha \in A$,
(i) $\phi[\alpha](0) \geq \phi[\alpha](x)$ for all $x \in X$;
(ii) $\phi[\alpha](y \star x) \geq min\{\phi[\alpha](z \star y), \phi[\alpha](z \star x)\}$ for all $x, y, z \in X$.

Example 4.2. Let $X = \mathbb{Z}$ defined by $x \star y = y - x$ for all $x, y \in X$ is a PMS-algebra under the usual subtraction. Let $A = \{\alpha_1, \alpha_2\}$ be the set of parameters, where $\alpha_1 =$ divisible by 2 and $\alpha_2 =$ divisible by 3.
For parameter α_1, a mapping $\phi[\alpha_1] : X \longrightarrow [0, 1]$ defined by

$$\phi[\alpha_1](x) = \begin{cases} 0.6 & \text{if } x \in <4> \\ 0.4 & \text{if } x \in <2> - <4> \\ 0 & \text{otherwise} \end{cases}$$

Hence $\phi[\alpha_1]$ is a fuzzy PMS-ideal over X.
For parameter α_2, a mapping $\phi[\alpha_2] : Z \longrightarrow [0, 1]$ defined by

$$\phi[\alpha_1](x) = \begin{cases} 0.7 & \text{if } x \in <2> \\ 0.5 & \text{if } x \in <3> - <2> \\ 0.3 & \text{if } x \notin <2> \ and\ x \notin <3> \end{cases}$$

; where $<4> = \{4t/t \in \mathbf{Z}\}$, $<2> = \{2t/t \in \mathbb{Z}\}$ and $<3> = \{3t/t \in \mathbb{Z}\}$. Hence $\phi[\alpha_2]$ is also a fuzzy PMS-ideal over X. Thus $\langle\phi, A\rangle$ is a fuzzy soft PMS-ideal over X.

Theorem 4.3. *Every fuzzy soft PMS-ideal of a PMS-algebra X is a fuzzy soft PMS-algebra.*

Proof. Let $\langle\phi, A\rangle$ be a fuzzy soft PMS-ideal of X.
Since $\phi[\alpha](x \star y) \geq min\{\phi[\alpha](0 \star x), \phi[\alpha](0 \star y)\} = min\{\phi[\alpha](x), \phi[\alpha](y)\}$. Hence $\phi[\alpha]$ is a fuzzy PMS-subalgebra of X, and so $\langle\phi, A\rangle$ is a fuzzy soft PMS-algebra over X. □

Theorem 4.4. *Let $\langle\phi, A\rangle$ be a Fuzzy soft PMS-ideal over a PMS-algebra X. If B is a subset of A, then $\langle\phi|_B, B\rangle$ is a fuzzy soft PMS-ideal over X.*

Proof. Suppose $\langle\phi, A\rangle$ is a fuzzy soft PMS-ideal over X. Let $B \subseteq A$ and $x \in X$. Then $\phi|_B[\alpha](x) \neq \emptyset$. Since $\phi[\alpha](x)$ is a fuzzy PMS-ideal of X for all $\alpha \in A$ and $\phi|_B(x) \subseteq \phi[\alpha](x), \forall\alpha \in B$, then $\phi|_B(x)$ is a fuzzy PMS-ideal of X. Hence, $\langle\phi|_B, B\rangle$ is a fuzzy soft PMS-ideal over X. □

Theorem 4.5. *Let $\langle\phi, A\rangle$ be a fuzzy soft PMS-ideal over X. Then $X_{\phi_\alpha} = \{x \in X : \phi[\alpha](x) = \phi[\alpha](0);\ \alpha \in A\}$ is a PMS-ideal of X.*

Proof. Suppose $\langle \phi, A\rangle$ is a fuzzy soft PMS-ideal over X. Since $x \star x = 0$, $\phi[\alpha](0) = \phi[\alpha](x \star x) \geq min\{\phi[\alpha](x), \phi[\alpha](x)\} = \phi[\alpha](x)$ for each $\alpha \in A$ and $x \in X$. Let $x, y, z \in X$ such that $z \star y \in X_{\phi_\alpha}$ and $z \star x \in X_{\phi_\alpha}$. Then $\phi[\alpha](z \star y) = \phi[\alpha](0)$ and $\phi[\alpha](z \star x) = \phi[\alpha](0)$. Since $\phi[\alpha]$ is a PMS-ideal, $\phi[\alpha](y \star x) \geq min\{\phi[\alpha](z \star y), \phi[\alpha](z \star x)\} = min\{\phi[\alpha](0), \phi[\alpha](0)\} = \phi[\alpha](0)$ and $\phi[\alpha](0) \geq [\alpha(y \star x)]$. Hence $\phi[\alpha](y \star x) = \phi[\alpha](0)$ and so X_{ϕ_α} is a PMS-ideal of X. □

Theorem 4.6. *Let $\langle \phi, A\rangle$ be a fuzzy soft set over a PMS-algebra X. Then $\langle \phi, A\rangle$ is a fuzzy soft PMS-ideal of X if and only if $\langle \phi^t, A\rangle$ is a soft PMS-ideal over X for each $t \in [0, 1]$.*

Proof. Suppose $\langle \phi, A\rangle$ is a fuzzy soft PMS-ideal over X. For each $t \in [0, 1]$ and $\alpha \in A$. Let $x \in \phi^t[\alpha]$. Then $\phi[\alpha](x) \geq t$, and since $\langle \phi, A\rangle$ is a fuzzy soft PMS-ideal, $\phi[\alpha](0) \geq \phi[\alpha](x)$ for all $x \in X$. It follows that $\phi[\alpha](0) \geq t$. So $0 \in \phi^t[\alpha]$. Let $x, y, z \in X$ such that $z \star y \in \phi^t[\alpha]$ and $z \star x \in \phi^t[\alpha]$. Then $\phi^t[\alpha](z \star y) \geq t$ and $\phi^t[\alpha](z \star x) \geq t$. Since $\langle \phi, A\rangle$ is a fuzzy soft PMS-ideal, $\phi[\alpha](y \star x) \geq min\{\phi[\alpha](z \star y), \phi[\alpha](z \star x)\} \geq t$. It follows that $y \star x \in \phi^t[\alpha]$. Hence $\phi^t[\alpha]$ is a PMS-ideal of X. Thus by Definition 2.4, $\langle \phi^t, A\rangle$ is a soft PMS-ideal over X for each $t \in [0, 1]$.
Conversely, assume that $\langle \phi^t[\alpha], A\rangle$ is a soft PMS-ideal over X for each $t \in A$. Let $\phi[\alpha](x) = t$. Since $\phi^t[\alpha]$ is a PMS-ideal of X, we have $0 \in \phi^t[\alpha]$. Hence $\phi[\alpha](0) \geq t = \phi[\alpha](x)$. For each $\alpha \in A$ and let $x, y, z \in X$ such that $\phi[\alpha](z \star y) = t_1$ and $\phi[\alpha](z \star x) = t_2$. Then $z \star y \in \phi^{t_1}[\alpha]$ and $z \star x \in \phi^{t_2}[\alpha]$. Take $t = min\{t_1, t_2\}$. Then $z \star y, z \star x \in \phi^t[\alpha]$. Since $\phi^t[\alpha]$ is a PMS-ideal of X, $y \star x \in \phi^t[\alpha]$. So $\phi[\alpha](y \star x) \geq t = min\{\phi[\alpha](z \star y), \phi[\alpha](z \star x)\}$. Hence $\phi[\alpha]$ is a fuzzy PMS-ideal of X. Therefore $\langle \phi, A\rangle$ is a fuzzy soft PMS-ideal over X. □

Theorem 4.7. *Let X be a PMS-algebra and let $\langle \phi, A\rangle$ and $\langle \phi, B\rangle$ be fuzzy soft PMS-ideals over X. If $A \cap B \neq \emptyset$, then $\langle \phi, A\rangle \,\widetilde{\cap}\, \langle \phi, B\rangle$ is a fuzzy soft PMS-ideal over X.*

Proof. Suppose $\langle \phi, A\rangle \,\widetilde{\cap}\, \langle \sigma, B\rangle = \langle \gamma, C\rangle$, where $C = A \cap B$. Let $\alpha \in C$. Then $\gamma[\alpha] = \phi[\alpha] \cap \sigma[\alpha]$. Let $x \in X$. Then

$$\begin{aligned}
\gamma[\alpha](0) &= (\phi[\alpha] \cap \sigma[\alpha])(0) \\
&= min\{\phi[\alpha](0), \sigma[\alpha](0)\} \\
&\geq min\{\phi[\alpha](x), \sigma[\alpha](x)\} \\
&= (\phi[\alpha] \cap \sigma[\alpha])(x) \\
&= \gamma[\alpha](x).
\end{aligned}$$

Hence $\gamma[\alpha](0) \geq \gamma[\alpha](x)$. Let $x, y, z \in X$.

$$\begin{aligned}
\gamma[\alpha](y \star x) &= (\phi[\alpha] \cap \sigma[\alpha])(y \star x) \\
&= min\{\phi[\alpha](y \star x), \sigma[\alpha](y \star x)\} \\
&\geq min\{min\{\phi[\alpha](z \star y), \phi[\alpha](z \star x)\}, min\{\sigma[\alpha](z \star y), \sigma[\alpha](z \star x)\}\} \\
&= min\{min\{\phi[\alpha](z \star y), \sigma[\alpha](z \star y)\}, min\{\phi[\alpha](z \star x), \sigma[\alpha](z \star x)\}\} \\
&= min\{(\phi[\alpha] \cap \sigma[\alpha])(z \star y), (\phi[\alpha] \cap \sigma[\alpha])(z \star x)\} \\
&= min\{\gamma[\alpha](z \star y), \gamma[\alpha](z \star x)\}
\end{aligned}$$

Hence $\langle \phi, A\rangle \,\widetilde{\cap}\, \langle \sigma, B\rangle = \langle \gamma, C\rangle$ is a fuzzy soft PMS-ideal over X. □

Theorem 4.8. *Let X be a PMS-algebra; and let $\langle \phi, A\rangle$ and $\langle \phi, B\rangle$ be fuzzy soft PMS-ideals over X. If $A \cap B = \emptyset$, then $\langle \phi, A\rangle \,\widetilde{\cup}\, \langle \phi, B\rangle$ is a fuzzy soft PMS-ideal over X.*

Proof. Define $\langle \phi, A\rangle \,\widetilde{\cup}\, \langle \sigma, B\rangle = \langle \gamma, C\rangle$, where $A \cup B = C$. Since $A \cap B = \emptyset$, either $\alpha \in A \backslash B$ or $\alpha \in B \backslash A$ for all $a \in C$. If $\alpha \in A \backslash B$, then $\gamma[\alpha] = \phi[\alpha]$ is a fuzzy PMS-ideal of X because $\langle \phi, A\rangle$ is a fuzzy soft PMS-ideal over X. If $\alpha \in B \backslash A$, then $\gamma[\alpha] = \sigma[\alpha]$ is a fuzzy PMS-ideal of X because $\langle \sigma, B\rangle$ is a fuzzy soft PMS-ideal over X. Hence $\langle \phi, A\rangle \,\widetilde{\cup}\, \langle \sigma, B\rangle = \langle \gamma, C\rangle$ is a fuzzy soft PMS-ideal over X. □

Theorem 4.9. *If $\langle \phi, A\rangle$ and $\langle \sigma, B\rangle$ are Fuzzy soft PMS-ideals over a PMS-algebra X, then the fuzzy soft set $\langle \phi, A\rangle \,\widetilde{\wedge}\, \langle \sigma, B\rangle$ is a fuzzy soft PMS-ideal over X.*

Proof. Define $\langle \phi, A\rangle \,\widetilde{\wedge}\, \langle \sigma, B\rangle = \langle \gamma, C\rangle$, where $C = A \times B$. Let $(\alpha_1, \alpha_2) \in A \times B$. Then $\gamma[\alpha_1, \alpha_2] = \phi[\alpha_1] \cap \sigma[\alpha_2]$. Let $x \in X$. Then

$$\begin{aligned}
\gamma[\alpha_1, \alpha_2](0) &= (\phi[\alpha_1] \cap \sigma[\alpha_2])(0) \\
&= min\{\phi[\alpha_1](0), \sigma[\alpha_2](0)\} \\
&\geq min\{\phi[\alpha_1](x), \sigma[\alpha_2](x)\} \\
&= (\phi[\alpha_1] \cap \sigma[\alpha_2])(x) \\
&= \gamma[\alpha_1, \alpha_2](x).
\end{aligned}$$

Let $x, y, z \in X$. Then

$$\begin{aligned}
\gamma[\alpha_1, \alpha_2](y \star x) &= (\phi[\alpha_1] \cap \sigma[\alpha_2])(y \star x) \\
&= min\{\phi[\alpha_1](y \star x), \sigma[\alpha_2](y \star x)\} \\
&\geq min\{min\{\phi[\alpha_1](z \star y), \phi[\alpha_1](z \star x)\}, min\{\sigma[\alpha_2](z \star y), \sigma[\alpha_2](z \star x)\}\} \\
&= min\{min\{\phi[\alpha_1](z \star y), \sigma[\alpha_2](z \star y)\}, min\{\phi[\alpha_1](z \star x), \sigma[\alpha_2](z \star x)\}\} \\
&= min\{\gamma[\alpha_1, \alpha_2](z \star y), \gamma[\alpha_1, \alpha_2](z \star x)\}.
\end{aligned}$$

Hence $\langle \phi, A\rangle \,\widetilde{\wedge}\, \langle \sigma, B\rangle = \langle \gamma, C\rangle$ is a fuzzy soft PMS-ideal of X, where $C = A \times B$. □

Theorem 4.10. *Let $f : X \longrightarrow Y$ be an epimorphism of PMS-algebras. If $\langle \phi, A\rangle$ is a fuzzy soft PMS-ideal over X with sup property, then $\langle f(\phi_1), A\rangle$ is a fuzzy soft PMS-ideal over Y.*

Proof. Let $\alpha \in A$ and Let $a, b, c \in X$ with $x \in f^{-1}(a), y \in f^{-1}(b), z \in f^{-1}(c)$ such that $\phi[\alpha](x) = sup\{\phi[\alpha](t) : t \in f^{-1}(a)\}, \phi[\alpha](y) = sup\{\phi[\alpha](t) : t \in f^{-1}(b)\}, \phi[\alpha](z) = sup\{\phi[\alpha](t) : t \in f^{-1}(c)\}$. Now

$$\begin{aligned}
f(\phi[\alpha])(0) &= sup\{\phi[\alpha](t) : t \in f^{-1}(0)\} \\
&= \phi[\alpha](0) \\
&\geq \phi[\alpha](x) \\
&= sup\{\phi[\alpha](t) : t \in f^{-1}(a)\} \\
&= f(\phi[\alpha])(a)
\end{aligned}$$

$$\begin{aligned}
f(\phi[\alpha])(b \star a) &= sup\{\phi[\alpha](t) : t \in f^{-1}(b \star a)\} \\
&= \phi[\alpha](y \star x) \\
&\geq min\{\phi[\alpha](z \star y), \phi[\alpha](z \star x)\} \\
&= min\{sup\{\phi[\alpha](t) : t \in f^{-1}(c \star b)\}, sup\{\phi[\alpha](t) : t \in f^{-1}(c \star a)\}\} \\
&= min\{f(\phi[\alpha])(c \star b), f(\phi[\alpha])(c \star a)\}
\end{aligned}$$

Hence $f(\phi[\alpha])$ is a fuzzy PMS-ideal of Y, and so $\langle f(\phi_1), A\rangle$ is a fuzzy soft PMS-ideal over Y. □

Theorem 4.11. *Let $f : X \longrightarrow Y$ be a homomorphism of PMS-algebras. If $\langle \phi, A\rangle$ is fuzzy soft PMS-ideal over Y, then $\left\langle f^{-1}(\phi_1), A\right\rangle$ is a fuzzy soft PMS-ideal over X.*

Proof. Let $\alpha \in A$ and let $x, y, z \in X$.
$f^{-1}(\phi[\alpha])(0) = \phi[\alpha](f(0)) = \phi[\alpha](0) \geq \phi[\alpha](f(x)) = f^{-1}(\phi[\alpha])(x)$.

$$\begin{aligned}
f^{-1}(\phi[\alpha])(y \star x) &= \phi[\alpha](f(y \star x)) \\
&= \phi[\alpha](f(y) \star f(x)) \\
&\geq min\{\phi[\alpha](f(z) \star f(y)), \phi[\alpha](f(z) \star f(x))\} \\
&= min\{\phi[\alpha](f(z \star y)), \phi[\alpha](f(z \star x))\} \\
&= min\{f^{-1}(\phi[\alpha](z \star y)), f^{-1}(\phi[\alpha](z \star x))\}
\end{aligned}$$

Hence $f^{-1}(\phi[\alpha])$ is a fuzzy PMS-ideal of X, and so $\left\langle f^{-1}(\phi_1), A\right\rangle$ is a fuzzy soft PMS-ideal over X. □

Theorem 4.12. *Let $f : X \longrightarrow Y$ be a homomorphism of PMS-algebras. If $\langle \phi, A\rangle$ is a fuzzy soft PMS-ideal over Y, then the fuzzy soft set $\left\langle \phi^f, A\right\rangle$ is a fuzzy soft PMS-ideal over X.*

Proof. Let $\alpha \in A$ and $x, y, z \in X$. Then
$\phi^f[\alpha](0) = \phi[\alpha](f(0)) = \phi[\alpha](0) \geq \phi[\alpha](f(x)) = \phi^f[\alpha](x)$

$$\begin{aligned}
\phi^f[\alpha](y \star x) &= \phi[\alpha](f(y \star x)) \\
&= \phi[\alpha](f(y) \star f(x)) \\
&\geq min\{\phi[\alpha](f(z) \star f(y)), \phi[\alpha](f(z) \star f(x))\} \\
&= min\{\phi[\alpha](f(z \star y)), \phi[\alpha](f(z \star x))\} \\
&= min\{\phi^f[\alpha](z \star y), \phi^f[\alpha](z \star x)\}
\end{aligned}$$

Hence $\phi^f[\alpha]$ is a fuzzy PMS-ideal of X, and so $\left\langle \phi^f, A\right\rangle$ is a fuzzy soft PMS-ideal over X. □

However, the converse of Theorem 4.12 is true if f is an epimorphism of PMS-algebras as shown in Theorem 4.13

Theorem 4.13. *Let $f : X \longrightarrow Y$ be an epimorphism of PMS-algebras. If $\left\langle \phi^f, A\right\rangle$ is a fuzzy soft PMS-ideal over X, then $\langle \phi, A\rangle$ is a fuzzy soft PMS-ideal over Y.*

Proof. Let $\alpha \in A$ and let $a, b, c \in Y$. Then there exists $x, y, z \in X$ such that $f(x) = a$ and $f(y) = b, f(z) = c$. Now
$\phi[\alpha](0) = \phi[\alpha](f(0)) = \phi^f[\alpha](0) \geq \phi^f[\alpha](x) = \phi[\alpha](f(x)) = \phi[\alpha](a)$.

$$\begin{aligned}
\phi[\alpha](b \star a) &= \phi[\alpha](f(y) \star f(x)) \\
&= \phi[\alpha](f(y \star x)) \\
&= \phi^f[\alpha](y \star x) \\
&\geq min\{\phi^f[\alpha](z \star y), \phi^f[\alpha](z \star x)\} \\
&= min\{\phi[\alpha](f(z \star y)), \phi[\alpha](f(z \star x))\} \\
&= min\{\phi[\alpha](f(z) \star f(y)), \phi[\alpha](f(z) \star f(x))\} \\
&= min\{\phi[\alpha](a), \phi[\alpha](b)\}
\end{aligned}$$

Hence $\phi[\alpha]$ is a fuzzy PMS-ideal of Y, and so $\langle \phi, A\rangle$ is a fuzzy soft PMS-ideal over Y. □

5 Conclusion remarks

A fuzzy soft set is a parameterized family of subsets of the universe in which the parameterized families are fuzzy subsets. Fuzzy soft sets are extensions of soft sets, obtained by merging fuzzy sets with soft sets. These sets are applied to various algebras to form their algebraic structures. In a similar approach, we have characterized fuzzy soft sets in PMS-algebras. We studied the structure of PMS-subalgebras, PMS-ideals, and their detailed properties within PMS-algebras by applying fuzzy soft sets. We also characterized the properties of fuzzy soft PMS-algebras and fuzzy soft PMS-ideals under homomorphisms. Furthermore, we proved the equivalence of classical soft PMS-algebras and fuzzy soft PMS-algebras by fixing the parameters. The study of fuzzy soft PMS-algebras is relevant for understanding uncertainties and can also be applied to other algebraic structures closely related to PMS-algebras. In our future work, we plan to extend soft homomorphisms and isomorphisms. Moreover, we can extend this concept to intuitionistic fuzzy soft PMS-algebras and characterize fuzzy soft PMS-algebras based on soft binary operations.

References

[1] D. Molodtsov, *Soft set theory first results*, Computers and mathematics with applications, **37(4-5)** , 19–31, (1999).

[2] P. Maji, A. R. Roy and R. Biswas, *An application of soft sets in a decision making problem*, Computers and Mathematics with Applications, **44(8-9)**, 1077–1083, (2002).

[3] A. R. Roy and P. Maji, *A fuzzy soft set theoretic approach to decision making problems*, Journal of computational and Applied Mathematics, **203(2)**, 412–418, (2007).

[4] P. K. Maji, R. Biswas and A. R. Roy, *Soft set theory*, Computers and mathematics with applications, **45(4-5)**, 555–562, (2003).

[5] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, *On some new operations in soft set theory*, Computers and Mathematics with Applications, **57(9)**, 1547–1553, (2009).

[6] H. Aktas and N. Cagman, *Soft sets and soft groups*, Information sciences, **177(13)**, 2726–2735, (2007).

[7] Y. B. Jun, *Soft BCK/BCI-algebras*, Computers and Mathematics with Applications, **56(5)**, 1408–1413, (2008).

[8] N. M. Kassahun, B. A. Alaba, Y. G. Wondifraw and Z. T. Wale *Soft PMS-algebras*, Korean Journal of Mathematics, **31(4)**, 465–477, (2023).

[9] L. A. Zadeh, *Fuzzy sets*, Information and control, **8(3)**, 338–353, (1965).

[10] A. Rosenfeld, *Fuzzy groups*, Journal of mathematical analysis and applications, **35(3)**, 512–517, (1971).

[11] P. S. Selvam and K. Nagalakshmi, *Fuzzy PMS-ideals in PMS-algebras*, Annals of pure and applied mathematics, **12(2)**, 153–159, (2016).

[12] S. Shaqaqha, *Characterizations of Artinian and Noetherian Gamma Rings in Terms of Homogeneous Complex Fuzzy Ideals*, Palistenian Journal of Mathematics, **11(1)**, 260–264, (2022).

[13] G. Muhiuddin, M. Al-Tahan, A. Mahboob, S. Hoskova-Mayerova, and S. Al-Kaseasbeh, *Linear Diophantine fuzzy set theory applied to BCK/BCI-Algebras*, Mathematics, **10(12)**, 2138, (2022).

[14] M. Al-Tahan, A. Rezaei, S. Al-Kaseasbeh, B. Davvaz, and M. Riaz, *Linear Diophantine fuzzy n-fold weak subalgebras of a BE-algebra*, Missouri Journal of Mathematical Sciences, **35(2)**, 136–148, (2023).

[15] M.Y. Abbasi, A.F. Talee, S.A. Khan, K. Hila, *A Hesitant Fuzzy Set Approach to Ideal Theory in Γ Semigroups*, Advances in Fuzzy Systems, **1**, 5738024, (2018).

[16] V. Torra, *Hesitant fuzzy sets*, International journal of intelligent systems, **25(6)**, 529–539, (2010).

[17] V.Torra, Y. Narukawa, *On hesitant fuzzy sets and decision*, In Proceedings of the IEEE International Conference on Fuzzy Systems, Jeju-do,Republic of Korea, 1378–1382, (2009).

[18] R. M.Rodriguez, L. Martinez, V. Torra, Z.S. Xu, F. Herrera, *Hesitant fuzzy sets: state of the art and future directions*, International journal of intelligent systems, **29(6)**, 495–524, (2014).

[19] A. Girum, B. Assaye, B. Tarekegn, B. Lamesgin, *Hesitant Fuzzy T-ideals of TM algebra*, Palestine Journal of Mathematics, **14(2)**, 653–661, (2025).

[20] P. K. Maji, R. Biswas, and A. Roy, *Fuzzy soft sets*, Journal of Fuzzy Mathematics, **9**, 589–602, (2001).

[21] A. Aygünolu and H. Aygün, *Introduction to fuzzy soft groups*, Computers and Mathematics with Applications, **58(6)**, 1279–1286, (2009).

[22] Y. B. Jun, K. Lee and C. H. Park, *Fuzzy soft set theory applied to BCK/BCI-algebras*, Computers and Mathematics with Applications, **59(9)**, 3180–3192, (2010).

[23] C. -F. Yang, *Fuzzy soft semigroups and fuzzy soft ideals*, Computers and Mathematics with Applications, **61(2)**, 255–261 (2011).

[24] A. Kharal and B. Ahmad, *Mappings on fuzzy soft classes*, Advances in fuzzy systems, **2009**, 407890, (2009).

[25] P. S. Sithar Selvam and K. Nagalakshmi, *On PMS-algebras*, Transylvanian Review, **24(10)**, 1622–1628, (2016).

[26] P. Sharma, *α-fuzzy subgroups*, International Journal of fuzzy Mathematics and systems, **3(1)**, 47–59, (2013).

Author information

Nibret Melese Kassahun, Department of Mathematics, Debre Markos University, Debre Markos, Ethiopia.
E-mail: nibret.melese20100@gmail.com

Berhanu Assaye Alaba, Department of Mathematics, Bahir Dar University, Bahir Dar,, Ethiopia.
E-mail: birhanu.assaye290113@gmail.com

Yohannes Gedamu Wondifraw, Department of Mathematics, Bahir Dar University, Bahir Dar,, Ethiopia.
E-mail: yohannesg27@gmail.com

Zelalem Teshome Wale, Department of Mathematics, Addis Ababa University, Addis Ababa,, Ethiopia.
E-mail: zelalem.teshome@aau.edu.et

Received: 2025-03-01
Accepted: 2025-07-11