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A note on meromorphic functions on a compact Riemann surface having poles at a single point

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Abstract. In this paper, we present a proof of the Weierstrass gap theorem for compact Riemann surfaces using the terminology of cohomology Groups. In the process, we identify an interesting combinatorial problem that naturally arises as a byproduct of the statement of the theorem. Finally, we include a brief note on Weierstrass points and obtain some consequences of the Weierstrass gap theorem.

1 Introduction

Let X be a compact Riemann surface of genus g. One of the important questions is the existence of a meromorphic function having a pole at a point P, and is holomorphic in $X \setminus \{P\}$.

The case $g = 0$ is trivial, as there is always a function on the sphere with a simple pole. In case of $g = 1$, there is no meromorphic function on X with a simple pole at P that is holomorphic in $X \setminus \{P\}$. This follows from the well-known fact that there is no doubly periodic meromorphic function having precisely a simple pole at a single point in any period parallelogram.

The aim of this article is to study the meromorphic functions with a pole of prescribed order at a single point on a compact Riemann surface of genus $g \geq 2$.

The following theorem is a simple consequence of the Riemann-Roch theorem proved in [2], which guarantees the existence of a meromorphic function at a point P, with the order of the pole is $\leq g + 1$.

Theorem 1.1. *Suppose X is a compact Riemann surface of genus g and P is a point on X. Then there is a non-constant meromorphic function $f \in \mathcal{M}(X)$ which has a pole of order $\leq g + 1$ at P and is holomorphic in $X \setminus \{P\}$.*

There are some exceptional points, known as Weierstrass points which play an important role in the existence of meromorphic functions with a pole of oder $\leq g$ and holomorphic in $X \setminus \{P\}$. See [2].

Theorem 1.2. *Suppose X is a compact Riemann surface of genus g and P is a point on X. Then there exists a non-constant meromorphic function on X which has a pole of order $\leq g$ at P and is holomorphic in $X \setminus \{P\}$ if and only if P is a Weierstrass point.*

In this note we prove the following theorem which provides more information about the order of a pole at a point P on X:

Theorem 1.3 (Weierstrass gap theorem). *Let X be a compact Riemann surface of genus $g \geq 1$. Suppose P is a point on X. Then there are precisely g orders n_k,*

$$1 = n_1 < n_2 < \cdots < n_g < 2g \tag{1.1}$$

such that there does not exist a meromorphic function on X with a pole of order n_k at P, and holomorphic in $X \setminus \{P\}$.

The numbers n_k, for $k = 1, \dots, g$ are called gaps at P and their complement in $\mathbb{N}$ are called non-gaps. There are precisely g non-gaps in $\{2, \dots, 2g\}$. Further, the gap sequence is uniquely determined by the point P.

In general, the proof of the Weierstrass gap theorem is proved as an application of Riemann-Roch theorem [1, 5, 6]. In [1], the proof of the Weierstrass gap theorem was derived as a special case of the Noether theorem. In this expository note, Theorem 1.3 will be proved in the spirit of the proof given in [6], using the terminology of sheaf cohomology, [3].

In Section 2, we mention some of the consequences of Riemann-Roch and Serre Duality theorems that are useful for comparing the dimensions of cohomology groups, [2]. The proof of the Weierstrass gap theorem is given in Section 3. In Section 4, we form a combinatorial problem that appears to be a byproduct of the statement of the Weierstrass gap theorem. In Section 5, we mention some remarks on Weierstrass points and conclude with some examples in Section 6.

2 Some Consequences of Riemann-Roch and Serre Duality Theorems

One can refer [2], for more detailed material presented in this section.

Definition 2.1. A divisor on X is a mapping

$$D : X \to \mathbb{Z}$$

such that for any compact subset $K \subset X$ there are only finitely many points $x \in K$ such that $D(x) \neq 0$. The set of all divisors on X is an abelian group with respect to addition which we denote it by $Div(X)$.

Definition 2.2. Suppose U is an open subset of X and f is a meromorphic function on U. For $P \in U$ define

$$ord_P(f) = \begin{cases} 0, \text{ if } f \text{ is holomorphic and non-zero at } P \\ m, \text{ if } f \text{ has a zero of order } m \text{ at } P \\ -m, \text{ if } f \text{ has a pole of order } m \text{ at } P \\ \infty \text{ if } f \text{ is identically zero in a neighborhood of } P \end{cases} \tag{2.1}$$

Thus for any meromorphic function $f \in \mathcal{M}(X) \setminus \{0\}$, the mapping $x \mapsto ord_x(f)$ is a divisor on X and is denoted by (f). The function f is said to be a multiple of the divisor D if $(f) \geq D$. If f is holomorphic then $(f) \geq 0$.

To define divisor of a meromorphic 1-form ω on U at a point $P \in U$, choose a coordinate chart (V, z) of P. Then on $V \cap U$, one may write $\omega = f dz$, where f is a meromorphic function. Set $ord_P(\omega) = ord_P(f)$. It can be verified that this is independent of the choice of the chart. For 1-forms $\omega \in (X) \setminus \{0\}$ the mapping $x \mapsto ord_x(\omega)$ is again a divisor on X, denoted by (ω).

A divisor $D \in Div(X)$ is called a *principal divisor* if there exists a non-zero meromorphic function $f \in \mathcal{M}(X) \setminus \{0\}$ such that $D = (f)$. The divisor of a meromorphic 1-form $\omega \in \mathcal{M}^1(X) \setminus \{0\}$ is called a *canonical divisor*.

Definition 2.3. (Degree of a Divisor) If X is a compact Riemann surface and D is a divisor on X then there are only finitely many $x \in X$ such that $D(x) \neq 0$. Hence one can define a mapping

$$\deg : Div(X) \to \mathbb{Z}$$

called the degree of D , by taking

$$\deg(D) = \sum_{x \in X} D(x)$$

Definition 2.4. Let D be a divisor on X. For any open set $U \subset X$, we define $\mathcal{O}_D(U)$ the set of all those meromorphic functions which are multiples of the divisor $-D$, i.e.,

$$\mathcal{O}_D(U) = \{f \in \mathcal{M}(U) : ord_x(f) + D(x) \geq 0 \text{ for every } x \in U\}.$$

It can be verified $\mathcal{O}_D$ is a sheaf. It denotes the sheaf of meromorphic functions which are multiples of the divisor $-D$. If $D = 0$, one has $\mathcal{O}_0 = \mathcal{O}$ which is the set of holomorphic functions on X.

To D, we can associate a line bundle L_D such that the sheaf of holomorphic cross sections of L_D is isomorphic to the sheaf $\mathcal{O}_D$ of meromorphic multiples of $-D$ and identified with $H^0(X, L_D)$. We prefer to use the notation $H^0(X, \mathcal{O}_D)$.

Theorem 2.5. *Let X be a compact Riemann surface and $D \in Div(X)$. If $\deg D < 0$ then $H^0(X, \mathcal{O}_D) = 0$.*

Proof. Suppose $f \in H^0(X, \mathcal{O}_D)$ with $f \neq 0$. Then $(f) \geq -D$ and thus

$$\deg(f) \geq -\deg D > 0$$

This contradicts the fact that X is compact and $\deg(f) = 0$. □

Theorem 2.6. *(The Riemann-Roch Theorem) Suppose D is a divisor on a compact Riemann surfaceX of genus g. Then $H^0(X, \mathcal{O}_D)$ and $H^1(X, \mathcal{O}_D)$ are finite dimensional vector spaces and*

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg D$$

We can use Serre-Duality theorem to obtain equality of dimensions:

$$\dim H^0(X, \Omega_{-D}) = \dim H^1(X, \mathcal{O}_D) \tag{2.2}$$

For $D = 0$ we obtain

$$\dim H^0(X, \Omega) = g = \dim H^1(X, \mathcal{O}) \tag{2.3}$$

Here $H^0(X, \Omega) = \Omega(X)$ which denotes the sheaf of holomorphic 1-forms on X. The following equation can also be obtained as an application of Seree Duality Theorem:

$$\dim H^1(X, \Omega) = \dim H^0(X, \mathcal{O}) = 1 \tag{2.4}$$

Remark 2.7. This is a known result that there are no non-constant holomorphic maps on a compact Riemann surface.

Suppose X is a compact Riemann surface of genus g and ω is a non-vanishing meromorphic 1-from on X. We denote $K = (\omega)$ which is a canonical divisor. Then $\deg(K) = 2g - 2$. Hence the degree of a canonical divisor $K = 2g - 2$.

In view of (2.2), we may also use the following form of the Riemann-Roch theorem in proof of theorem 1.3.

$$\dim H^0(X, \mathcal{O}_D) - \dim H^0(X, \Omega_{-D}) = 1 - g + \deg D \tag{2.5}$$

Lemma 2.8. *Suppose X is a compact Riemann surface of genus g and D is a divisor on X. Then*

$$H^0(X, \Omega_{-D}) = 0 \text{ whenever } > 2g - 2 \tag{2.6}$$

Proof. Suppose ω is a non-vanishing meromorphic 1-form on X and K is its canonical divisor. Then there is an isomorphism $\Omega_{-D} \cong \mathcal{O}_{K-D}$. Thus

$$H^0(X, \Omega_{-D}) \cong H^0(X, \mathcal{O}_{K-D})$$

If $\deg D > 2g - 2$, then $\deg(K - D) < 0$ and $H^0(X, \mathcal{O}_{K-D}) = 0$. □

3 Proof of Theorem 1.3 (Weierstrass Gap Theorem)

Proof. Suppose $P \in X$. If D is a zero divisor, then $\dim H^1(X, \mathcal{O}) = g$ and $\deg D = 0$.

By the Riemann Roch theorem $\dim H^0(X, \mathcal{O}) = 1$. Therefore, there are no non-constant holomorphic functions on X.

Define the divisor D_n such that

$$D_n(x) = \begin{cases} n \text{ if } x = P \\ 0 \text{ if } x \neq P \end{cases}$$

For $n = 1$, we have deg $D_1 = 1$. Once again by the Riemann -Roch theorem,

$$\dim H^0(X, \mathcal{O}_{D_1}) = 2 - g + \dim H^0(X, \Omega_{-D_1}).$$

If $\dim H^0(X, \Omega_{-D_1}) = g$, then $\dim H^0(X, \mathcal{O}_{D_1}) = 2$, hence there exists $f \in \mathcal{M}(X)$ which has a simple pole at P and is holomorphic in $X \setminus \{P\}$.

If $\dim H^0(X, \Omega_{-D_1}) = g - 1$, then $\dim H^0(X, \mathcal{O}_{D_1}) = 1$, hence there is no meromorphic function which has a simple pole at P and is holomorphic in $X \setminus \{P\}$.

Now we want to see the effect of changing the divisor from D_{n-1} to D_n. By the Riemann - Roch Theorem

$$\dim H^0(X, \mathcal{O}_{D_{n-1}}) = n - g + \dim H^0(X, \Omega_{-D_{n-1}})$$

and

$$\dim H^0(X, \mathcal{O}_{D_n}) = n + 1 - g + \dim H^0(X, \Omega_{-D_n})$$

If

$$\dim H^0(X, \Omega_{-D_{n-1}}) = \dim H^0(X, \Omega_{-D_n})$$

then

$$\dim H^0(X, \mathcal{O}_{D_{n-1}}) + 1 = \dim H^0(X, \mathcal{O}_{D_n}).$$

So there exists a meromorphic function $f \in \mathcal{M}(X)$ with a pole of order n at P and is holomorphic in $X \setminus \{P\}$.

If

$$\dim H^0(X, \Omega_{-D_{n-1}}) = \dim H^0(X, \Omega_{-D_n}) + 1$$

then

$$\dim H^0(X, \mathcal{O}_{D_n}) = \dim H^0(X, \mathcal{O}_{D_{n-1}}).$$

So there will not exist a function with a pole of order n at P and is holomorphic in $X \setminus \{P\}$.

So if $\dim H^0(X, \Omega_{-D_n})$ remains the same as n increases by 1, then a new linearly independent function is added in going from the sheaf $\mathcal{O}_{D_{n-1}}$ to $\mathcal{O}_{D_n}$.

From (2.3), we have $\dim H^0(X, \Omega) = g$. Suppose ω is a non-vanishing meromorphic 1-form on a compact Riemann surface of genus g, and K is its divisor, then $\deg(\omega) = 2g - 2$.

By Lemma (2.8)

$$\dim H^0(X, \Omega_{-D_{2g-1}}) = 0.$$

Therefore, the number of times $\dim H^0(X, \Omega_{-D_n})$ does not remain the same must be g times and at each change it decreases by 1.

This completes the proof. □

4 Analyzing gaps and non-gaps

Example 4.1. Suppose X is a compact Riemann surface of genus $g = 3$. We see the possible gap sequences are

$\{1, 3, 5\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$

and corresponding non-gap sequences in $\{2, \ldots, 2g\}$ are:

$\{2, 4, 6\}$, $\{4, 5, 6\}$, $\{3, 5, 6\}$, $\{3, 4, 6\}$.

Note that $2g$ is always a non-gap.

Suppose $P \in X$. If f has a pole of order s at P, and g has a pole of order t at P, then fg has a pole of order $s + t$ at P. Hence, the following problem may be of some interest to see the number of possible gaps and non-gaps $\leq 2g$ on X. The reader must note that this problem has nothing to do with the proof of the theorem.

Problem 1. *Write the numbers 2 to $2g - 1$ into two (disjoint) parts $G = \{n_1, n_2, \ldots, n_{g-1}\}$ and $G' = \{m_1, m_2, \ldots, m_{g-1}\}$ such that no number in G is a sum of any combination of numbers in G'. How many pairs of such G and G' exist?*

Clearly then $\{1\} \cup G$ gives possible gap sequence and $G' \cup \{2g\}$ gives corresponding non-gap sequence in $\{2, \ldots, 2g\}$ at a point P on X.

5 Weierstrass points And Hyperelliptic Riemann surfaces

We define a Weierstrass point by using gap sequence as follows:

Definition 5.1. Suppose $P \in X$ and

$$0 < n_1 < n_2 < \dots n_g < 2g$$

be the gap sequence at P. In terms of the gap sequence we define the weight of the point P, denoted by $\omega(P)$ by

$$\omega(P) = \sum_{i=1}^{g} (n_i - i)$$

Note that $w(P) \geq 0$ for all $P \in X$.

Definition 5.2. A point $P \in X$ is called a Weierstrass point if $\omega(P) > 0$.

One can compute the number of Weierstrass points counted according to their weights on a compact Riemann surface X of genus g. It is equal to $g^3 - g = (g-1)g(g+1)$.

It follows that there are no Weierstrass points on the surfaces of genus $g = 0$ and genus $g = 1$. Also from the theorem 1.3, it follows that there is no non-constant meromorphic function on torus ($g = 1$ surface) with a single simple(=order 1) pole.

The following theorem gives the bounds for Weierstrass points on a compact Riemann surface of genus $g \geq 2$.

Theorem 5.3. *Suppose X is a compact Riemann surface of genus $g \geq 2$. let $W(X)$ denotes the number of Weierstrass points on X. Then*

$$2g + 2 \leq W(X) \leq g^3 - g \tag{5.1}$$

A point p is called a hyperelliptic Weierstrass point if the non-gap sequence starts with 2 and the hyperelliptic Riemann surfaces are characterized by the gap sequence at the Weierstrass points:

$$P = \{1, 3, \dots, 2g-1\} \tag{5.2}$$

hence the non-gaps are Q={2,4, ..., 2g}

Let X be a hyperelliptic Riemann surface and p be a Weierstrass point on X. Then we can find

$$\omega(p) = [1 + 3 + \dots + (2g-1)] - [1 + 2 + \dots + g] = \frac{g(g-1)}{2} \tag{5.3}$$

Therefore, from (5.3) we can see that there are precisely $2g + 2$ Weierstrass points on a hyperelliptic Riemann surface.

In [4], Jenkins proved that if h is a first non-gap at P and $(h, k) = 1$, then k is a gap if

$$g > \frac{(h-1)(k-1)}{2}.$$

Furthermore, he notes that hyperelliptic Riemann surfaces are a special case whose first non-gap sequence begins with a prime number. For the Exceptional Riemann surfaces, the gap sequence G and the non-gap sequence G' at each Weierstrass point are given by

$$G = \{1, 2, 3, \dots, g-1, g+1\} \text{ and } G' = \{g, g+2, \dots, 2g-1\}. \tag{5.4}$$

6 Concluding remarks

Sometimes we can apply Theorem 1.3 to find more information on the possible order of a pole of a meromorphic function at a point P, which is holomorphic in $X \setminus \{P\}$. The following example illustrates this:

Example 6.1. Suppose X is a hyperelliptic Riemann surface of genus 6 and P is a Weierstrass point. Then there exist meromorphic functions with orders 2, 4, and 6 at P which are non-gaps of a hyperelliptic gap sequence. But at the same time, if P is not a Weierstrass point, there is no meromorphic function with a pole of order < 7 at P, and holomorphic in $X \setminus \{P\}$.

Another important application of the Weierstrass gap sequence consists in proving Automorphism group of a compact Riemann surface of genus $g \geq 2$ is finite. This is a well-known result proved by Schwarz. The proof follows from the observation that if σ is an automorphism of a compact Riemann surface of genus $g \geq 2$, then the Wierstrass points map to Weierstrass points. This follows again because the gap sequence at a point P and at $\sigma(P)$ is the same. Hurwitz found that the number of automorphisms does not exceed $84(g-1)$.

References

[1] *H.M. Farkas and I. Kra,* Riemann surfaces*, GTM, Springer, Second Edition, (1991).*

[2] *O. Forster,* Lectures on Riemann surfaces*, GTM 81, Springer-Verlag, New York, (1993).*

[3] *Gollakota V V Hemasundar,* A note on meromorphic functions on a compact Riemann surface having poles at a single point*, arXiv:2407.18286v2[math.cv], (2025)*

[4] *James A Jenkins,* Some Remarks on the Weierstrass points*, Proceedings of the American Mathematical Society ,* **V 44***, Number 1, May (1974).*

[5] *R.C. Gunning,* Lectures on Riemann surfaces*, Princeton University Press, second printing (1968).*

[6] *George Springer,* Introduction to Riemann surfaces*, Addison Wesley Publishing Company, Inc, (1957).*

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