Palestine Journal of Mathematics

Vol 15(1)(2026) , 273–283

Family of surfaces with common geodesic in a Walker 3-manifold

El Hadji Baye Camara and Ameth Ndiaye

Communicated by: Zafar Ahsan

MSC 2020 Classifications: Primary 53A10; Secondary 53C42.

Keywords and phrases: Ruled surface, Walker manifold, Curvature, Mean curvature, Gauss curvature.

The authors would like to express their sincere thanks to the Editor and the anonymous Reviewers for their helpful comments and suggestions.

Corresponding Author: Ameth Ndiaye

Abstract. The topic of creating a family of surfaces from a given curve was examined in this study. We obtain the necessary and sufficient conditions for the coefficients to meet the geodesic and isoparametric requirements by expressing the family of surfaces as a linear combination of the components of the Frenet trihedron frame of the curve in Walker 3-manifold. Then, we present some results regarding these family of surfaces being flat and minimal.

1 Introduction

It is commonly recognized that the dynamics of Lagrangians can be explained in large part by current differential geometry. Additionally, the dynamics of Hamiltonians can be developed within a good framework provided by contemporary differential geometry. In this work the ambiant space we will consider is a Lorentzian three-manifold admitting a parallel null vector field called strict Walker manifold. It is known that Walker metrics have served as a powerful tool of constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics, see [1, 5].

It is well-known that semi-Riemannian geometry has an important tool to describe spacetime events. Therefore, solutions of some structures about 4-Walker manifold can be used to explain spacetime singularities. In [7], the authors present complex and paracomplex analogues of Lagrangian and Hamiltonian mechanical systems on 4-Walker manifold. Finally, they discuss the geometrical-physical results related to complex (paracomplex) mechanical systems.

In [2], the author considered a Walker 3-manifold M_f^3 where f is a smooth function and he first characterized the Killing vector fields, aiming to obtain the corresponding Killing magnetic curves. When the manifold is endowed with a unitary spacelike vector field ξ, the author proved that after a reparameterization, any lightlike curve normal to ξ is a lightlike geodesic. He also showed that on M_f^3, equipped with a Killing vector field V, any arc length parameterized spacelike or timelike curve, normal to V, is a magnetic trajectory associated to V.

The relativistic explanation of gravity relies heavily on geodesics. Geodesics are the pathways of freely falling particles in a certain space, according to Einstein's principle of equivalence. In [3] the authors investigate the family of surfaces with common geodesic in Minkowski 3-space. For a given curve $\alpha = \alpha(s)$, they determine the parametric representation of the surface $\varphi(s,t)$. Using the Frenet trihedron frame from differential geometry, they determine the necessary constraints on the coefficients of vectors of the frame to satisfy the geodesic and isoparametric requirements, as well as the necessary and sufficient conditions for the correct parametric representation of the surface $\varphi(s,t)$ when the parameter s is the arc-length of the curve $\alpha = \alpha(s)$.

The study of differential geometry of surfaces captured many researchers' attention, see [4] for more informations. In [8], the authors construct two special families of ruled surfaces in a three dimensional strict Walker manifold. The local degeneracy (resp. non-degeneracy) to one of this family has a strong consequence on the geometry of the ambiant Walker manifold. In [9] the same authors study minimal translations surfaces in a strict Walker 3-manifold. Based on the existence of two isometries, they classify minimal translation surfaces on this class of manifold. See the paper [8] for further study.

Motivated by the above work, in this paper we study the geometric properties of some type of surfaces called family of surfaces with common geodesic in a Walker 3−dimensional manifold.

2 Preliminaries

2.1 Three Dimensional Walker Spaces

General Walker manifolds are pseudo-Riemannian manifolds (M, g^ϵ, D) with a distribution D on which g^ϵ is zero (a lightlike distribution) and that is parallel with respect to the Levi-Civita connection of g^ϵ. In the 3-dimensional case, Walker 3-manifolds have the specific feature that all geometric data is encoded in a single function. As a general fact, a canonical form for a $(2r+1)$-dimensional pseudo-Riemannian manifold M admitting a parallel field of null r-dimensional planes D is given by the metric tensor in matrix form:

$$g_{ij} = \begin{pmatrix} 0 & 0 & I_r \\ 0 & \varepsilon & 0 \\ I_r & 0 & B \end{pmatrix}$$

where I_r is the $r \times r$ identity matrix, B is a symmetric $r \times r$ matrix whose entries are functions of the coordinates $x_1, \ldots, x_{2r+1}$, and $\varepsilon = \pm 1$ [1]. Therefore, any 3-dimensional Walker manifold is locally isometric to the manifold M whose metric has the matrix:

$$g_w = g_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & w \end{pmatrix}$$

with respect to the natural frame $\{\partial_x, \partial_y, \partial_z\}$, where $w = w(x, y, z)$ is a smooth function defined on an open subset $O \subseteq \mathbb{R}^3$. Thus,

$$g_w = 2\, dx\, dz + \varepsilon\, dy^2 + w(x, y, z)\, dz^2. \tag{2.1}$$

The manifold M has signature $(1, 2)$ if $\varepsilon = 1$ and $(2, 1)$ if $\varepsilon = -1$. If $w(x, y, z) = w(y, z)$, then M is called a strict Walker manifold. The Levi-Civita connection ∇ is well-known, and the non-zero components of the Christoffel symbols are:

$$\nabla_{\partial_x} \partial_z = \frac{1}{2} w_x\, \partial_x, \quad \nabla_{\partial_y} \partial_z = \frac{1}{2} w_y\, \partial_x, \quad \nabla_{\partial_z} \partial_z = \frac{1}{2}(w w_x + w_z)\, \partial_x - \frac{\varepsilon}{2} w_y\, \partial_y - \frac{1}{2} w_x\, \partial_z.$$

The components of the curvature of the Walker 3-manifold (M, g_w) are given by

$$\begin{aligned} R(\partial y, \partial z)\partial z &= \frac{1}{2} w w_{xy} \partial x - \frac{\varepsilon}{2} w_{yy} \partial y - \frac{1}{2} w_{xy} \partial z; \\ R(\partial x, \partial z)\partial x &= \frac{1}{2} w_{xx} \partial x \quad ; \quad R(\partial x, \partial z)\partial y = \frac{1}{2} w_{yx} \partial x; \\ R(\partial x, \partial z)\partial z &= \frac{1}{2} w w_{xx} \partial x - \frac{\varepsilon}{2} w_{xy} \partial y - \frac{1}{2} w_{xx} \partial z; \\ R(\partial y, \partial z)\partial x &= \frac{1}{2} w_{yx} \partial x \quad ; \quad R(\partial y, \partial z)\partial y = \frac{1}{2} w_{yy} \partial x. \end{aligned} \tag{2.2}$$

Remark 2.1. If (M, g_w) is a strict Walker then the non zero components in (2.2) are

$$R(\partial y, \partial z)\partial y = \frac{1}{2} w_{yy} \partial x \quad ; \quad R(\partial y, \partial z)\partial z = -\frac{\varepsilon}{2} w_{yy} \partial y. \tag{2.3}$$

The cross product $\times_w$, with the property $g_w(U \times_w V, W) = \det(U, V, W)$ for $U, V, W \in \mathbb{R}^3$, is given by:

$$U \times_w V = \Big(u_1 v_2 - u_2 v_1 - (u_2 v_3 - u_3 v_2) w, -\varepsilon(u_1 v_3 - u_3 v_1), u_2 v_3 - u_3 v_2\Big). \tag{2.4}$$

2.2 Geometry of Surfaces in a Walker 3-Manifold

In this section, we study the differential geometry of surfaces in a Walker manifold. Let U be an open subset of the plane $\mathbb{R}^2$ where horizontal or vertical lines intersect U in intervals (if at all). A two-parameter map is a smooth map $\varphi : U \to M$. Thus, φ is composed of two interwoven families of parameter curves:

(i) The s-parameter curves $t = t_0$ of φ is $s \mapsto \varphi(s, t_0)$.

(ii) The t-parameter curves $s = s_0$ of φ is $t \mapsto \varphi(s_0, t)$.

The partial velocities $\varphi_s = d\varphi(\partial_s)$ and $\varphi_t = d\varphi(\partial_t)$ are vector fields on φ. If φ lies in the domain of a coordinate system $(x_1, \ldots, x_n)$, then its coordinate functions $x_i \circ \varphi$ $(1 \leq i \leq n)$ are real-valued functions on U, and:

$$\varphi_s = \sum_i \frac{\partial x_i}{\partial s} \partial x_i, \quad \varphi_t = \sum_i \frac{\partial x_i}{\partial t} \partial x_i.$$

Assume now that M is a pseudo-Riemannian manifold. If Z is a smooth vector field on φ, its partial covariant derivatives are:

$$Z_s = \nabla_{\partial_s} Z, \quad Z_t = \nabla_{\partial_t} Z,$$

where $Z_s(s_0, t_0)$ is the covariant derivative at s_0 of the vector field $s \mapsto Z(s, t_0)$ on the curve $s \mapsto \varphi(s, t_0)$. In coordinates, $Z = \sum_i Z^i \partial x_i$, where each $Z^i = Z(x^i)$ is a real-valued function. Then:

$$Z_s = \sum_k \left(\frac{\partial Z^k}{\partial s} + \sum_{i,j} \Gamma_{ij}^k Z^i \frac{\partial x^j}{\partial s} \right) \partial x^k. \tag{2.5}$$

In the case $Z = \varphi_s$, the derivative $Z_s = \varphi_{ss}$ gives the accelerations of the s-parameter curves, while φ_{tt} gives the t-parameter accelerations. In coordinates:

$$\varphi_{st} = \sum_k \left(\frac{\partial^2 x^k}{\partial s \partial t} + \sum_{i,j} \Gamma_{ij}^k \frac{\partial x^i}{\partial s} \frac{\partial x^j}{\partial t} \right) \partial x^k. \tag{2.6}$$

Next, assume that φ is an isometric immersion. The first fundamental form of the immersion φ is given by:

$$E = g_w(\varphi_s, \varphi_s), \quad F = g_w(\varphi_s, \varphi_t), \quad G = g_w(\varphi_t, \varphi_t). \tag{2.7}$$

The coefficients of the second fundamental form of φ are:

$$\begin{cases} e &= \varepsilon g_w(\varphi_{ss}, \xi), \\ f &= \varepsilon g_w(\varphi_{st}, \xi), \\ g &= \varepsilon g_w(\varphi_{tt}, \xi) \end{cases} \tag{2.8}$$

where $\varepsilon = g_w(\xi, \xi)$ is the sign of the unit normal ξ along φ. Finally, the mean curvature H of the surface φ is given by:

$$H = \frac{\varepsilon}{2} \frac{eG - 2fF + gE}{EG - F^2}. \tag{2.9}$$

For a surface Σ in (M, g_w), the Gauss equation relates the sectional curvature $K^M(\partial_s, \partial_t)$ of Σ to the sectional curvature of (M, g_w) as:

$$K(\partial_s, \partial_t) = K^M(\partial_s, \partial_t) + \varepsilon \frac{eg - f^2}{EG - F^2}. \tag{2.10}$$

3 Surfaces with common geodesic in Walker 3-manifold

Let α be a regular speed curve defined by

$$\alpha = \alpha(s), \quad \alpha : I \subset \mathbb{R} \longmapsto (M, g_w), \quad 0 \leq s \leq L. \tag{3.1}$$

Then we have the Frenet equations

$$\left\{ \begin{array}{lllll} T'(s) & = & \nabla_T T & = & \varepsilon_2 \kappa(s) N(s) \\ N'(s) & = & \nabla_T N & = & -\varepsilon_1 \kappa(s) T(s) - \varepsilon_3 \tau(s) B(s) \\ B'(s) & = & \nabla_T B & = & \varepsilon_2 \tau(s) N(s) \end{array} \right. \quad \text{avec} \quad \left\{ \begin{array}{l} \varepsilon_1 = g_w(T, T) \\ \varepsilon_2 = g_w(N, N) \\ \varepsilon_3 = g_w(B, B), \end{array} \right. \tag{3.2}$$

where $(T(s), N(s), B(s))$ is the Frenet frame of the curve.
We define the surface $\varphi(s,t)$ by the curve α as:

$$\varphi(s,t) = \alpha(s) + a(s,t)T(s) + b(s,t)N(s) + c(s,t)B(s), \tag{3.3}$$

where $(T(s), N(s), B(s))$ is the Frenet frame of $\alpha(s)$ and $a(s,t)$, $b(s,t)$ et $c(s,t)$ are functions of class C^1.
Let us consider the local pseudo-orthonormal fields frame (e_1, e_2, e_3) on M given by

$$e_1 = \sqrt{w}\partial x - \frac{\sqrt{w}}{w}\partial z, \quad e_2 = \partial y \quad \text{et} \quad e_3 = -\frac{\sqrt{w}}{w}\partial z, \tag{3.4}$$

where w is a positive function, with $g_w(e_1; e_1) = -1$; $g_w(e_2; e_2) = \varepsilon$ and $g_w(e_3; e_3) = 1$.
Thus, the signature of the metric g_w is $(-1; \varepsilon; 1)$ and we put

$$\left\{ \begin{array}{l} \alpha(s) = \alpha_1(s)e_1 + \alpha_2(s)e_2 + \alpha_3(s)e_3 \\ \\ T(s) = T_1(s)e_1 + T_2(s)e_2 + T_3(s)e_3 \\ \\ N(s) = N_1(s)e_1 + N_2(s)e_2 + N_3(s)e_3 \\ \\ B(s) = B_1(s)e_1 + B_2(s)e_2 + B_3(s)e_3 \end{array} \right. \tag{3.5}$$

where $T_i, N_i, B_i,\ i = 1, 2, 3$ are some functions. Then we have

$$\begin{aligned} \varphi(s,t) = & \Big(\alpha_1(s) + a(s,t)T_1(s) + b(s,t)N_1(s) + c(s,t)B_1(s)\Big)e_1 \\ & + \Big(\alpha_2(s) + a(s,t)T_2(s) + b(s,t)N_2(s) + c(s,t)B_2(s)\Big)e_2 \\ & + \Big(\alpha_3(s) + a(s,t)T_3(s) + b(s,t)N_3(s) + c(s,t)B_3(s)\Big)e_3 \end{aligned} \tag{3.6}$$

where $a(s,t)$, $b(s,t)$ et $c(s,t)$ are some functions of class C^1.
For the following definitions, one can see [3].

Definition 3.1. A curve lying in a surface is called geodesic if only if the normal vector $N(s)$ of the curve and the normal vector $n(s,t)$ of the surface are parallels.

We recall that, for the surface given in (3.3), the normal vector is expressed as

$$n(s,t) = \varphi_s \times_w \varphi_t. \tag{3.7}$$

Definition 3.2. A curve $\alpha(s)$ lying on the surface $\varphi(s,t)$ is isoparametric if for a rank $t = t_0$, we have: $\varphi(s,t_0) = \alpha(s)$. That is $a(s,t_0) = b(s,t_0) = c(s,t_0) = 0$.

Definition 3.3. A curve on a surface is isogeodesic if and only if it is geodesic and isoparametric.

By using the equation in (3.6) and (3.2) we have:

$$\varphi_s = \Big(\alpha_1'(s) + a_s T_1(s) + a(s,t)T_1'(s) + b_s N_1(s) + b(s,t)N_1'(s) + c_s B_1(s) + C(s,t)B_1'(s)\Big)e_1$$
$$+ \Big(\alpha_2'(s) + a_s T_2(s) + a(s,t)T_2'(s) + b_s N_2(s) + b(s,t)N_2'(s) + c_s B_2(s) + C(s,t)B_2'(s)\Big)e_2$$
$$+ \Big(\alpha_3'(s) + a_s T_3(s) + a(s,t)T_3'(s) + b_s N_3(s) + b(s,t)N_3'(s) + c_s B_3(s) + C(s,t)B_3'(s)\Big)e_3$$
$$= \Big(\big(1 + a_s - \varepsilon_1 b(s,t)\kappa(s)\big)T_1(s) + \big(\varepsilon_2 a(s,t)\kappa(s) + b_s + \varepsilon_2 c(s,t)\tau(s)\big)N_1(s) + \big(c_s - \varepsilon_3 b(s,t)\tau(s)\big)B_1(s)\Big)e_1$$
$$+ \Big(\big(1 + a_s - \varepsilon_1 b(s,t)\kappa(s)\big)T_2(s) + \big(\varepsilon_2 a(s,t)\kappa(s) + b_s + \varepsilon_2 c(s,t)\tau(s)\big)N_2(s) + \big(c_s - \varepsilon_3 b(s,t)\tau(s)\big)B_2(s)\Big)e_2$$
$$+ \Big(\big(1 + a_s - \varepsilon_1 b(s,t)\kappa(s)\big)T_3(s) + \big(\varepsilon_2 a(s,t)\kappa(s) + b_s + \varepsilon_2 c(s,t)\tau(s)\big)N_3(s) + \big(c_s - \varepsilon_3 b(s,t)\tau(s)\big)B_3(s)\Big)e_3$$

If we put

$$\begin{cases} X = 1 + a_s - \varepsilon_1 b(s,t)\kappa(s) \\ Y = \varepsilon_2 a(s,t)\kappa(s) + b_s + \varepsilon_2 c(s,t)\tau(s) \\ Z = c_s - \varepsilon_3 b(s,t)\tau(s) \end{cases} \tag{3.8}$$

then we obtain

$$\varphi_s = \Big(XT_1(s)+YN_1(s)+ZB_1(s)\Big)e_1+\Big(XT_2(s)+YN_2(s)+ZB_2(s)\Big)e_2+\Big(XT_3(s)+YN_3(s)+ZB_3(s)\Big)e_3. \tag{3.9}$$

and the computation gives

$$\varphi_t = \Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)e_1+\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big)e_2+\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)e_3. \tag{3.10}$$

Now we have the following result:

Theorem 3.4. *Let (M, g_w) be a Walker 3-manifold. The $\alpha : I \subset \mathbb{R} \to M$ is geodesic on the surface $\varphi(s,t)$ if and only if*

(i) $\Big(\big(1+a_s-\varepsilon_1 b(s,t)\kappa(s)\big)T_1(s)+\big(\varepsilon_2 a(s,t)\kappa(s)+b_s+\varepsilon_2 c(s,t)\tau(s)\big)N_1(s)+\big(c_s-\varepsilon_3 b(s,t)\tau(s)\big)B_1(s)\Big)$ $\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big)-\Big[\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)+w\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)\Big]\Big[\big(1 + a_s - \varepsilon_1 b(s,t)\kappa(s)\big)T_2(s) + \big(\varepsilon_2 a(s,t)\kappa(s) + b_s + \varepsilon_2 c(s,t)\tau(s)\big)N_2(s) + \big(c_s - \varepsilon_3 b(s,t)\tau(s)\big)B_2(s)\Big] - w\Big[\big(1 + a_s - \varepsilon_1 b(s,t)\kappa(s)\big)T_3(s) + \big(\varepsilon_2 a(s,t)\kappa(s) + b_s + \varepsilon_2 c(s,t)\tau(s)\big)N_3(s) + \big(c_s - \varepsilon_3 b(s,t)\tau(s)\big)B_3(s)\Big]\Big(a_tT_2(s) + b_tN_2(s) + c_tB_2(s)\Big) = 0;$

(ii) $-\varepsilon\Big[\Big(\big(1+a_s\big)T_1(s)+\big(\varepsilon_2 a(s,t)\kappa(s)+b_s+\varepsilon_2 c(s,t)\tau(s)\big)N_1(s)+\big(c_s-\varepsilon_3 b(s,t)\tau(s)\big)B_1(s)\Big)$ $\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)-\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)\Big(\big(1+a_s-\varepsilon_1 b(s,t)\kappa(s)\big)T_3(s)$ $+ \big(\varepsilon_2 a(s,t)\kappa(s) + b_s + \varepsilon_2 c(s,t)\tau(s)\big)N_3(s) + \big(c_s - \varepsilon_3 b(s,t)\tau(s)\big)B_3(s)\Big)\Big] \neq 0;$

(iii) $\Big[\Big(1+a_s-\varepsilon_1 b(s,t)\kappa(s)\Big)T_2(s)+\Big(\varepsilon_2 a(s,t)\kappa(s)+b_s+\varepsilon_2 c(s,t)\tau(s)\Big)N_2(s)+\Big(c_s-\varepsilon_3 b(s,t)\tau(s)\Big)B_2(s)\Big]$ $\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)-\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big)\Big[\Big(1+a_s-\varepsilon_1 b(s,t)\kappa(s)\Big)T_3(s)$ $+ \Big(\varepsilon_2 a(s,t)\kappa(s) + b_s + \varepsilon_2 c(s,t)\tau(s)\Big)N_3(s) + \Big(c_s - \varepsilon_3 b(s,t)\tau(s)\Big)B_3(s)\Big] = 0.$

Proof. By using the formula of the normal vector (3.7) in the relations (3.9) and (3.10), we obtain

$$n(s,t)=\varphi_s\times_w\varphi_t=\begin{pmatrix} \Big(XT_1(s)+YN_1(s)+ZB_1(s)\Big)\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big) \\ -\Big(XT_2(s)+YN_2(s)+ZB_2(s)\Big)\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big) \\ -w\Big[\Big(XT_2(s)+YN_2(s)+ZB_2(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big) \\ -\Big(XT_3(s)+YN_3(s)+ZB_3(s)\Big)\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big)\Big] \\ \\ -\varepsilon\Big[\Big(XT_1(s)+YN_1(s)+ZB_1(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big) \\ -\Big(XT_3(s)+YN_3(s)+ZB_3(s)\Big)\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)\Big] \\ \\ \Big(XT_2(s)+YN_2(s)+ZB_2(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big) \\ -\Big(XT_3(s)+YN_3(s)+ZB_3(s)\Big)\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big) \end{pmatrix}^t.$$

If we put

$$\begin{cases} \varphi_1(s,t) &= \Big(XT_1(s)+YN_1(s)+ZB_1(s)\Big)\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big) \\ &-\Big(XT_2(s)+YN_2(s)+ZB_2(s)\Big)\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big) \\ &-w\Big[\Big(XT_2(s)+YN_2(s)+ZB_2(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big) \\ &-\Big(XT_3(s)+YN_3(s)+ZB_3(s)\Big)\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big)\Big] \\ \\ \varphi_2(s,t) &= -\varepsilon\Big[\Big(XT_1(s)+YN_1(s)+ZB_1(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big) \\ &-\Big(XT_3(s)+YN_3(s)+ZB_3(s)\Big)\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)\Big] \\ \\ \varphi_3(s,t) &= \Big(XT_2(s)+YN_2(s)+ZB_2(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big) \\ &-\Big(XT_3(s)+YN_3(s)+ZB_3(s)\Big)\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big) \end{cases}$$

we get

$$n(s,t)=\varphi_1(s,t)e_1+\varphi_2(s,t)e_2+\varphi_3(s,t)e_3.$$

Consequently $N(s)$ is parallel to $n(s,t)$ if only if,

$$\varphi_1(s,t)=0\,;\qquad \varphi_2(s,t)\neq 0\qquad \text{et}\qquad \varphi_3(s,t)=0.$$

Then the theorem is proved. □

By using this result and the definition of isogeodesic, we have the following corollary.

Corollary 3.5. *If the curve $\alpha(s)$ is isogeodesic on the surface $\varphi(s,t)$, then*

(i) $a(s,t_0)=b(s,t_0)=c(s,t_0)=0$;

(ii) $$\Big((1+a_s)T_1(s)+b_sN_1(s)+c_sB_1(s)\Big)\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big) \\ -\Big((1+a_s)T_2(s)+b_sN_2(s)+c_sB_2(s)\Big)\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big) \\ -w\Big[\Big((1+a_s)T_2(s)+b_sN_2(s)+c_sB_2(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big) \\ -\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big)\Big]=0;$$

(iii) $$-\varepsilon\Big[\Big((1+a_s)T_1(s)+b_sN_1(s)+c_sB_1(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big) \\ -\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)\Big]\neq 0;$$

(iv) $\Big((1+a_s)T_2(s)+b_sN_2(s)+c_sB_2(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)$
$-\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big)=0.$

4 Geometric properties of the surfaces with common geodesic in Walker 3-manifold

Let φ the surface given in (3.6). Using the equation in (2.7)and the Corollary 3.5, then the components of the first fundamental form are given by

$$\begin{aligned}E=\varepsilon\Big((1+a_s)T_2(s)+b_sN_2(s)+c_sB_2(s)\Big)^2+w\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)^2\\ +\ 2\Big((1+a_s)T_1(s)+b_sN_1(s)+c_sB_1(s)\Big)\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\end{aligned} \quad (4.1)$$

$$\begin{aligned}F=&\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)\\ &+\ w\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\\ &+\ \varepsilon\Big((1+a_s)T_2(s)+b_sN_2(s)+c_sB_2(s)\Big)\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big)\\ &+\ \Big((1+a_s)T_1(s)+b_sN_1(s)+c_sB_1(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)\end{aligned} \quad (4.2)$$

$$\begin{aligned}G=&2\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)\\ &+\ w\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)^2+\varepsilon\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big)^2.\end{aligned} \quad (4.3)$$

Let ξ be the unit normal vector of the surface φ defined by

$$\xi=\frac{\varphi_s\times_w\varphi_t}{\sqrt{|\ \varphi_s\times_w\varphi_t\ |}}.$$

By an easy computation and using the Corollary 3.5, we have

$$\begin{aligned}\varphi_s\times_w\varphi_t=\Big(0,-\varepsilon\Big[&\Big((1+a_s)T_1(s)+b_sN_1(s)+c_sB_1(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)\\ &-\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)\Big],0\Big).\end{aligned}$$

Let us putting

$$\begin{aligned}\mathcal{A}=-\varepsilon\Big[&\Big((1+a_s)T_1(s)+b_sN_1(s)+c_sB_1(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)\\ &-\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)\Big],\end{aligned} \quad (4.4)$$

then we have

$$\xi=\frac{\varphi_s\times_w\varphi_t}{\sqrt{|\ \varphi_s\times_w\varphi_t\ |}}=\frac{\Big(0,\mathcal{A},0\Big)}{\sqrt{|\ \mathcal{A}\ |}}.$$

Before we compute the coefficients of the second fundamental form, we give the expression of the covariant derivatives of φ. Now if we put

$$\begin{aligned}x^1&=\Big(\alpha_1(s)+a(s,t)T_1(s)+b(s,t)N_1(s)+c(s,t)B_1(s)\Big)\\ x^2&=\Big(\alpha_2(s)+a(s,t)T_2(s)+b(s,t)N_2(s)+c(s,t)B_2(s)\Big)\\ x^3&=\Big(\alpha_3(s)+a(s,t)T_3(s)+b(s,t)N_3(s)+c(s,t)B_3(s)\Big)\end{aligned}$$

then the surface given in (3.6) becomes

$$\varphi(s,t) = x^1 e_1 + x^2 e_2 + x^3 e_3. \tag{4.5}$$

Using the equation (2.6), (4.5) and the Corollary 3.5 we get

$$\varphi_{ss} = \begin{pmatrix} \Big(a_{ss} - 2\varepsilon_1\kappa(s)b_s\Big)T_1(s) + \Big(\varepsilon_2\kappa(s) + b_{ss} + 2\varepsilon_2\kappa(s)a_s + 2\varepsilon_2\tau(s)c_s\Big)N_1(s) \\ +\Big(c_{ss} - 2\varepsilon_3\tau(s)b_s\Big)B_1(s) + \frac{1}{2}(ww_x + w_z)\Big((1+a_s)T_3(s) + b_sN_3(s) + c_sB_3(s)\Big)^2 \\ +w_x\Big((1+a_s)T_1(s) + b_sN_1(s) + c_sB_1(s)\Big)\Big((1+a_s)T_3(s) + b_sN_3(s) + c_sB_3(s)\Big) \\ +w_y\Big((1+a_s)T_2(s) + b_sN_2(s) + c_sB_2(s)\Big)\Big(1+a_s)T_3(s) + b_sN_3(s) + c_sB_3(s)\Big) \\ \\ \Big(a_{ss} - 2\varepsilon_1\kappa(s)b_s\Big)T_2(s) + \Big(\varepsilon_2\kappa(s) + b_{ss} + 2\varepsilon_2\kappa(s)a_s + 2\varepsilon_2\tau(s)c_s\Big)N_2(s) \\ +\Big(c_{ss} - 2\varepsilon_3\tau(s)b_s\Big)B_2(s) - \frac{\varepsilon}{2}w_y\Big((1+a_s)T_3(s) + b_sN_3(s) + c_sB_3(s)\Big)^2 \\ \\ \Big(a_{ss} - 2\varepsilon_1\kappa(s)b_s\Big)T_3(s) + \Big(\varepsilon_2\kappa(s) + b_{ss} + 2\varepsilon_2\kappa(s)a_s + 2\varepsilon_2\tau(s)c_s\Big)N_3(s) \\ +\Big(c_{ss} - 2\varepsilon_3\tau(s)b_s\Big)B_3(s) - \frac{1}{2}w_x\Big((1+a_s)T_3(s) + b_sN_3(s) + c_sB_3(s)\Big)^2 \end{pmatrix},$$

$$\varphi_{st} = \begin{pmatrix} \Big(a_tT_3(s) + b_tN_3(s) + c_tB_3(s)\Big)\Big[\frac{1}{2}\Big(ww_x + w_z\Big)\Big((1+a_s)T_3(s) + b_sN_3(s) + c_sB_3(s)\Big) \\ +w_x\Big((1+a_s)T_1(s) + b_sN_1(s) + c_sB_1(s)\Big) + w_y\Big((1+a_s)T_2(s) + b_sN_2(s) + c_sB_2(s)\Big)\Big] \\ -\varepsilon_1\kappa(s)b_tT_1(s) + \varepsilon_2\big(\kappa(s)a_t + \tau(s)c_t\big)N_1(s) - \varepsilon_3\tau(s)b_tB_1(s) \\ \\ -\frac{\varepsilon}{2}w_y\Big((1+a_s)T_3(s) + b_sN_3(s) + c_sB_3(s)\Big)\Big(a_tT_3(s) + b_tN_3(s) + c_tB_3(s)\Big) \\ -\varepsilon_1\kappa(s)b_tT_2(s) + \varepsilon_2\Big(\kappa(s)a_t + \tau(s)c_t\Big)N_2(s) - \varepsilon_3\tau(s)b_tB_2(s) \\ \\ -\frac{1}{2}w_x\Big((1+a_s)T_3(s) + b_sN_3(s) + c_sB_3(s)\Big)\Big(a_tT_3(s) + b_tN_3(s) + c_tB_3(s)\Big) \\ -\varepsilon_1\kappa(s)b_tT_3(s) + \varepsilon_2\Big(\kappa(s)a_t + \tau(s)c_t\Big)N_3(s) - \varepsilon_3\tau(s)b_tB_3(s) \end{pmatrix}$$

and

$$\varphi_{tt} = \begin{pmatrix} \dfrac{1}{2}\Big(ww_x + w_z\Big)\Big(a_tT_3(s) + b_tN_3(s) + c_tB_3(s)\Big)^2 + a_{tt}T_1(s) + b_{tt}N_1(s) + c_{tt}B_1(s) \\ +w_x\Big(a_tT_1(s) + b_tN_1(s) + c_tB_1(s)\Big)\Big(a_tT_3(s) + b_tN_3(s) + c_tB_3(s)\Big) \\ +w_y\Big(a_tT_2(s) + b_tN_2(s) + c_tB_2(s)\Big)\Big(a_tT_3(s) + b_tN_3(s) + c_tB_3(s)\Big) \\ \\ a_{tt}T_2(s) + b_{tt}N_2(s) + c_{tt}B_2(s) - \frac{\varepsilon}{2}w_y\Big(a_tT_3(s) + b_tN_3(s) + c_tB_3(s)\Big)^2 \\ \\ a_{tt}T_3(s) + b_{tt}N_3(s) + c_{tt}B_3(s) - \frac{1}{2}w_x\Big(a_tT_3(s) + b_tN_3(s) + c_tB_3(s)\Big)^2 \end{pmatrix}.$$

Then we have the coefficients of the second fundamental forms

$$e = \frac{\varepsilon^2\mathcal{A}}{\sqrt{|\,\mathcal{A}\,|}}\Big[\big(a_{ss} - 2\varepsilon_1\kappa(s)b_s\big)T_2(s) + \big(\varepsilon_2\kappa(s) + b_{ss} + 2\varepsilon_2\kappa(s)a_s + 2\varepsilon_2\tau(s)c_s\big)N_2(s)$$
$$+ \big(c_{ss} - 2\varepsilon_3\tau(s)b_s\big)B_2(s) - \frac{\varepsilon}{2}w_y\big((1+a_s)T_3(s) + b_sN_3(s) + c_sB_3(s)\big)^2\Big],$$

$$f = \frac{\varepsilon^2 \mathcal{A}}{\sqrt{|\mathcal{A}|}}\Big[-\frac{\varepsilon}{2}w_y\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)$$
$$-\varepsilon_1\kappa(s)b_tT_2(s)+\varepsilon_2\big(\kappa(s)a_t+\tau(s)c_t\big)N_2(s)-\varepsilon_3\tau(s)b_tB_2(s)\Big]$$

and

$$g = \frac{\varepsilon^2 \mathcal{A}}{\sqrt{|\mathcal{A}|}}\Big[\quad a_{tt}T_2(s)+b_{tt}N_2(s)+c_{tt}B_2(s)-\tfrac{\varepsilon}{2}w_y\big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\big)^2 \quad\Big].$$

Using the equation (2.9) we have

$$H = \frac{\varepsilon^3 \mathcal{A}}{2\sqrt{|\mathcal{A}|}(EG-F^2)}\Big(C_1-2C_2+C_3\Big), \tag{4.6}$$

where

$$C_1 = \Big[\big(a_{ss}-2\varepsilon_1\kappa(s)b_s\big)T_2(s)+\big(\varepsilon_2\kappa(s)+b_{ss}+2\varepsilon_2\kappa(s)a_s+2\varepsilon_2\tau(s)c_s\big)N_2(s)$$
$$+\big(c_{ss}-2\varepsilon_3\tau(s)b_s\big)B_2(s)-\frac{\varepsilon}{2}w_y\big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\big)^2\Big]$$
$$\Big[2\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)$$
$$+w\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)^2+\varepsilon\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big)^2\Big] \tag{4.7}$$

$$C_2 = \Big[-\frac{\varepsilon}{2}w_y\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)$$
$$-\varepsilon_1\kappa(s)b_tT_2(s)+\varepsilon_2\big(\kappa(s)a_t+\tau(s)c_t\big)N_2(s)-\varepsilon_3\tau(s)b_tB_2(s)\Big]$$
$$\Big[\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)$$
$$+w\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)$$
$$+\varepsilon\Big((1+a_s)T_2(s)+b_sN_2(s)+c_sB_2(s)\Big)\Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big)\Big] \tag{4.8}$$

$$C_3 = \Big[a_{tt}T_2(s)+b_{tt}N_2(s)+c_{tt}B_2(s)-\frac{\varepsilon}{2}w_y\big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\big)^2\Big]$$
$$\Big[\varepsilon\Big((1+a_s)T_2(s)+b_sN_2(s)+c_sB_2(s)\Big)^2+w\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)^2$$
$$+2\Big((1+a_s)T_1(s)+b_sN_1(s)+c_sB_1(s)\Big)\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big]. \tag{4.9}$$

Theorem 4.1. *The family of surfaces given in (3.6) is* ***minimal*** *if and only if*

$$C_1-2C_2+C_3=0,$$

where C_1, C_2 *et* C_3 *are given by the equations (4.7), (4.8) and (4.9).*

Proof. The surface is minimal if $H=0$. If we take (4.6) we see that
$H=0 \Longleftrightarrow C_1-2C_2+C_3=0$ since $\dfrac{\varepsilon^3\mathcal{A}}{2\sqrt{|\mathcal{A}|}(EG-F^2)}\neq 0.$ □

Theorem 4.2. *The family of surfaces with common geodesic in a strict Walker 3-manifold is flat iff*

$$\frac{1}{2}L_2J_3w_{yy}\big(L_2J_3-\varepsilon^2L_3J_2\big)+\varepsilon\big(eg-f^2\big)=0.$$

Proof. By the Corollary (3.5) and the relation (3.4), the equations (3.9) and (3.10) become respectively:

$$\varphi_s = \sqrt{w}\Big((1+a_s)T_1(s)+b_sN_1(s)+c_sB_1(s)\Big)\partial x + \Big((1+a_s)T_2(s)+b_sN_2(s)+c_sB_2(s)\Big)\partial y - \frac{\sqrt{w}}{w}\Big[\Big((1+a_s)T_1(s)+b_sN_1(s)+c_sB_1(s)\Big)+\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big]\partial z. \tag{4.10}$$

and

$$\varphi_t = \sqrt{w}\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)\partial x + \Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big)\partial y - \frac{\sqrt{w}}{w}\Big[\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)+\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)\Big]\partial z. \tag{4.11}$$

By putting

$$\begin{cases} L_1 = \sqrt{w}\Big((1+a_s)T_1(s)+b_sN_1(s)+c_sB_1(s)\Big) \\ L_2 = \Big((1+a_s)T_2(s)+b_sN_2(s)+c_sB_2(s)\Big) \\ L_3 = -\frac{\sqrt{w}}{w}\Big[\Big((1+a_s)T_1(s)+b_sN_1(s)+c_sB_1(s)\Big)+\Big((1+a_s)T_3(s)+b_sN_3(s)+c_sB_3(s)\Big)\Big] \end{cases}. \tag{4.12}$$

and

$$\begin{cases} J_1 = \sqrt{w}\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big) \\ J_2 = \Big(a_tT_2(s)+b_tN_2(s)+c_tB_2(s)\Big) \\ J_3 = -\frac{\sqrt{w}}{w}\Big[\Big(a_tT_1(s)+b_tN_1(s)+c_tB_1(s)\Big)+\Big(a_tT_3(s)+b_tN_3(s)+c_tB_3(s)\Big)\Big]. \end{cases}. \tag{4.13}$$

Then the equations (4.10) and (4.11) become

$$\varphi_s = L_1\partial x + L_2\partial y - L_3\partial z \tag{4.14}$$

and

$$\varphi_t = J_1\partial x + J_2\partial y - J_3\partial z. \tag{4.15}$$

Using the equation (2.3), we have

$$R(\varphi_s,\varphi_t)\varphi_s = -\frac{1}{2}L_2^2J_3w_{yy}\partial x - \frac{\varepsilon}{2}L_2L_3J_3w_{yy}\partial y. \tag{4.16}$$

So we get

$$g_w\big(R(\varphi_s,\varphi_t)\varphi_s,\varphi_t\big) = \frac{1}{2}L_2^2J_3^2w_{yy} - \frac{\varepsilon^2}{2}L_2L_3J_2J_3w_{yy}. \tag{4.17}$$

Furthemore we have by using (2.10)

$$K = \frac{\frac{1}{2}L_2J_3w_{yy}\big(L_2J_3-\varepsilon^2L_3J_2\big)+\varepsilon\big(eg-f^2\big)}{EG-F^2}. \tag{4.18}$$

The surface is flat if $K=0$.
Since $EG-F^2\neq 0$, then $K=0 \iff \frac{1}{2}L_2J_3w_{yy}\Big(L_2J_3-\varepsilon^2L_3J_2\Big)+\varepsilon\Big(eg-f^2\Big)=0.$ □

References

[1] M. Brozos-Vazquez, E. Garcia-Rio, P. Gilkey, S. Nikcevic, R. Vazquez-Lorenzo, *The Geometry of Walker Manifolds*, G . Krantz (Ed.), Synthesis Lectures on Mathematics and Statistics, Washington University, St. Louis, Morgan and Claypool Publishers, Williston, VT, (5) (2009).

[2] C. L. Bejan, S. L. Drută-Romaniuc, *Walker manifolds and Killing magnetic curves*, Differential Geometry and its Applications, **35** (Supplement) (2014) 106-116.

[3] E. Kasap, F. Talay Akyildiz, *Surfaces with common geodesic in Minkowski 3-space*, Applied Mathematics and Computation, **177 (1)** (2006) 260-270.

[4] M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Inc., Englewood Cliffs, N.J., viii+503 (1976) 190-191.

[5] A. S. Diallo, A. Ndiaye and A. Niang, *Minimal graphs on three-dimensional Walker manifolds*. Proceedings of the First NLAGA-BIRS Symposium, Dakar, Senegal, Trends Math., Birkhauser/Springer, Cham, (2020) 425-438.

[6] M. A. Dramé, A. Ndiaye and A.S. Diallo, *2-ruled hypersurfaces in a Walker 4-manifold*, Facta Univ. Ser. Math. Inform. **39 (1)** (2024) 67-86.

[7] M. Tekkoyun, *A Survey on Geometric Dynamics of 4-Walker Manifold*, Journal of Modern Physics, Article ID: 8636, DOI:10.4236/jmp.2011.211163, **2 (11)** (2011) 1318-1323.

[8] A. Niang, A. Ndiaye and A. S. Diallo, *A Classification of Strict Walker 3-Manifold*, Konuralp J. Math., **9 (1)** (2021) 148-153.

[9] A. Niang, A. Ndiaye, and A. S. Diallo, *Minimals translation surfaces in a strict Walker 3-manifold*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **73 (2)** (2024) 554-568.

Author information

El Hadji Baye Camara, Département de Mathematiques et Informatique, FST, UCAD, Dakar, Sénégal.
E-mail: camaraelhadjibaye@gmail.com

Ameth Ndiaye, Département de Mathematiques, FASTEF, UCAD, Sénégal.
E-mail: ameth1.ndiaye@ucad.edu.sn

Received: 2025-03-04
Accepted: 2025-06-02