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MINIMAL PRIME IDEALS IN COMMUTATIVE TERNARY SEMIRINGS

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Abstract This paper introduces and explores the concepts of minimal prime ideals (completely prime ideals) associated with an ideal, compressed ideals, and the Thierrin radical of an ideal within a commutative ternary semiring. We provide a detailed study of these notions and establish the relationship between the Thierrin radical and the algebraic radical of an ideal in a commutative ternary semiring.

1 Introduction

There is a large literature dealing with ternary algebraic systems introduced by Lehmer [7], in 1932. In 1971, Lister [8] introduced and developed the theory of ternary rings. Dutta and Kar [3] introduced the notion of ternary semiring in 2003, which generalizes the notions of ternary ring and semiring.
A nonempty set S is a commutative ternary semiring if it satisfies the following conditions:

(i) S is a commutative additive semigroup,

(ii) S is a commutative ternary semigroup with a ternary multiplication denoted by juxtaposition and

(iii) $x_1x_2(x_3 + x_4) = x_1x_2x_3 + x_1x_2x_4$, for all $x_1, x_2, x_3, x_4 \in S$.

Throughout this paper, S will denote a commutative ternary semiring unless otherwise stated.

Dutta and Kar credited with pioneering the notions of ideals, prime ideals, semiprime ideals, and their associated radicals within the context of ternary semirings ([3], [4], [5]). Recently, the concepts of S-prime ideals and primary ideals in commutative rings have been further developed and investigated by various authors such as Pathak, Goswami ([10]), and Groenewald ([9]).

An additive subsemigroup I of S is an ideal in S if $x_1x_2i \in I$, for all $x_1, x_2 \in S, i \in I$. An ideal I in S is proper if $I \neq S$. Let P be a proper ideal in S, then P is the prime, if for any three ideals X_1, X_2, X_3 of S, $X_1X_2X_3 \subseteq P$ implies at least one of $X_1 \subseteq P$ or $X_2 \subseteq P$ or $X_3 \subseteq P$. P is completely prime if whenever $x_1x_2x_3 \in P$, for all $x_1, x_2, x_3 \in S$ implies that at least one of $x_1 \in P$ or $x_2 \in P$ or $x_3 \in P$.

In 2012, Daddi and Pawar [2] defined the ternary A-semiring and provided characterizations that extend the concept of the A-semiring established by Allen, Neggers, and Kim[1] . Further, author carries out a detailed study of the radical of an ideal in a ternary A-semiring.

S is said to be a ternary A-semiring if every proper ideal of S is contained in a prime ideal of S. If I is a proper ideal of a ternary A-semiring S, then the radical of I is the intersection of all prime ideals containing I denoted by $\sqrt{I}$. I is a semi-prime ideal of S, if $I = \sqrt{I}$. S with an identity e is a ternary A-semiring.

Theorem 1.1. *The following assertions hold for the proper ideals I and J of a ternary A-semiring S.*

(i) $\sqrt{I} = \{x \in S :$ *there exists an odd positive integer* n *such that* $x^n \in I\}$.

(ii) If $I \subseteq J$, *then* $\sqrt{I} \subseteq \sqrt{J}$.

(iii) If $I \cap J \neq \emptyset$, *then* $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

(iv) $\sqrt{\sqrt{I}} = \sqrt{I}$.

(v) If I is a prime ideal of S, then I is semi-prime.

This paper aims to develop the ideal theory for ternary A-semiring and to connect this theory with the theory developed by Dutta and Kar ([3], [4], [5]). Also, we introduce the concepts of minimal prime ideals associated with an ideal, compressed ideals in a ternary semiring, and examine their properties in detail. Further, we develop Iseki's [6] theory of the Thierrin radical of an ideal in arbitrary semirings by introducing and studying the Thierrin radical specifically in the context of ternary semirings. Finally, we establish that, in a ternary A-semiring, the Thierrin radical of an ideal coincides with the algebraic radical of that ideal.

2 Minimal Prime Ideal

In this section, we introduce and study the concept of a minimal prime ideal (completely prime ideal) associated with an ideal in S. We begin with the following:

Definition 2.1. A prime ideal M in S is a minimal prime ideal (completely prime ideal; respectively) associated with an ideal I in S if $I \subset M$ and there is a prime (completely prime; respectively) ideal P in S such that $I \subset P \subseteq M$ implies $P = M$.

Example 2.2. In the commutative ternary semiring Z_0^- of all non-positive integers, $M = \{2k : k \in Z_0^-\}$ is a minimal prime ideal associated with an ideal $I = \{4k : k \in Z_0^-\}$ in Z_0^-. $M = \{2k : k \in Z_0^-\}$ and $N = \{3k : k \in Z_0^-\}$ are minimal prime ideals associated with an ideal $J = \{6k : k \in Z_0^-\}$ in Z_0^-.

Example 2.3. Consider the commutative ternary semiring Z_e of all positive even integers. Let $M = \{x \in Z_e / x > 2\}$. M is a maximal ideal of Z_e, but not a prime. Hence M is not contained in any prime ideal of Z_e. This shows that Z_e is not a ternary A-semiring. Hence, there does not exist a minimal prime ideal associated with M.

The following theorem deals with the existence of a minimal prime ideal associated with an ideal in S.

Theorem 2.4. *If S is a ternary A-semiring, then for every ideal I in S, there exists a minimal prime ideal associated with I.*

Proof. Let I be a proper ideal in S. As S is a ternary A-semiring, then there exists a prime ideal P in S such that $I \subset P$. Let $\mathcal{P} = \{Q_i\}$ be the family of prime ideals in S such that $I \subset Q_i \subseteq P$. We have $P \in \mathcal{P}$, therefore $\mathcal{P} \neq \emptyset$. Evidently $\mathcal{P}$ forms a partially ordered set under the inclusion of sets.
Let $\{Q_i\}_{i\in\Delta}$ be the chain in $\mathcal{P}$, where Δ is any indexing set. Let $Q = \cap_{i\in\Delta} Q_i$, then Q is an ideal in S such that $I \subset Q \subseteq P$. Now we have to show that Q is prime. Let $x, y, z \in Q \subset S$ be such that $xyz \in Q$, then $xyz \in Q_i$ for each $i \in \Delta$. Suppose that there exists $n \in \Delta$ such that $x, y \notin Q_n$. Hence $z \in Q_n$, since Q_n is prime.
For $Q_i \in \{Q_i\}_{i\in\Delta}$,
Case (i) If $Q_i \subseteq Q_n$, then $z \in Q_i$. As Q_i is prime, it follows that $x \in Q_i$ or $y \in Q_i$. Hence $x \in Q_i \subseteq Q_n$ or $y \in Q_i \subseteq Q_n$, which is a contradiction.
Case (ii) If $Q_n \subseteq Q_i$, then $z \in Q_i$, for each $i \in \Delta$. As $\{Q_i\}_{i\in\Delta}$ is a chain, $z \in \cap_{i\in\Delta} Q_i = Q$.
Consequently $\cap_{i\in\Delta} Q_i = Q$ is prime. Hence, by Zorn's lemma, there exists a maximal element, say M in $\mathcal{P}$. Therefore, if J is a prime ideal in S such that $I \subset J \subseteq M \subseteq P$, then $J = M$. This shows that M is a minimal prime ideal associated with an ideal I in S. □

The following theorem characterizes the radical of an ideal in terms of the minimal prime ideals associated with an ideal in a ternary A-semiring.

Theorem 2.5. *If S is a ternary A-semiring, then the radical of a proper ideal P in S is the intersection of all minimal prime ideals associated with P.*

Proof. Let $\mathcal{P} = \{P_i\}_{i\in\Delta}$ be the family of all prime ideals in S containing P and $\{P_i\}_{i\in\Delta^*}$ be the family of all minimal prime ideals associated with an ideal P in S, where Δ is any indexing set and $\Delta^* \subseteq \Delta$. Let $x \in \sqrt{P}$. Then $x \in \cap_{i\in\Delta} P_i$ .Thus $x \in P_i$ for any $i \in \Delta$ and hence $x \in P_i$ for any $i \in \Delta^*$. Hence $x \in \cap_{i\in\Delta^*} P_i$. This shows that $\sqrt{P} \subseteq \cap_{i\in\Delta^*} P_i$. Conversely assume that $\cap_{i\in\Delta^*} P_i \nsubseteq \sqrt{P}$. Then there exists $x \in \cap_{i\in\Delta^*} P_i$ such that $x \notin \sqrt{P}$. Thus, there exists $P_n \in \mathcal{P}$ such that $x \notin P_n$. Hence, by Theorem 2.4, there exists $P_j \in \{P_i\}_{i\in\Delta^*}$ such that $P \subset P_j \subset P_n$. However, $P_j \in \{P_i\}_{i\in\Delta^*}$ and $x \in \cap_{i\in\Delta^*} P_i$, implies that $x \in P_j \subset P_n$; which is a contradiction. This shows that $\cap_{i\in\Delta^*} P_i \subseteq \sqrt{P}$. Hence, we get $\sqrt{P} = \cap_{i\in\Delta^*} P_i$. □

Corollary 2.6. *A proper ideal P in a ternary A-semiring S is a minimal prime ideal associated with P, if $\sqrt{P} = P$.*

Now, we discuss some characterizations of primary ideals in a ternary A-semiring.

Definition 2.7. A proper ideal P in a commutative ternary semiring S is primary, if for $x_1, x_2, x_3 \in S$ such that $x_1x_2x_3 \in P$ and $x_1 \notin P, x_2 \notin P$ implies $x_3^n \in P$, for some odd positive integer n.

In a commutative ternary semiring, while every prime ideal is primary, the converse is not necessarily true. We illustrate this in the following example.

Example 2.8. Consider the ideal $I = \{8k : k \in Z_0^-\}$ in the commutative ternary semiring Z_0^-, as $(-3)(-2)(-4) = -24 \in P$, but $(-3) \notin P, (-2) \notin P$ and $(-4) \notin P$. Thus I is primary, but not a prime ideal of Z_0^-.

The following theorem gives the useful characterization of a primary ideal in ternary A-semiring.

Theorem 2.9. *In a ternary A-semiring S, a proper ideal P in S is primary if and only if the following conditions are equivalent for $x_1, x_2, x_3 \in S$ and ideals X, Y, Z of S :*

(i) If $x_1x_2x_3 \in P$ and $x_1 \notin P, x_2 \notin P$, then $x_3 \in \sqrt{P}$,

(ii) If $x_1x_2x_3 \in P$ and $x_1 \notin P, x_2 \notin \sqrt{P}$, then $x_3 \in P$,

(iii) If $x_1x_2x_3 \in P$ and $x_1 \notin \sqrt{P}, x_2 \notin \sqrt{P}$, then $x_3 \in P$,

(iv) If $XYZ \subseteq P$ and $X \nsubseteq P, Y \nsubseteq P$, then $Z \subseteq \sqrt{P}$,

(v) If $XYZ \subseteq P$and $X \nsubseteq P, Y \nsubseteq \sqrt{P}$, then $Z \subseteq P$,

(vi) If $XYZ \subseteq P$ and $X \nsubseteq \sqrt{P}, Y \nsubseteq \sqrt{P}$, then $Z \subseteq P$.

Theorem 2.10. *The radical of a primary ideal in a ternary A-semiring S is prime.*

Proof. Let P be a primary ideal in S. Take $x, y, z \in S$ such that $xyz \in \sqrt{P}$ and $x \notin P, y \notin P$. As $xyz \in \sqrt{P}$, then by Theorem 1.1, there exists an odd positive integer n such that $(xyz)^n \in P$. As S is a ternary A-semiring, it follows that $x^n y^n z^n \in P$ and $x^n \notin P, y^n \notin P$. Now P is primary, now by using Theorem 2.9, $z^n \in P$. Thus $(z^n)^m \in P$ for some odd positive integer m. Clearly $mn = p$ is an odd positive integer and $(z^n)^m = z^{mn} = z^p \in P$. Therefore $z \in \sqrt{P}$ implies that $\sqrt{P}$ is prime. □

The following is an immediate consequence of Theorem 2.10.

Corollary 2.11. *The radical of a primary ideal P in a ternary A-semiring S is primary.*

Corollary 2.12. *The radical of a primary ideal P in a ternary A-semiring S is the unique minimal prime ideal associate with P.*

The intersection of any two primary ideals in the ternary A-semiring need not be primary. We illustrate this in the following example.

Example 2.13. The ideals $I = \{2k : k \in Z_0^-\}$ and $J = \{3k : k \in Z_0^-\}$ are primary ideals in the commutative ternary semiring Z_0^-. Here $I \cap J = \{6k : k \in Z_0^-\} = \{0, -6, -12, -18, \dots\}$ is not a primary ideal in Z_0^-. As $(-1)(-2)(-3) = -6 \in I \cap J$, but neither $(-1) \in I \cap J, (-2) \in I \cap J, (-3) \in I \cap J$, nor there exists any odd positive integers n such that $(-1)^n \in I \cap J, (-2)^n \in I \cap J, (-3)^n \in I \cap J$.

A sufficient condition for the intersection of primary ideals in a ternary A-semiring to be primary is established in the following theorem.

Theorem 2.14. *Let $P_1, P_2, \dots, P_n$ be primary ideals in a ternary A-semiring S. If $\sqrt{P_i} = P$, for each $i = 1, 2, \dots, n$, then $\cap_{i=1}^n P_i$ is primary and $\sqrt{\cap_{i=1}^n P_i} = P$.*

Proof. Let $x \in P$, then $\sqrt{P_i} = P$, implies that $x^{m_i} \in P_i$, for each $i = 1, 2, \dots, n$ and for some positive integer m_i. Let $m = max\{m_1, m_2, m_3, ..., m_n\}$. Since each P_i is an ideal and $x^{m_i} \in P_i$, it follows that $x^m = x^{m-m_i} x^{m_i} \in P_i$ for each i. Thus $x^m \in \cap_{i=1}^n P_i$. Hence $x \in \sqrt{\cap_{i=1}^n P_i}$.
Conversely, as $\cap_{i=1}^n P_i \subseteq P_i$, hence by Theorem 1.1, $\sqrt{\cap_{i=1}^n P_i} \subseteq \sqrt{P_i} = P$. Thus, $\sqrt{\cap_{i=1}^n P_i} = P$. Now,let $x, y, z \in S$ be such that $xyz \in \cap_{i=1}^n P_i$ and $x \notin P, y \notin P$. Since each P_i is primary, $xyz \in P$ and $x \notin P = \sqrt{P_i}$,$y \notin P = \sqrt{P_i}$, it follows that $z \in P_i$ for each i. Thus, $z \in \cap_{i=1}^n P_i$. Therefore, $\cap_{i=1}^n P_i$ is primary. □

3 Thierrin radical of an ideal

At the outset, we define

Definition 3.1. An ideal I in S is said to be compressed, if $x_1^3 x_2^3 ... x_n^3 \in I$ implies $x_1 x_2 ... x_n \in I$, where $x_1, x_2, ..., x_n \in S$ and n is an odd positive integer.

Theorem 3.2. *In a commutative ternary semiring S, we get*

(i) Every completely prime ideal is a compressed ideal.

(ii) Every compressed ideal is completely semi-prime.

(iii) The intersection of any family of compressed ideals is a compressed ideal if it is nonempty.

Remark 3.3. Every completely prime ideal in S is a compressed ideal,but the converse need not be true. That is, every compressed ideal need not be completely prime ideal. The following example shows that there exists a compressed ideal that is not completely a prime ideal.

Example 3.4. Consider the ternary semiring Z_0^+. Here $I = \{6n : n \in Z_0^+\}$ is a compressed ideal in Z_0^+. As $12 \in I$, but $2 \notin I$ and $3 \notin I$. Therefore I is not completely prime ideal.

Theorem 3.5. *If I is a compressed ideal in S and $x_1, x_2, ..., x_n \in S$, for an odd positive integers n, then there exist odd positive integers $m_1, m_2, \dots, m_n$ such that $x_1^{m_1} x_2^{m_2} ... x_n^{m_n} \in I$ if and only if $x_1 x_2 ... x_n \in I$.*

Proof. First assume that $x_1 x_2 ... x_n \in I$. As I is a compressed ideal in S and every compressed ideal is completely semi-prime. Hence for any odd positive integer m_1, we have $x_1^{m_1} x_2 ... x_n \in I$ and $x_2 ... x_n x_1^{m_1} \in I$. By the same argument, $x_2^{m_2} x_3 ... x_n x_1^{m_1} \in I$ and $x_3 ... x_n x_1^{m_1} x_2^{m_2} \in I$, we have $x_1^{m_1} x_2^{m_2} ... x_n^{m_n} \in I$, for any odd positive integers $n, m_1, m_2, \dots, m_n$. Conversely, let $x_1^{m_1} x_2^{m_2} ... x_n^{m_n} \in I$. Then for $m_n = 3$, for all n, we have $x_1^3 x_2^3 \dots x_n^3 \in I$. Since I is compressed, we have $x_1 x_2 \dots x_n \in I$. □

Now, we define the Thierrin radical of an ideal in a ternary semiring as follows.

Definition 3.6. For an ideal I in S, if there exists $x \in S$ such that $x = x_1 x_2 ... x_n$ implies $x_1^3 x_2^3 \dots x_n^3 \in I$, for some odd positive integer n and $x_1, x_2, ..., x_n \in S$, then x is called a T-element for I.

Let us denote $T^1(I)$ as the set of all T-elements for I and $T_1(I)$ as the ideal generated by $T^1(I)$. By induction, for $m > 1$, $T_m(I)$ is defined as $T_m(I) = T_1(T_{m-1}(I))$.

Definition 3.7. The Thierrin radical $T^*(I)$ of I is defined as $T^*(I) = \bigcup_{m=1}^{\infty} T_m(I)$.

It is clear that every element in I is a T-element for I and each $T_m(I)$ is an ideal in S, $T_m(I) \subseteq T_{m+1}(I)$.

Example 3.8. For an ideal $I = \{9k : k \in Z_0^-\}$ is in Z_0^-, $T^1(I) = \{3k : k \in Z_0^-\}$.

Theorem 3.9. *The Thierrin radical $T^*(I)$ of an ideal I in S is the intersection of all compressed ideal containing I and hence $T^*(I)$ is a compressed ideal in S.*

Proof. Clearly, $I \subseteq T^*(I)$ and $T^*(I)$ is an ideal. Let $x_1^3 x_2^3 ... x_n^3 \in T^*(I)$, for any odd positive integers n, then we have $x_1^3 x_2^3 ... x_n^3 \in T_m(I)$, for some integer m. Hence $x_1 x_2 \ldots x_n \in T_{m+1}(I)$. Therefore $x_1 x_2 \ldots x_n \in T^*(I)$. Hence $T^*(I)$ is compressed. Now let J be any compressed ideal containing I. Then we shall show that $T^*(I) \subseteq J$. Let $x \in T_1(I)$, then for any odd positive integers n, $x = x_1 x_2 \ldots x_n$ such that $x_1^3 x_2^3 \ldots x_n^3 \in I$. Since $I \subseteq J$ and J is compressed, hence $x = x_1 x_2 \ldots x_n \in J$. Therefore $T_1(I) \subseteq J$. Then by recursively we have $T_m(I) \subseteq J$. Therefore $T^*(I) \subseteq J$. Consequently, $T^*(I)$ is the intersection of all compressed ideal containing I. By Theorem 3.2, $T^*(I)$ is a compressed ideal in S. □

As every compressed ideal in S is a completely prime ideal, hence we get

Corollary 3.10. *The Thierrin radical $T^*(I)$ of an ideal I in S is the intersection of all minimal completely prime ideals associated with I.*

The following theorem shows that, in a ternary A-semiring the Thierrin radical of an ideal coincides with the algebraic radical of an ideal.

Theorem 3.11. *If S is a ternary A-semiring, then $T^*(I) = \sqrt{I}$, for every proper ideal I in S.*

Proof. The notion of prime ideal and completely prime ideal are equivalent in a ternary A-semiring S. Thus, the collection of all minimal prime ideals associated with I is identical to the collection of all minimal completely prime ideals associated with I. Thus, by using Theorem 3.9, we get $T^*(I) = \sqrt{I}$. □

4 Conclusion remarks

This study advances ideal theory in the context of ternary A-semirings by establishing connections with prior work by Dutta and Kar. The introduction and characterization of minimal prime ideal associated with an ideal and compressed ideal in a ternary semiring. Furthermore, by adapting Iseki's Thierrin radical and proving its equivalence to the algebraic radical, the study extends classical radical theory and demonstrates the orderly behavior of ideals in ternary A-semirings.

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