

SPECTRA AND ENERGY OF CONJUGATE SKEW GAIN GRAPHS

Shahul Hameed K, Ramakrishnan K O and Biju K

Communicated by: Madeleine Al Tahan

MSC 2010 Classifications: Primary 05C22; Secondary 05C50.

Keywords and phrases: Conjugate skew gain graphs, Adjacency matrix, Spectrum, Energy.

The authors would like to thank the reviewers and the editor for their constructive comments and valuable suggestions that improved the quality of our paper.

Abstract A conjugate skew gain graph (abbreviated as csg) is a graph with its oriented edges having weights called skew gains from the multiplicative group \mathbb{C}^\times of complex numbers, such that they get conjugated when we reverse the orientation. In this paper, we deal with the spectral properties and energy of csgs, where the energy of a csg is the sum of the absolute values of the eigenvalues of its adjacency matrix. We discuss some bounds for the energy and compute the energies of certain special types of csgs.

1 Introduction

In this article, we discuss some of the spectral properties of a conjugate skew gain graph (which is abbreviated as csg from now on) and its energy. A csg [14] $G^\varphi = (G, \varphi)$ is a graph G with edges being oriented and given skew gains from the multiplicative group \mathbb{C}^\times of non-zero complex numbers on them, such that the skew gains get conjugated on the reversal of orientation. More formally, for each oriented edge \vec{uv} , there is a non-zero complex number $\varphi(\vec{uv})$ such that the reverse oriented edge \vec{vu} has the edge skew gain $\varphi(\vec{vu}) = \overline{\varphi(\vec{uv})}$. We denote the set of oriented edges of G^φ by \vec{E} . Graphs, signed graphs, and complex unit gain graphs are special cases of csgs.

If there are n vertices v_1, v_2, \dots, v_n in G , the adjacency matrix [14] of a csg G^φ , denoted by $A(G^\varphi)$ or simply by A , is defined as the hermitian matrix (a_{ij}) of order n , given by

$$a_{ij} = \begin{cases} \varphi(\vec{v_i v_j}), & \text{if } \vec{v_i v_j} \in \vec{E}, \\ 0, & \text{otherwise.} \end{cases} \quad (1.1)$$

Note that by definition, $a_{ji} = \overline{a_{ij}}$ and hence the adjacency matrix A is hermitian, whereby all the eigenvalues of A are real numbers. If $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ are these n eigenvalues, then the energy of G^φ is defined as $\mathcal{E}(G^\varphi) = \sum_{i=1}^n |\lambda_i|$. The energy of a graph [4] was defined by Ivan Gutman and it has many applications in chemical graph theory. As such, several researchers later carried forward this idea, producing a large body of work on it. Moreover, several extensions of energy to various other discrete structures have appeared in the literature; for instance, one may refer to the energy of signed graphs in [5, 10], the energy of a complex unit gain graph [11], the energy of mixed graphs [8], etc. A good book of reference for the energy of various discrete structures is [7].

2 Spectral properties of csgs

Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ be a given matrix. Then we use the notation $|A|$ for the real matrix $|A| = (|a_{ij}|)$. If $A \in \mathbb{C}^{n \times n}$ is a Hermitian matrix with the eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, then the spectral radius $\rho(A)$ of A is defined as $\rho(A) = \max\{|\lambda_i| : 1 \leq i \leq n\} = \max\{\lambda_1, -\lambda_n\}$.

For a hermitian matrix $B = (b_{ij})$ with zeros on the diagonal, we denote the sum $\sum_{1 \leq i < j \leq n} |b_{ij}|^2$ by M . In the case of the adjacency matrix A of a csg G^φ , this quantity will be $M = \sum_{e \in E(G)} |\varphi(\vec{e})|^2$.

The absolute-value degree of a vertex v [14] is $d_a(v) = \sum_{e \sim v} |\varphi(\vec{e})|$, and the maximum absolute-value degree Δ_a of a csg G^φ is defined by $\Delta_a = \max_{v \in V(G)} d_a(v)$.

We also define the squared-absolute-value degree of a vertex v in a csg as $d_{2a}(v) = \sum_{e \sim v} |\varphi(\vec{e})|^2$, and correspondingly, the maximum squared-absolute-value degree Δ_{2a} is defined as $\Delta_{2a} = \max_{v \in V(G)} d_{2a}(v)$. Clearly, $\sum_{v \in V} d_{2a}(v) = 2M$ and $2M \leq n\Delta_{2a}$, where n is the order of G .

Note that, in the case of graphs, signed graphs, and complex unit gain graphs, since $|\varphi(\vec{e})| = 1$ for all the edges, $d_a(v) = d_{2a}(v) = d(v)$ (the usual degree of a vertex), and $\Delta_a = \Delta_{2a} = \Delta$ (the maximum degree of the underlying graph).

We denote the set $\{z \in \mathbb{C}^n : |z| = k\}$ by $\mathbb{T}(k)$, and when $k > 1$, $\mathbb{T}^{>1}(k)$ denotes the set $\{z \in \mathbb{C}^n : |z| = k > 1\}$. Similarly, the notation $\mathbb{T}^{<1}(k)$ stands for the set $\{z \in \mathbb{C}^n : |z| = k < 1\}$. Recall that $z \in \mathbb{C}^n : |z| = 1 = \mathbb{T}(1) = \mathbb{T}$.

Now we provide a basic lemma, the results of which will be used in the subsequent discussions very often.

Lemma 2.1. *If $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of the adjacency matrix $A = (a_{ij})$ of a csg, then*

$$\sum_{i=1}^n \lambda_i^2 = 2 \sum_{1 \leq i < j \leq n} |a_{ij}|^2 = \sum_{v \in V(G)} \sum_{e \in E(G) \sim v} |\varphi(\vec{e})|^2 = \sum_{v \in V(G)} d_{2a}(v) = 2M$$

and

$$\sum_{i < j} \lambda_i \lambda_j = -M$$

Proof. For the first part, we just have to notice that $\sum_{i=1}^n \lambda_i^2 = \text{tr}(A^2) = 2 \sum_{1 \leq i < j \leq n} |a_{ij}|^2 = 2M$ and the other equalities arising from it are as per the discussions given above. For the second result, we note that $\sum_{i < j} \lambda_i \lambda_j = 1/2 \left(\left(\sum_{i=1}^n \lambda_i \right)^2 - \sum_{i=1}^n \lambda_i^2 \right) = 1/2(0 - 2M) = -M$. □

It is proved in [14] that the spectral radius satisfies the inequality $\rho(A(G^\phi)) \leq \Delta_a$. We now provide some more bounds for the spectral radius of the adjacency matrix of a csg. For this, we begin with a known result applicable to all matrices with complex entries.

Theorem 2.2 ([12], Theorem 8.1). *For any matrix $A \in \mathbb{C}^{n \times n}$, $\rho(A) \leq \rho(|A|)$*

Theorem 2.3. *Let G^φ be a csg. Then $\rho(A(G^\phi)) \leq \rho(A(G^{|\phi|}))$*

Proof. Since $|A(G^\varphi)| = A(G^{|\phi|})$, the result follows from Theorem 2.2. □

Theorem 2.4 ([12], Theorem 2.1). *If $A \in \mathbb{C}^{n \times n}$ is a hermitian matrix with real eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and if $p = \text{tr}(A)/n, q = \sqrt{\text{tr}(A^2)/n - p^2}$, then*

$$p - q(n - 1)^{\frac{1}{2}} \leq \lambda_n \leq p - q/(n - 1)^{\frac{1}{2}}$$

and

$$p + q/(n - 1)^{\frac{1}{2}} \leq \lambda_1 \leq p + q(n - 1)^{\frac{1}{2}}$$

Theorem 2.5. *If G^φ is a csg with n vertices, then*

$$-\sqrt{\frac{2M(n - 1)}{n}} \leq \lambda_n \leq -\sqrt{\frac{2M}{n(n - 1)}}$$

and

$$\sqrt{\frac{2M}{n(n - 1)}} \leq \lambda_1 \leq \sqrt{\frac{2M(n - 1)}{n}}$$

. Hence the spectral radius $\rho(A(G^\varphi))$ satisfies the bounds given by

$$\sqrt{\frac{2M}{n(n-1)}} \leq \rho(A(G^\varphi)) \leq \sqrt{\frac{2M(n-1)}{n}}$$

Proof. We apply Theorem 2.4 to prove the results here for which the values of the parameters therein are $p = \text{tr}(A(G^\varphi))/n = 0$; $\text{tr}(A^2(G^\varphi))$ as it is equal to $2M$ which makes $q = 2M/n$. Substituting these values in Theorem 2.4 and noting the fact $\rho(A(G^\varphi)) = \max\{\lambda_1, -\lambda_n\}$, the proof is complete. \square

The largest eigenvalue λ_1 of the adjacency matrix of a csg G^φ is often referred to as the index of that csg. We provide a lower bound for the index in the following corollary.

Corollary 2.6. *If λ_1 is the index of a connected csg G^φ , then $\lambda_1 \geq \sqrt{\frac{2\delta_{2a}}{(n-1)}}$ where $\delta_{2a} = \min\{d_{2a}(v) : v \in v(G)\}$.*

Proof. This follows from Theorem 2.5. \square

We now require one or two other types of degrees for the vertices of a csg G^φ . If the image set of all oriented edges is taken as $\varphi(\vec{E}) = \{\varphi(\vec{e}) = z \in \mathbb{C}^\times : \text{for some } \vec{e} \in \vec{E}(G)\}$, we define z -degree of a vertex v for a given complex number z in the image set as $d^z(v) = |\{u \in V(G) : u \sim v, \varphi(\vec{vu}) = z\}|$. The net-degree of a vertex v is $d^{net}(v) = \sum_{z \in \varphi(\vec{E})} z d^z(v)$. This can also be viewed as $d^{net}(v) = \sum_{e \sim v} \varphi(\vec{e})$. Note that the usual degree $d(v) = \sum_{z \in \varphi(\vec{E})} d^z(v)$.

Lemma 2.7. (i) *The sum $\sum_{i=1}^n d^{net}(v_i) = M_1$ is a real number given by*

$$M_1 = \sum_{i=1}^n d^{net}(v_i) = 2\Re\left(\sum_{1 \leq i < j \leq n: i \sim j} \varphi(\vec{e}_{ij})\right)$$

$$(ii) M_2 = \sum_{i=1}^n |d^{net}(v_i)|^2 = 2M + 2\Re\left(\sum_{i,j < k} \varphi(\vec{e}_{ij})\varphi(\vec{e}_{jk})\right)$$

Proof. (i) By definition, $d^{net}(v_i) = \sum_{v_j \in N(v_i)} \varphi(\vec{v_i v_j})$. Therefore,

$$\begin{aligned} \sum_{i=1}^n d^{net}(v_i) &= \sum_{v_i} \sum_{v_j \in N(v_i)} \varphi(\vec{v_i v_j}) = \sum_{1 \leq i < j \leq n: i \sim j} (\varphi(\vec{v_i v_j}) + \varphi(\vec{v_j v_i})) \\ &= 2\Re\left(\sum_{1 \leq i < j \leq n: i \sim j} \varphi(\vec{e}_{ij})\right). \end{aligned}$$

Also, (ii)

$$\begin{aligned} |d^{net}(v_i)|^2 &= d^{net}(v_i)\overline{d^{net}(v_i)} = \sum_j \varphi(\vec{e}_{ij}) \cdot \sum_k \overline{\varphi(\vec{e}_{ik})} \\ &= \sum_{j,k} \varphi(\vec{e}_{ij})\overline{\varphi(\vec{e}_{ik})} = \sum_j |\varphi(\vec{e}_{ij})|^2 + \sum_{k \neq j} \varphi(\vec{e}_{ki})\varphi(\vec{e}_{ij}) \\ &= \sum_j |\varphi(\vec{e}_{ij})|^2 + 2\Re \sum_{k < j} \varphi(\vec{e}_{ki})\varphi(\vec{e}_{ij}) \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i=1}^n |d^{net}(v_i)|^2 &= \sum_i \sum_j |\varphi(\vec{e}_{ij})|^2 + \sum_i 2\Re \sum_{k < j} \varphi(\vec{e}_{ki})\varphi(\vec{e}_{ij}) \\ &= 2M + 2\Re\left(\sum_{i,j < k} \varphi(\vec{e}_{ij})\varphi(\vec{e}_{jk})\right) \end{aligned}$$

\square

The following result is often referred to as Rayleigh-Ritz theorem used for computing the largest and smallest eigenvalues of a hermitian matrix.

Lemma 2.8 ([12], Theorem 4.2.2). *If $A \in \mathbb{C}^n$ is a hermitian matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, then*

$$\lambda_1 = \max_{\mathbf{x} \neq 0 \in \mathbb{C}^n} \frac{\mathbf{x}^* A \mathbf{x}}{\mathbf{x}^* \mathbf{x}} = \max_{\mathbf{x}^* \mathbf{x} = 1} \mathbf{x}^* A \mathbf{x} \tag{2.1}$$

and

$$\lambda_n = \min_{\mathbf{x} \neq 0 \in \mathbb{C}^n} \frac{\mathbf{x}^* A \mathbf{x}}{\mathbf{x}^* \mathbf{x}} = \min_{\mathbf{x}^* \mathbf{x} = 1} \mathbf{x}^* A \mathbf{x} \tag{2.2}$$

Theorem 2.9. *For a csg G^φ , the largest eigenvalue (or the index) λ_1 and the smallest eigenvalue λ_n satisfy the inequalities:*

$$(i) \lambda_n \leq \frac{1}{n} \sum_{i=1}^n d^{net}(v_i) \leq \lambda_1$$

$$(ii) \lambda_n \leq \sqrt{\frac{1}{n} \sum_{i=1}^n |d^{net}(v_i)|^2} \leq \lambda_1$$

Proof. Take $\mathbf{j} = (1, 1, \dots, 1)$ and define $M_1 = \mathbf{j}^* A \mathbf{j} / \mathbf{j}^* \mathbf{j}$, $M_2 = \mathbf{j}^* A^2 \mathbf{j} / \mathbf{j}^* \mathbf{j}$. Using Lemma 2.8, we get the results, since $M_1 = \frac{1}{n} \sum_{i=1}^n d^{net}(v_i)$ and $M_2 = \frac{1}{n} \|A \mathbf{j}\|^2 = \frac{1}{n} \sum_{i=1}^n |d^{net}(v_i)|^2$. While computing the bounds, note the fact that, if λ is an eigenvalue of A , then λ^2 will be an eigenvalue of A^2 . □

Now we deal with the spectra of those csgs for which the underlying graphs are bipartite.

Theorem 2.10. *If the underlying graph G of the csg G^φ is bipartite, then whenever λ is an eigenvalue of $A = A(G^\varphi)$, $-\lambda$ is also an eigenvalue.*

Proof. Assuming that $X = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue λ , as the adjacency matrix can be written in the block partitioned form $A(G^\varphi) = \begin{pmatrix} O & B \\ B^* & O \end{pmatrix}$, the equation $A X = \lambda X$ implies $B^* \mathbf{x} = \lambda \mathbf{y}$ and $B \mathbf{y} = \lambda \mathbf{x}$. Then

$$A \begin{pmatrix} \mathbf{x} \\ -\mathbf{y} \end{pmatrix} = \begin{pmatrix} O & B \\ B^* & O \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ -\mathbf{y} \end{pmatrix} = \begin{pmatrix} -B \mathbf{y} \\ B^* \mathbf{x} \end{pmatrix} = \begin{pmatrix} -\lambda \mathbf{x} \\ \lambda \mathbf{y} \end{pmatrix} = -\lambda \begin{pmatrix} \mathbf{x} \\ -\mathbf{y} \end{pmatrix},$$

proving the result that $-\lambda$ is also an eigenvalue of A . □

The above theorem shows that spectra of a csg for which the underlying graph is bipartite is symmetric about origin.

3 Energy of csgs

We begin with certain bounds for the energy of csgs which are already established in the cases of graphs [4, 7], signed graphs [5, 10] and mixed graphs [8] and we prove the one for csgs, having a few major differences as shown below. The first one is often referred to as McClelland’s bound for energy. Since the proof closely resembles that of the original theorem appeared in [9] with a bit of changes for csgs, we provide only a brief sketch, omitting many of the details.

Theorem 3.1. *If G^φ is a csg with order n , then the energy $\mathcal{E}(G^\varphi)$ satisfy the inequality:*

$$\sqrt{2M + n(n - 1)(\det A)^{2/n}} \leq \mathcal{E}(G^\varphi) \leq \sqrt{2Mn} \leq n\sqrt{\Delta_{2a}}$$

Proof.

$$\begin{aligned}
 (\mathcal{E}(G^\varphi))^2 &= \left(\sum_{i=1}^n |\lambda_i|\right)^2 \\
 &= \sum_{i=1}^n \lambda_i^2 + \sum_{i \neq j} |\lambda_i||\lambda_j| \\
 &\geq 2M + n(n-1)(\det A)^{2/n}.
 \end{aligned}$$

Also, using Cauchy-Schwartz inequality applied to the vector $(|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|)$ consisting of the eigenvalues and the vector $\mathbf{j} = (1, 1, \dots, 1)$,

$$\begin{aligned}
 \mathcal{E}(G^\varphi) &= \sum_{i=1}^n |\lambda_i| \leq \sqrt{\sum_{i=1}^n \lambda_i^2} \sqrt{\sum_{i=1}^n 1} = \sqrt{2Mn} \\
 &\leq \sqrt{(n\Delta_{2a})n} = n\sqrt{\Delta_{2a}} \quad \square
 \end{aligned}$$

The following corollary adapts the proof technique [1].

Corollary 3.2. $\mathcal{E}(G^\varphi) = n\sqrt{\Delta_{2a}}$ if and only if $A^*A = \Delta_{2a}I_n$ where I_n is the identity matrix of order n .

Proof. Equality $\mathcal{E}(G^\varphi) = n\sqrt{\Delta_{2a}}$ holds if and only if the Schwarz’s inequality becomes equality and this is true if and only if $2M = n\Delta_{2a}$. This in turn is true if and only if there is a real constant α such that $|\lambda_i|^2 = \alpha$ for all i and G^φ is Δ_{2a} -regular. This is possible if and only $A^*A = \alpha I_n$ and $\alpha = \Delta_{2a}$. \square

A lower bound for the energy in terms of the spectral radius is as follows.

Theorem 3.3. $\mathcal{E}(G^\varphi) \geq 2\rho(A(G^\varphi))$

Proof. If $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of $A(G^\varphi)$, as $\sum_{i=1}^n \lambda_i = 0$, we get $\lambda_1 = -\sum_{i=2}^n \lambda_i$. Therefore, $|\lambda_1| \leq \sum_{i=2}^n |\lambda_i|$. Therefore, $2|\lambda_1| \leq \sum_{i=1}^n |\lambda_i|$. In this way all the eigenvalues satisfy this inequality which establishes the bound $\mathcal{E}(G^\varphi) \geq 2\rho(A(G^\varphi))$. \square

Theorem 3.4. For a csg G^φ with average square-absolute-degree $\frac{2M}{n} \geq 1$, its energy $\mathcal{E}(G^\varphi)$ satisfy the inequality $2\sqrt{M} \leq \mathcal{E}(G^\varphi) \leq 2M$

Proof.

$$\begin{aligned}
 (\mathcal{E}(G^\varphi))^2 &= \sum_{i=1}^n \lambda_i^2 + 2 \sum_{i \neq j} |\lambda_i||\lambda_j| = 2M + 2 \sum_{i \neq j} |\lambda_i \lambda_j| \\
 &\geq 2M + 2 \left| \sum_{i \neq j} \lambda_i \lambda_j \right| \geq 4M
 \end{aligned}$$

For the upper bound we have the additional assumption that $\frac{2M}{n} \geq 1$ i.e., $n \leq 2M$. With this assumption together with the result proved in Theorem 3.1, namely $\mathcal{E}(G^\varphi) \leq \sqrt{2Mn}$ provides the required upper bound. \square

In fact, all csgs for which the edge skew gains are chosen from $\mathbb{T}^{>1}(k)$ will satisfy the property that $2M/n > 1$.

We now provide an integral formula for the energy of a csg which was established for graphs by Coulson [2].

Theorem 3.5. If $\Phi(\lambda)$ is the characteristic polynomial of a csg G^φ , then its energy $\mathcal{E}(G^\varphi) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(n - \mathbf{i}\lambda \frac{\Phi'(\mathbf{i}\lambda)}{\Phi(\mathbf{i}\lambda)} \right) d\lambda$ where $\int_{-\infty}^{\infty} f(\lambda) d\lambda$ represents the principal value of the integral.

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the roots of the characteristic polynomial $\Phi(\lambda)$ so that $\Phi(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i)$ and $\Phi'(\lambda) = \sum_{i=1}^n \prod_{\substack{\ell=1 \\ \ell \neq i}}^n (\lambda - \lambda_\ell)$. Therefore, $\mathbf{i}\lambda \frac{\Phi'(\mathbf{i}\lambda)}{\Phi(\mathbf{i}\lambda)} = \mathbf{i}\lambda \sum_{i=1}^n \frac{1}{\mathbf{i}\lambda - \lambda_i}$. Now,

$$\begin{aligned} |\lambda_i| &= |\lambda_i| + \mathbf{i}0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\lambda_i^2}{\lambda_i^2 + \lambda^2} d\lambda + \mathbf{i} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\lambda_i \lambda}{\lambda_i^2 + \lambda^2} d\lambda \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\lambda_i^2 + \lambda_i \lambda}{\lambda_i^2 + \lambda^2} d\lambda = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(1 - \frac{\mathbf{i}\lambda}{\mathbf{i}\lambda - \lambda_i} \right) d\lambda. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{E}(G^\varphi) &= \sum_{i=1}^n |\lambda_i| = \frac{1}{\pi} \int_{-\infty}^{\infty} \sum_{i=1}^n \left(1 - \frac{\mathbf{i}\lambda}{\mathbf{i}\lambda - \lambda_i} \right) d\lambda \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(n - \mathbf{i}\lambda \frac{\Phi'(\mathbf{i}\lambda)}{\Phi(\mathbf{i}\lambda)} \right) d\lambda \quad \square \end{aligned}$$

As a corollary we have another form using integrals for the computation of energy of csgs, the proof of which is excluded as it can be adapted from that in [2, 8].

Corollary 3.6. If G^φ is a csg of order n , $\mathcal{E}(G^\varphi) = \frac{2}{\pi} \int_0^\infty \frac{1}{\lambda^2} \log \left| \lambda^n \Phi \left(\frac{\mathbf{i}}{\lambda} \right) \right| d\lambda$.

4 Spectra and energy of some classes of csgs

Before we delve into the details, as we require a few basic notions of balance theory, we briefly sketch only very essential concepts in the following lines. The notion of balance theory for csgs is discussed in [14] extending the same from the realm of signed graphs and complex unit gain graphs. Given an oriented cycle \vec{C} in a csg, its edge skew gain $\varphi(\vec{C})$ is defined as the product of the edge skew gains and a cycle C is said to be balanced if $\varphi(\vec{C}) = |\varphi(\vec{C})|$. A csg G^φ itself is said to be balanced if every cycle in it satisfies this condition. Many other details such as the switching and its impact on the balance can be had from [14]. Note that two switching equivalent csgs will have the same energy as their adjacency matrices will be similar and hence cospectral. This section is devoted to give the formula for the spectra and energy of certain specified csgs. First of all, we compute the energy of cycles with the edge skew gains on the oriented edges from a selected subset of the complex plane. We use the notation $C_n^\varphi(\theta)$ to denote a csg with the underlying graph as the cycle C_n , $\varphi(\vec{E}(C_n^\varphi)) \subseteq \mathbb{T}(k)$ and $\arg(\varphi(C_n)) = \theta$ ($0 \leq \theta < 2\pi$). First we deal with the energy of the csg $C_n^\varphi(\theta)$ which becomes an absolute-degree $2k$ -regular csg when $\varphi(\vec{E}(C_n^\varphi(\theta))) \subseteq \mathbb{T}(k)$. Using a switching function (the details of which again can be had from [14]), the csg $C_n^\varphi(\theta)$ can be switched to the one with all oriented edges having the edge skew gains k except for the edge $v_1 v_2$ for which the edge skew gain will be $ke^{i\theta}$.

Theorem 4.1 ([14], Corollary 3.4). *The adjacency spectrum of $C_n^\varphi(\theta)$ is given by*

$$\lambda_j(A(C_n^\varphi(\theta))) = 2k \cos \left(\frac{\theta + 2\pi j}{n} \right), \quad j = 1, 2, \dots, n \quad (4.1)$$

The following result can be easily obtained from basic trigonometric summation of cosine series and we give only a brief sketch of the proof.

Lemma 4.2. $\sum_{j=1}^r \cos((2\pi j + \theta)/n) = \cos(((r+1)\pi + \theta)/n) \sin(\pi r/n) / \sin(\pi/n)$ and $\sum_{j=1}^r \sin((2\pi j + \theta)/n) = \sin(((r+1)\pi + \theta)/n) \sin(\pi r/n) / \sin(\pi/n)$

Proof. Note that the required sums are the real and imaginary parts of $\sum_{j=1}^r e^{i(2\pi j + \theta)/n}$. Now,

$$\begin{aligned} \sum_{j=1}^r e^{i(2\pi j + \theta)/n} &= e^{i(\theta/n)} \sum_{j=1}^r (e^{i(2\pi/n)})^j = e^{i(\theta/n)} \times e^{2\pi i/n} \left(\frac{e^{2\pi r i/n} - 1}{e^{2\pi i/n} - 1} \right) \\ &= e^{i(\theta + \pi(r+1))/n} \times \frac{2 \sin(\pi r/n)}{2 \sin(\pi/n)} \end{aligned}$$

Hence separating into real and imaginary parts, we obtain the two sums of cosine and sine series. □

The notation $SV(n, \theta)$, in the following lemma and in the discussion the comes later on, is used to denote the set $\{j \in [1, n] \cap \mathbb{N} : \cos((\theta + 2\pi j)/n) < 0\}$. In the following lemma we find a precise expression for this set in the form of $[a \ b] \cap \mathbb{N}$ as we require these values of a and b for the computation of the energy of $C_n^\varphi(\theta)$. Actually, $\mathcal{E}(C_n^\varphi) = 2 \sum_{j \in SV(n, \theta)} (-\lambda_j)$ as $\lambda_j < 0$

when $j \in SV(n, \theta)$ and because $\sum_{j=1}^n \lambda_j = 0$. Therefore, with these values of a and b in hand, $\mathcal{E}(C_n^\varphi) = 2 \sum_{j=a}^{j=b} (-\lambda_j) = -4k \sum_{j=a}^{j=b} \cos((\theta + 2\pi j)/n)$.

Lemma 4.3. *Let*

$$[a \ b] = \begin{cases} \begin{cases} [p \ 3p - 1] & \text{if } \theta \neq 0 \\ [p + 1 \ 3p - 1] & \text{if } \theta = 0 \end{cases} & \text{for } n = 4p, \\ \begin{cases} [p + 1 \ 3p] & \text{if } 0 \leq \theta \leq \pi/2 \\ [p \ 3p] & \text{if } \pi/2 < \theta < 3\pi/2 \\ [p \ 3p - 1] & \text{if } 3\pi/2 \leq \theta < 2\pi \end{cases} & \text{for } n = 4p + 1, \\ \begin{cases} [p + 1 \ 3p + 1] & \text{if } 0 \leq \theta < \pi \\ [p + 1 \ 3p] & \text{if } \theta = \pi \\ [p \ 3p] & \text{if } \pi < \theta < 2\pi \end{cases} & \text{for } n = 4p + 2, \\ \begin{cases} [p \ 3p - 1] & \text{if } 0 \leq \theta < \pi/2 \\ [p \ 3p - 2] & \text{if } \pi/2 \leq \theta \leq 3\pi/2 \\ [p - 1 \ 3p - 2] & \text{if } 3\pi/2 < \theta < 2\pi \end{cases} & \text{for } n = 4p - 1. \end{cases} \tag{4.2}$$

For any fixed positive integer n and for $\theta \in [0, 2\pi)$, if $SV(n, \theta) = [a \ b] \cap \mathbb{N}$, then $\cos(\frac{2\pi j + \theta}{n}) < 0$ for any positive integer $j \leq n$ if and only if $j \in SV(n, \theta)$.

Proof. First of all, note that $\cos(\theta) < 0$ if and only if $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ over the interval $[0 \ 2\pi)$. Therefore, for any positive integer n and for $j = 1, 2, \dots, n$, $\cos(\frac{2\pi j + \theta}{n}) < 0$ will occur if and only if $(2\pi j + \theta)/n \in (\frac{\pi}{2}, \frac{3\pi}{2})$. i.e., if and only if $j \in SV(n, \theta) = (\frac{n}{4} - \frac{\theta}{2\pi}, \frac{3n}{4} - \frac{\theta}{2\pi}) \cap \mathbb{N}$ where,

by choice of θ we have $0 \leq \frac{\theta}{2\pi} < 1$. Now depending on the values of n , we have the following four main cases within which contains their subcases also.

Case(1): $n = 4p$, for some $p \in \mathbb{N}$ and $0 \leq \theta < 2\pi$. Then as per the descriptions given above, $SV(4p, \theta) = (p - \frac{\theta}{2\pi} \ 3p - \frac{\theta}{2\pi}) \cap \mathbb{N} = [p \ 3p - 1] \cap \mathbb{N}$ when $\theta \neq 0$ and when $\theta = 0$, $SV(4p, \theta) = [p + 1 \ 3p - 1] \cap \mathbb{N}$

Case(2): $n = 4p + 1$, for some $p \in \mathbb{N}$ and $0 \leq \theta < 2\pi$.

Here, $SV(4p + 1, \theta) = (p + (\frac{1}{4} - \frac{\theta}{2\pi}) \ 3p + (\frac{3}{4} - \frac{\theta}{2\pi})) \cap \mathbb{N}$. Take $\epsilon_\theta = (\frac{1}{4} - \frac{\theta}{2\pi})$. Then, $-\frac{3}{4} < \epsilon_\theta \leq \frac{1}{4}$. We consider the following subcases.

Subcase 2(a): $0 \leq \epsilon_\theta \leq \frac{1}{4}$. This gives $0 \leq \theta \leq \frac{\pi}{2}$ and hence $SV(4p + 1, \theta) = (p + \epsilon_\theta \ 3p + \frac{1}{2} + \epsilon_\theta) \cap \mathbb{N} = [p + 1 \ 3p] \cap \mathbb{N}$.

Subcase 2(b): $-\frac{1}{2} < \epsilon_\theta < 0$ i.e., $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Then $SV(4p + 1, \theta) = (p + \epsilon_\theta \ 3p + \frac{1}{2} + \epsilon_\theta) \cap \mathbb{N} = [p \ 3p] \cap \mathbb{N}$.

Subcase 2(c): $-\frac{3}{4} < \epsilon_\theta \leq -\frac{1}{2}$ i.e., when $\frac{3\pi}{2} \leq \theta < 2\pi$. In this case, $SV(4p + 1, \theta) = (p + \epsilon_\theta \ 3p + \frac{1}{2} + \epsilon_\theta) \cap \mathbb{N} = [p \ 3p - 1] \cap \mathbb{N}$

Case(3): $n = 4p + 2$, for some $p \in \mathbb{N}$

Here $SV(4p + 2, \theta) = (p + (\frac{1}{2} - \frac{\theta}{2\pi}) \ 3p + (\frac{3}{2} - \frac{\theta}{2\pi})) \cap \mathbb{N}$. Let $\epsilon_\theta = (\frac{1}{2} - \frac{\theta}{2\pi})$. Then $-\frac{1}{2} < \epsilon_\theta \leq \frac{1}{2}$. We consider the following subcases here.

Subcase 3(a): When $0 < \epsilon_\theta \leq \frac{1}{2}$ or equivalently $0 \leq \theta < \pi$. Then $SV(4p + 2, \theta) = (p + \epsilon_\theta \ 3p + 1 + \epsilon_\theta) \cap \mathbb{N} = [p + 1 \ 3p + 1] \cap \mathbb{N}$.

Subcase 3(b): $\epsilon_\theta = 0$. As such, $\theta = \pi$ and hence $SV(4p + 2, \pi) = (p \ 3p + 1) \cap \mathbb{N} = [p + 1 \ 3p] \cap \mathbb{N}$.

Subcase 3(c): $-\frac{1}{2} < \epsilon_\theta < 0$. i.e., $\pi < \theta < 2\pi$. This provides $SV(4p + 2, \theta) = (p + \epsilon_\theta \ 3p + 1 + \epsilon_\theta) \cap \mathbb{N} = [p \ 3p] \cap \mathbb{N}$.

Case(4): $n = 4p - 1$, for some $p \in \mathbb{N}$. In this case, $SV(4p - 1, \theta) = n(p - (\frac{1}{4} + \frac{\theta}{2\pi}) \ 3p - (\frac{3}{4} + \frac{\theta}{2\pi})) \cap \mathbb{N}$. Taking $\epsilon_\theta = (\frac{1}{4} + \frac{\theta}{2\pi})$, we obtain $\frac{1}{4} \leq \epsilon_\theta < \frac{5}{4}$. This leads to the following subcases.

Subcase 4(a): $\frac{1}{4} \leq \epsilon_\theta < \frac{1}{2}$, or $0 \leq \theta < \frac{\pi}{2}$. This provides $SV(4p - 1, \theta) = (p - \epsilon_\theta \ 3p - (\frac{1}{2} + \epsilon_\theta)) \cap \mathbb{N} = [p \ 3p - 1] \cap \mathbb{N}$.

Subcase 4(b): $\frac{1}{2} \leq \epsilon_\theta \leq 1$ or equivalently when $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, Here $SV(4p - 1, \theta) = (p - \epsilon_\theta \ 3p - (\frac{1}{2} + \epsilon_\theta)) \cap \mathbb{N} = [p \ 3p - 2] \cap \mathbb{N}$.

Subcase 4(c): $1 < \epsilon_\theta < \frac{5}{4}$ or when $\frac{3\pi}{2} < \theta < 2\pi$. This makes $SV(4p - 1, \theta) = (p - \epsilon_\theta \ 3p - (\frac{1}{2} + \epsilon_\theta)) \cap \mathbb{N} = [p - 1 \ 3p - 2] \cap \mathbb{N}$. □

Theorem 4.4. If $\varphi(\vec{E}(C_n^\varphi(\theta))) \subseteq \mathbb{T}(k)$,

$$\mathcal{E}(C_n^\varphi(\theta)) = \begin{cases} k_1 \cos(\frac{\theta - \pi}{n}) & \text{if } n \equiv 0 \pmod{4}, \\ \begin{cases} k_1 \cos(\frac{\theta}{n}) & \text{for } 0 \leq \theta \leq \pi, \\ k_1 \cos(\frac{2\pi - \theta}{n}) & \text{for } \pi < \theta < 2\pi \end{cases} & \text{if } n \equiv 2 \pmod{4}, \\ \begin{cases} k_2 \cos(\frac{\theta}{n}) & \text{for } 0 \leq \theta \leq \pi/2, \\ k_2 \cos(\frac{\theta - \pi}{n}) & \text{for } \pi/2 < \theta < 2\pi, \end{cases} & \text{if } n \equiv 1 \pmod{4}, \\ \begin{cases} k_2 \cos(\frac{\theta}{n}) & \text{for } 0 \leq \theta < \pi/2, \\ k_2 \cos(\frac{\theta - \pi}{n}) & \text{for } \pi/2 \leq \theta \leq 3\pi/2, \\ k_2 \cos(\frac{2\pi - \theta}{n}) & \text{for } 3\pi/2 < \theta < 2\pi \end{cases} & \text{if } n \equiv -1 \pmod{4}. \end{cases} \tag{4.3}$$

where the constants $k_1 = 4k \csc\left(\frac{\pi}{n}\right)$ and $k_2 = 2k \csc\left(\frac{\pi}{2n}\right)$

Proof. Using Equation (4.1), $\mathcal{E}(C_n^\varphi) = 2k \sum_{j=1}^n \left| \cos\left(\frac{\theta + 2\pi j}{n}\right) \right|$. Now we compute this value for different values of n as follows. We omit the descriptions about $SV(n, \theta)$ in each cases as it has been given in detail in Lemma 4.3. Also, as already described, when the values of a and b are known from the set $SV(n, \theta) = [a, b] \cap \mathbb{N}$, $\mathcal{E}(C_n^\varphi) = -4k \sum_{j=a}^b \cos\left(\frac{\theta + 2\pi j}{n}\right)$. We use Lemma 4.2 for the computation of cosine sum.

Case 1: $n = 4p$. We have two subcases here.

Subcase 1(a): $\theta = 0$ i.e., when the cycle is balanced. Note that here $a = p + 1$ and $b = 3p - 1$. Hence,

$$\begin{aligned} \mathcal{E}(C_n^\varphi(\theta)) &= -4k \sum_{j=p+1}^{3p-1} \cos\left(\frac{2\pi j}{n}\right) = -4k \sum_{j=1}^{2p-1} \cos\left(\frac{2\pi(p+j)}{n}\right) \\ &= 4k \sum_{j=1}^{2p-1} \sin\left(\frac{2\pi j}{n}\right) = 4k \cot(\pi/n) \end{aligned}$$

Subcase 1(b): $\theta \neq 0$ i.e., the cycle is unbalanced.

Here $a = p$ and $b = 3p - 1$. Hence, when $\theta \neq 0$,

$$\begin{aligned} \mathcal{E}(C_n^\varphi(\theta)) &= -4k \sum_{j=p}^{3p-1} \cos\left(\frac{\theta + 2\pi j}{n}\right) = -4k \sum_{j=1}^{2p} \cos\left(\frac{\theta + 2\pi(p-1+j)}{n}\right) \\ &= 4k \sum_{j=1}^{2p} \sin\left(\frac{\theta + 2\pi(j-1)}{n}\right) = 4k \sum_{j=0}^{2p-1} \sin\left(\frac{\theta + 2\pi j}{n}\right) \\ &= 4k \frac{\cos\left(\frac{\theta-\pi}{n}\right)}{\sin(\pi/n)} \end{aligned}$$

This value is even true for $\theta = 0$, so we shall combine the result together and thus $\mathcal{E}(C_n^\varphi(\theta)) = 4k \frac{\cos\left(\frac{\theta-\pi}{n}\right)}{\sin(\pi/n)}$ when $n = 4p$.

Case 2: $n = 4p + 2$. We have three subcases here.

Subcase 2(a): $0 \leq \theta < \pi$: In this subcase, $a = p + 1$ and $b = 3p + 1$. Hence,

$$\begin{aligned} \mathcal{E}(C_n^\varphi(\theta)) &= -4k \sum_{j=p+1}^{3p+1} \cos\left(\frac{\theta + 2\pi j}{n}\right) = -4k \sum_{j=1}^{2p+1} \cos\left(\frac{\theta + 2\pi(p+j)}{n}\right) \\ &= 4k \frac{\sin\left(\frac{\theta+(2p+1)\pi}{n}\right)}{\sin(\pi/n)} = 4k \frac{\cos(\theta/n)}{\sin(\pi/n)} \end{aligned}$$

Subcase 2(b): $\theta = \pi$: Here $a = p + 1$ and $b = 3p$. Hence,

$$\begin{aligned} \mathcal{E}(C_n^\varphi(\theta)) &= -4k \sum_{j=p+1}^{3p} \cos\left(\frac{\pi(2j+1)}{n}\right) = -4k \sum_{j=1}^{2p} \cos\left(\frac{\pi(2p+1+2j)}{n}\right) \\ &= 4k \sum_{j=1}^{2p} \sin\left(\frac{2\pi j}{n}\right) = 4k \cot(\pi/n), \end{aligned}$$

which ceases to be a particular case of 2(a). **Subcase 2(c):** $\pi < \theta < 2\pi$: $a = p$ and $b = 3p$. So,

$$\begin{aligned} \mathcal{E}(C_n^\varphi(\theta)) &= -4k \sum_{j=p}^{3p} \cos\left(\frac{\theta + 2\pi j}{n}\right) = -4k \sum_{j=0}^{2p} \cos\left(\frac{\theta + 2\pi(p+j)}{n}\right) \\ &= 4k \frac{\cos((2\pi - \theta)/n)}{\sin(\pi/n)} \end{aligned}$$

Case 3: $n = 4p + 1$. Here there are three subcases as given below.

Subcase 3(a): $0 \leq \theta \leq \pi/2$. For this subcase, $a = p + 1$ and $b = 3p$. Hence,

$$\begin{aligned} \mathcal{E}(C_n^\varphi(\theta)) &= -4k \sum_{j=p+1}^{3p} \cos\left(\frac{\theta + 2\pi j}{n}\right) = -4k \sum_{j=1}^{2p} \cos\left(\frac{\theta + 2\pi(p+j)}{n}\right) \\ &= 4k \frac{\cos(\theta/n) \sin(2p\pi/n)}{\sin(\pi/n)} = 4k \frac{\cos(\theta/n) \cos(\pi/2n)}{\sin(\pi/n)}. \end{aligned}$$

Subcase 3(b): $\pi/2 < \theta < 3\pi/2$. In this case, the variation for j is from p to $3p$ to give,

$$\begin{aligned} \mathcal{E}(C_n^\varphi(\theta)) &= -4k \sum_{j=p}^{3p} \cos\left(\frac{\theta + 2\pi j}{n}\right) = -4k \sum_{j=0}^{2p} \cos\left(\frac{\theta + 2\pi(p+j)}{n}\right) \\ &= 4k \frac{\cos((\theta + 4p\pi)/n) \cos(\pi/2n)}{\sin(\pi/n)} = 4k \frac{\cos((\pi - \theta)/n) \cos(\pi/2n)}{\sin(\pi/n)}. \end{aligned}$$

Subcase 3(c): $3\pi/2 \leq \theta < 2\pi$. Here, j varies from p to $3p - 1$ to make,

$$\begin{aligned} \mathcal{E}(C_n^\varphi(\theta)) &= -4k \sum_{j=p}^{3p-1} \cos\left(\frac{\theta + 2\pi j}{n}\right) = -4k \sum_{j=0}^{2p-1} \cos\left(\frac{\theta + 2\pi(p+j)}{n}\right) \\ &= 4k \frac{\cos((\theta + 4p\pi)/n) \cos(\pi/2n)}{\sin(\pi/n)} = 4k \frac{\cos((\pi - \theta)/n) \cos(\pi/2n)}{\sin(\pi/n)}. \end{aligned}$$

Case 4: $n = 4p - 1$. This needs the following subcases.

Subcase 4(a): $0 \leq \theta < \frac{\pi}{2}$, then $a = p$ and $b = 3p - 1$.

$$\begin{aligned} \mathcal{E}(C_n^\varphi(\theta)) &= -4k \sum_{j=p}^{3p-1} \cos\left(\frac{\theta + 2\pi j}{n}\right) = -4k \sum_{j=1}^{2p} \cos\left(\frac{\theta + 2\pi(p+j-1)}{n}\right) \\ &= 4k \frac{\cos(\theta/n) \cos(\pi/2n)}{\sin(\pi/n)}. \end{aligned}$$

Subcase 4(b): $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, then j varies from p to $3p - 2$

$$\begin{aligned} \mathcal{E}(C_n^\varphi(\theta)) &= -4k \sum_{j=p}^{3p-2} \cos\left(\frac{\theta + 2\pi j}{n}\right) = -4k \sum_{j=1}^{2p-1} \cos\left(\frac{\theta + 2\pi(p+j-1)}{n}\right) \\ &= 4k \frac{\cos((\pi - \theta)/n) \cos(\pi/2n)}{\sin(\pi/n)}. \end{aligned}$$

Subcase 4(c): $\frac{3\pi}{2} < \theta < 2\pi$, then j varies from $p - 1$ to $3p - 2$

$$\begin{aligned} \mathcal{E}(C_n^\varphi(\theta)) &= -4k \sum_{j=p-1}^{3p-2} \cos\left(\frac{\theta + 2\pi j}{n}\right) = -4k \sum_{j=1}^{2p} \cos\left(\frac{\theta + 2\pi(p+j-2)}{n}\right) \\ &= 4k \frac{\cos((2\pi - \theta)/n) \cos(\pi/2n)}{\sin(\pi/n)}. \end{aligned}$$

□

As the energy depends on the value of θ , it is natural to discuss the monotonicity of $\mathcal{E}(C_n^\varphi(\theta))$ for a given n . In other words, given $\theta_1, \theta_2 \in [0, 2\pi)$ for a given n , which of the values $\mathcal{E}(C_n^\varphi(\theta_1))$ or $\mathcal{E}(C_n^\varphi(\theta_2))$ is bigger or smaller. We address that problem below.

Corollary 4.5. *The energy $\mathcal{E}(C_n^\varphi(\theta))$ decreases in: $[\pi, 2\pi)$, when $n = 4p$;*

- $[0, \pi]$, when $n = 4p + 2$;
 - $[0, \frac{\pi}{2}] \cup [\pi, 2\pi)$, when $n = 4p + 1$;
 - $[0, \pi/2] \cup [\pi, 3\pi/2]$, when $n = 4p - 1$
- and increases in:
- $[0, \pi]$, when $n = 4p$;
 - $[\pi, 2\pi)$, when $n = 4p + 2$;
 - $[\pi/2, \pi]$, when $n = 4p + 1$;
 - $[\pi/2, \pi] \cup [3\pi/2, 2\pi)$, when $n = 4p - 1$.

Proof. By Theorem 4.4, monotonicity of the energy depends on the factors $\cos(\theta/n)$, $\cos((\pi - \theta)/n)$ and $\cos((2\pi - \theta)/n)$, since all the energy values are scalar multiple of these factors for a fixed n . Now by the well known first derivative test, $\cos(\theta/n)$ is decreasing on $[0, 2\pi)$, the function $\cos((\pi - \theta)/n)$ is increasing on $[0, \pi]$ and decreasing on $[\pi, 2\pi)$ and finally $\cos((2\pi - \theta)/n)$ is increasing on $[0, 2\pi)$, for each $n > 1$. Now when we consider the intersecting part of these intervals with the interval of the energy variation in each of the cases established in Theorem 4.4, we get the results as stated. □

Now we deal with the csGs with the underlying graph as a sun graph which is obtained by adding pendant edges to each of the vertices of a cycle. Let C_n^φ be the cycle csG as described in the previous example to which pendant edges are added on each of its vertices v_1, v_2, \dots, v_n so that the other end vertices of the edges added be u_1, u_2, \dots, u_n with $\varphi(\vec{v_j u_j}) = ke^{i\theta_j}$. Let us denote such a sun csG by S_n^φ . An example of a sun csG S_3^φ is given in Fig. 1. We compute the spectra and energy of S_n^φ in the following theorem.

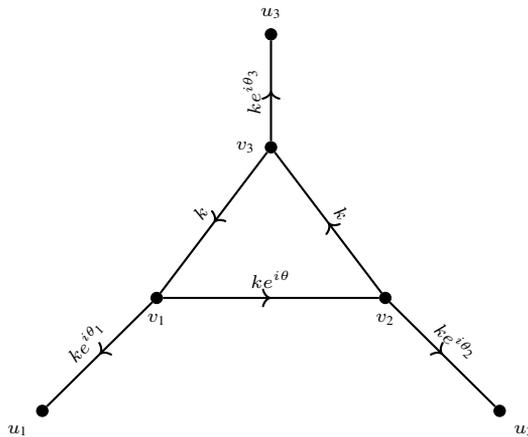


Figure 1. Sun graph S_3^φ .

Theorem 4.6. The spectra of S_n^φ is given by $\mu_j = (\lambda_j + \sqrt{\lambda_j^2 + 4k^2})/2$ and $\mu'_j = (\lambda_j - \sqrt{\lambda_j^2 + 4k^2})/2$ for $j = 1, 2, \dots, n$ where $\lambda_j = 2k \cos((\theta + 2\pi j)/n)$ are the eigenvalues of C_n^φ . The energy is given by $\mathcal{E}(S_n^\varphi) = 2k \sum_{j=1}^n \sqrt{1 + \cos^2((\theta + 2\pi j)/n)}$.

Proof. The adjacency matrix $A(S_n^\varphi)$ will be of the form $A(S_n^\varphi) = \begin{bmatrix} A(C_n^\varphi) & D \\ D^* & O \end{bmatrix}$ where D is the diagonal matrix $\text{diag}(ke^{i\theta_1}, ke^{i\theta_2}, \dots, ke^{i\theta_n})$. Taking $X = \begin{pmatrix} X_j \\ Y \end{pmatrix}$ as the eigenvector corresponding to the eigenvalue μ of $A(S_n^\varphi)$ where X_j is the eigenvector corresponding to λ_j of $A(C_n^\varphi)$. The equation $A(S_n^\varphi)X = \mu X$, gives $A(C_n^\varphi)X_j + DY = \mu X_j$ and $D^*X_j = \mu Y$. This on simplification becomes $X_j = \mu DY$ and thus $\mu A(C_n^\varphi)X_j + \mu DD^*Y = \mu^2 X_j$. So that $\mu^2 - \lambda_j \mu - k^2 = 0$ as $DD^* = k^2 I$ and $X_j \neq 0$. Solving which we get the $2n$ eigenvalues of $A(S_n^\varphi)$

as in the statement of the theorem. Now to compute the energy $\mathcal{E}(S_n^\varphi) = \sum_{j=1}^n |\mu_j| + \sum_{j=1}^n |\mu'_j|$, we

have the following claims:

Claim(1): $\mu_j \geq 0$ and $\mu'_j \leq 0$ for all j :

To prove this claim, we notice that for any real number x , $|x| \leq \sqrt{x^2 + 4k^2}$. As such $\mu_j = (\lambda_j + \sqrt{\lambda_j^2 + 4k^2})/2 \geq 0$, even if $\lambda_j < 0$. In the case of $\mu'_j = (\lambda_j - \sqrt{\lambda_j^2 + 4k^2})/2$ also the same observations hold making $\mu'_j \leq 0$.

Claim(2): $\sum_{j=1}^n |\mu_j| = \sum_{j=1}^n |\mu'_j|$: The proof of this claim follows from claim(1) and the fact that

$\sum_{j=1}^n \mu_j + \sum_{j=1}^n \mu'_j = 0$, this sum being the sum of all eigenvalues of S_n^φ .

Thus, $\mathcal{E}(S_n^\varphi) = 2 \sum_{j=1}^n \mu_j = 2 \sum_{j=1}^n (\lambda_j + \sqrt{\lambda_j^2 + 4k^2})/2$. Since $\lambda_j = 2k \cos((\theta + 2\pi j)/n)$ and

$\sum_{j=1}^n \lambda_j = 0$, $\mathcal{E}(S_n^\varphi) = \sum_{j=1}^n \sqrt{\lambda_j^2 + 4k^2} = 2k \sum_{j=1}^n \sqrt{1 + \cos^2((\theta + 2\pi j)/n)}$ □

References

- [1] C. Adiga, R. Balakrishnan, W. So, The skew energy of a digraph. *Linear Algebra Appl.*, **432** (2010) 1825–1835.
- [2] C. Coulson, On the calculation of the energy in unsaturated hydrocarbon molecules. *Proc. Cambridge Philos. Soc.*, **36** (1940) 201–203.
- [3] D. M. Cvetković, M. Doob, H. Sachs, **Spectra of Graphs: Theory and Application**, VEB Deutscher Verlag der Wissenschaften, Berlin, and Academic Press, New York, 1980.
- [4] I. Gutman, The energy of a graph, *10th Steiermärkisches Mathematisches Symposium (Stift Rein, Graz, 1978). Ber. Math.-Statist. Sect. Forschungszentrums Graz*, **103** (1978) 1–22.
- [5] K. A. Germina, S. Hameed K, T. Zaslavsky, On products and line graphs of signed graphs, their eigenvalues and energy. *Linear Algebra Appl.*, **435** (2011) 2432–2450.
- [6] J. Hage, T. Harju, The size of switching classes with skew gains. *Discrete Math.* **215** (2000) 81–92.
- [7] X. Li, Y. Shi, I. Gutman, **Graph Energy**, Springer, 2010.
- [8] J. Liu, X. Li, Hermitian-adjacency matrices and Hermitian energies of mixed graphs. *Linear Algebra Appl.*, **466** (2015) 182–207.
- [9] B. McClelland, Properties of the latent roots of a matrix: The estimation of π -electron energies. *J. Chem. Phys.*, **54** (1971) 640–643.
- [10] N. G. Nayak, Spectra and energy of signed graphs. *Int. J. Math. Comb.*, **1** (2017) 10–21.
- [11] A. Samanta, M. Rajesh Kannan, Bounds for the energy of a complex unit gain graph. *Linear Algebra Appl.*, **612** (2019) 1–29.
- [12] R. A. Horn, C. R. Johnson, **Matrix Analysis**, Cambridge University Press, Cambridge, 1990.
- [13] N. Reff, Spectral properties of complex unit gain graphs. *Linear Algebra Appl.*, **436** (2012) 3165–3176.
- [14] Shahul Hameed Koombail, K. O. Ramakrishnan, Balance theory: An extension to conjugate skew gain graphs. *Comm. Combin. Optim.*, **9** (2) (2024) 253–262.
- [15] T. Zaslavsky, Signed graphs. *Discrete Appl. Math.* **4** (1982) 47–74. Erratum *Discrete Appl. Math.* **5** (1983) 248.
- [16] T. Zaslavsky, A mathematical bibliography of signed and gain graphs and allied areas, VII Edition, *Electron. J. Combin.* **8** (1998), Dynamic Surveys, 124 pp.

Author information

Shahul Hameed K, Research Suervisor, Dr. Hermann Gundert Central Library, Kannur University, Kannur, Kerala, India.

E-mail: shabrennen@gmail.com

Ramakrishnan K O, Department of Mathematics, Government Brennen College, Thalassery, Kerala Research Scholar, Kannur University, India.

E-mail: ramkomaths@gmail.com

Biju K, Department of Mathematics, PRNSS College, Mattannur, Kerala, India Research Scholar, Kannur University, India.

E-mail: bijukaronnon@gmail.com

Received: 2025-03-11

Accepted: 2026-01-01