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INTERVAL-VALUED NEUTROSOPHIC SOFT MULTISET MATRICES AND THEIR APPLICATIONS

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Abstract In this paper, we introduce interval-valued neutrosophic soft multiset matrices to represent interval-valued neutrosophic soft multisets, and we establish some of their properties and examples. Additionally, we introduce AND, OR operators and IVNSMS-max-min decision function, and we propose an algorithm to solve interval-valued neutrosophic soft multisets. Later, we consider a real life application of interval-valued neutrosophic soft multisets and utilize our proposed algorithm to get the most favorable results. Finally, we generate a MATLAB code for our algorithm to execute and find scores of alternatives.

1 Introduction

In order to deal with uncertainty and ambiguity in the real world, L.A. Zadeh [29] introduced fuzzy sets and fuzzy logic in 1965. Fuzzy sets, in contrast to classical sets, use membership functions to represent elements by partial membership values ranging from 0 to 1. K. Atanassov[2] developed intuitionistic fuzzy sets as an extension of fuzzy sets by introducing a non membership function to represent the falsity nature of an element. Interval valued intuitionistic fuzzy sets were also initiated by K. Atanassov [3]. They provide a more flexible way to handle uncertainty by allowing the membership and non membership degrees to be expressed as intervals rather than single value. M.G. Thomson [26] introduced fuzzy matrix and Im et al. [14] developed intuitionistic fuzzy matrix as a generalization of Thomson's fuzzy matrix. Madhumangal Pal and Sultana K. Khan defined interval valued intuitionistic fuzzy matrices and their fundamental properties in [18].

In order to handle incomplete and ambiguous data, F. Smarandache [23] invented the mathematical framework known as neutrosophic sets in 1995. They generalize fuzzy sets and intuitionistic fuzzy sets by adding membership functions T, I and F that vary from 0 to 1. H. Wang et al. [27] established interval-valued neutrosophic sets as an expansion of neutrosophic sets by using intervals as membership values rather than a single value.

Molodtsov [22] introduced soft sets as a new mathematical framework to handle uncertainty and ambiguity rather than classical sets and fuzzy sets. With a set of parameters, soft sets can be used to characterize uncertain data in various kind of ways. Soft sets are widely extended by using other frameworks that result in hybrid structures such as fuzzy soft sets [20], intuitionistic fuzzy soft sets [19], interval valued intuitionistic fuzzy soft sets [17], vague soft sets [28] and soft rough sets [13]. Sujit Das et al. [9] defined the concept of interval valued intuitionistic fuzzy soft matrix (IVIFSM), and they presented an IVIFSM based group decision making algorithm.

Çağman-Enginoğlu [7] introduced soft matrices to represents soft sets and store soft sets in computer memory, and they introduced soft matrix operators, soft max-min decision function and a soft max-min decision making algorithm.

Neutrosophic soft sets [21] are a better mathematical tool to handle uncertainty and indeterminate data, which is an extension of soft sets and neutrosophic sets. I. Deli [11] introduced interval-valued neutrosophic soft sets as an improved version of neutrosophic soft sets which were created to manage uncertainty and indeterminate data with more flexibility. I. Deli and S. Broumi [10] established neutrosophic soft matrices and studied some of their properties. They also devised an NSM-decision making function in order to handle group decision making problems. S. Broumi et al. [5] developed a MATLAB toolbox for computational operations on single-valued neutrosophic soft matrices. They even developed a MATLAB toolbox for interval-valued neutrosophic matrices in [6].

A generalization of soft sets known as soft multisets was first presented by Alkhazaleh et al. [1] in 2011. Later, H.M. Balami and D. W. Mshelia [4] proposed operations of soft multisets. I. Deli and S. Broumi et al. [12] combined soft multisets and neutrosophic sets, and they introduced the new concept called neutrosophic soft multisets and their fundamental properties. A.E. Coskun et al. [8] constructed soft multiset matrices and presented a decision making algorithm using the idea of soft matrices on soft multisets. Recently some generalizations namely NeutroAlgebras and NeutroGeometry have been introduced in [24] and [25]. J. Jayasudha et al. [15] introduced interval-valued neutrosophic soft multisets (IVNSMS) as an expansion of neutrosophic soft multisets by integrating interval-valued neutrosophic sets and soft multisets. They also introduced the concept of neutrosophic soft multiset matrices and proposed an algorithm to solve group decision making in [16].

The novelty of this paper is to introduce an interval-valued neutrosophic soft multiset matrix and a decision making algorithm to present an application of interval-valued neutrosophic soft multisets. The rest of the paper is organized as follows: In section 2, we recollect the basic notions of soft multisets, interval-valued neutrosophic soft multisets. In section 3, we define an interval-valued neutrosophic soft matrix on U_i-IVNSMS-part of interval-valued neutrosophic soft multisets. Also we study the properties of Interval valued neutrosophic soft multiset matrices, AND, OR operators and IVNSMS-max-min decision function. In Section 4, we introduce an algorithm based on the AND operator and discuss an application of IVNSMS. Further a MATLAB code has been developed and employed to verify the results of the suggested algorithm. Section 5 deals with the conclusion of the paper.

2 Preliminaries

In this section, we revisit the fundamental notions which serve as the foundation for creating new notions and operations.

Definition 2.1. [27] Let U be a space of points (objects), with a generic element in U denoted by u. An interval value neutrosophic set($IVN-set$) A in U is characterized by truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. For each point $u \in U$; $T_A,\ I_A$ and $F_A \subseteq [0,1]$.
Thus, an $IVN-set$ over U can be represented by the set

$$A = \{\langle T_A(u), I_A(u), F_A(u)\rangle/u : u \in U\}.$$

Here, $(T_A(u), I_A(u), F_A(u))$ is called interval value neutrosophic number for all $u \in U$ and all interval value neutrosophic numbers over U will be denoted by $IVN(U)$.

Definition 2.2. [11] Let U be an initial universe set, $IVN(U)$ denotes the set of all interval-valued neutrosophic sets of U and E be a set of parameters that describe the elements of U. An interval-valued neutrosophic soft sets over U is a set defined by a set valued function Υ_K representing a mapping

$$\upsilon_K : E \longrightarrow IVN(U).$$

It can be written as a set of ordered pairs

$$\Upsilon_K = \{(x, \upsilon_K(x)) : x \in E\}.$$

Here, an interval-valued neutrosophic set υ_K is called approximate function of the interval-valued neutrosophic (ivn)-soft sets Υ_K. And $\upsilon_K(x)$ is called x-approximate value of $x \in E$.

Definition 2.3. [1] Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i\in I} U_i = \phi$, $\{E_{U_i} : i \in I\}$be a collection of sets of parameters, $U = \prod_{i\in I} P(U_i)$ where $P(U_i)$ denotes the powerset of U_i, $E = \prod_{i\in I} E_{U_i}$ and $A \subseteq E$.
A pair (I, A) is called a soft multiset over U given by the mapping $I : A \longrightarrow U$.

Definition 2.4. [15] Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i\in I} U_i = \phi$, $\{E_{U_i} : i \in I\}$be a collection of sets of parameters, $U = \prod_{i\in I} IVN(U_i)$ where $IVN(U_i)$ denotes the set of all Interval-valued neutrosophic sets of U_i, $E = \prod_{i\in I} E_{U_i}$ and $A \subseteq E$.
An Interval-valued neutrosophic soft multiset over U is the pair (I, A) given by the mapping $I : A \longrightarrow U$. It can be represented by,
$(I, A) = \{(a_k, \langle[infT_I(u), supT_I(u)], [infI_I(u), supI_I(u)], [infF_I(u), supF_I(u)]\rangle) :$
$a_k \in A \subseteq E, u \in U\}$
where, $a_k = (e_{U_1,j}, e_{U_2,j}, \cdots e_{U_i,j})$, $u = (u_{1,j}, u_{2,j}, \cdots u_{i,j})$ and
$\langle[infT_I(u), supT_I(u)], [infI_I(u), supI_I(u)], [infF_I(u), supF_I(u)]\rangle =$
$(\langle[infT_I(u_{1,j}), supT_I(u_{1,j})], [infI_I(u_{1,j}), supI_I(u_{1,j})], [infF_I(u_{1,j}), supF_I(u_{1,j})]\rangle,$
$\langle[infT_I(u_{2,j}), supT_I(u_{2,j})], [infI_I(u_{2,j}), supI_I(u_{2,j})], [infF_I(u_{2,j}), supF_I(u_{2,j})]\rangle,$
$\cdots \langle[infT_I(u_{i,j}), supT_I(u_{i,j})], [infI_I(u_{i,j}), supI_I(u_{i,j})], [infF_I(u_{i,j}), supF_I(u_{i,j})]\rangle).$

3 Interval-Valued Neutrosophic Soft Multiset Matrix

In this section, we define an interval-valued neutrosophic soft matrix on IVNSMS-part of interval-valued neutrosophic soft multisets. Further, we extend the same to an interval-valued neutrosophic soft multiset matrix. Additionally, we introduce the union, intersection of two IVNSMS matrices, the AND and OR operators and study their essential properties along with some examples.

Definition 3.1. Let (I_A, E) be an interval-valued neutrosophic soft multiset (IVNSMS) over U and $(e_{U_i,j}, I_A(e_{U_i,j}))$ be an U_i-interval-valued neutrosophic soft multiset part (U_i-IVNSMS-part) of IVNSMS (I_A, E) where $A_i \subseteq E_{U_i}$. Then a subset of $IVN(U_i) \times E_{U_i}$,

$$R_{IVN(U_i)} = \{(f_{IVN}(e_{U_i,j}), e_{U_i,j}) : \ e_{U_i,j} \in E_{U_i}, \ f_{IVN}(e_{U_i,j}) \in IVN(U_i)\}$$

uniquely defines a relation form of U_i-IVNSMS-part of (I_A, E).
The characteristic function of $R_{IVN(U_i)}$ is uniquely defined as

$$\Theta_{R_{IVN(U_i)}} : IVN(U_i) \times E_{U_i} \longrightarrow [0, 1] \times [0, 1] \times [0, 1]$$

where
$\Theta_{R_{IVN(U_i)}} = \langle[infT_I(u_{(i,j)}), supT_I(u_{(i,j)})], [infI_I(u_{(i,j)}), supI_I(u_{(i,j)})], [infF_I(u_{(i,j)}), supF_I(u_{(i,j)})]\rangle$
is an interval-valued neutrosophic number of $u_{(i,j)} \in U_i$.

Definition 3.2. Let U_i be the universe and E_{U_i} be the parameter set that characterize the universe U_i. The interval-valued neutrosophic soft matrix of U_i-IVNSMS-part for parameter set E_{U_i} is defined by

$$[\hat{a}^i_{lk}] = \{\Theta_{R_{IVN(U_i)}}(u_{(i,j)}, e_{U_i,j}) : u_{(i,j)} \in U_i, \ e_{U_i,j} \in E_{U_i}\}.$$

If $\hat{a}_{ij} = \Theta_{R_{IVN(U_i)}}(u_{(i,j)}, e_{U_i,j})$ then $[\hat{a}^i_{lk}] = [\hat{a}_{ij}]_{m_i\times n_i}$ where $1 \le l \le m_i, \ 1 \le k \le n_i$ is a $m_i \times n_i$ interval-valued neutrosophic soft matrix of U_i-IVNSMS-part for IVNSMS (I_A, E).
The interval-valued neutrosophic soft matrix of U_i-IVNSMS-part for the parameter set E_{U_r} is defined as

$$[\hat{0}_{lk}]_{m_i\times n_r} = [\langle[0, 0], [1, 1], [1, 1]\rangle]_{m_i\times n_r}$$

as every element $u_{(i,j)} \in U_i$ attains the characteristic function value $\Theta_{R_{IVN(U_i)}}(u_{(i,j)}) =$
$[\langle[0, 0], [1, 1], [1, 1]\rangle]$ for all the parameters $e_{U_r,j}$ from the set E_{U_r} which does not characterize the universe U_i.
Then the interval-valued neutrosophic soft multiset matrix (IVNSMS-matrix) of (I_A, E) is de-

fined as

$$[\hat{A}_{lk}]_{m\times n} = \begin{bmatrix} [\hat{a}^1_{lk}]_{m_1\times n_1} & [\hat{0}_{lk}]_{m_1\times n_2} & \cdots & [\hat{0}_{lk}]_{m_1\times n_N} \\ [\hat{0}_{lk}]_{m_2\times n_1} & [\hat{a}^2_{lk}]_{m_2\times n_2} & \cdots & [\hat{0}_{lk}]_{m_2\times n_N} \\ \vdots & \vdots & \cdots & \vdots \\ [\hat{0}_{lk}]_{m_N\times n_1} & [\hat{0}_{lk}]_{m_N\times n_2} & \cdots & [\hat{a}^N_{lk}]_{m_N\times n_N} \end{bmatrix}_{m\times n}$$

where $m = m_1 + m_2 + \cdots + m_N,\ n = n_1 + n_2 + \cdots + n_N,\ m_i =\mid U_i \mid$ and $n_i =\mid E_{U_i} \mid$. Based on the above definition, an interval-valued neutrosophic soft multiset (I_A, E) is uniquely characterized by the matrix $[\hat{A}_{lk}]_{m\times n}$. The matrix $\hat{N}_{m\times n}$ will denote the set of all $m \times n$ IVNSMS-matrices over $IVN(U)$. We shall use $[\hat{A}_{lk}]$ instead of $[\hat{A}_{lk}]_{m\times n}$ as $[\hat{A}_{lk}] \in \hat{N}_{m\times n}$ and it is obvious that $[\hat{A}_{lk}]$ is an $m \times n$ IVNSMS-matrix for $1 \leq l \leq m_i,\ 1 \leq k \leq n_i$ and $i = 1, 2, \cdots N$.

Example 3.3. Let U_1, U_2 and U_3 be the universes under consideration. Mrs.X wants to start a small fashion boutique with her savings as investment. Let (I, A) be an $IVNSMS(U)$ which describes "rental shops", "companies for raw materials"and "assistants to be appointed"respectively that Mrs.X is considering for a good rental shop, fabric company for supplying clothes and an assistant to help her. Let $U_1 = \{u_{(1,1)}, u_{(1,2)}, u_{(1,3)}\}$ be the universe for rental shops, $U_2 = \{u_{(2,1)}, u_{(2,2)}, u_{(2,3)}\}$ be the universe for fabric company and $U_3 = \{u_{(3,1)}, u_{(3,2)}\}$ be the universe for available assistants. Let $\{E_{U_1}, E_{U_2}, E_{U_3}\}$ be a collection of parameters which describes above universes, where

$E_{U_1} = \{e_{U_1,1} = annual\ rent,\ e_{U_1,2} = monthly\ rent,\ e_{U_1,3} = mid\ town\}$,
$E_{U_2} = \{e_{U_2,1} = good\ quality,\ e_{U_2,2} = ontime\ delivery\}$ and
$E_{U_3} = \{e_{U_3,1} = experienced\ person,\ e_{U_3,2} = fresher\}$.
Let $U = \prod_{i=1}^{3} IVNS(U_i),\ E = \prod_{i=1}^{3} E_{U_i}$ and $A \subseteq E$ such that
$A = \{a_1 = (e_{U_1,1}, e_{U_2,1}, e_{U_3,1}),\ a_2 = (e_{U_1,2,}, e_{U_2,2}, e_{U_3,2}),\ a_3 = (e_{U_1,3}, e_{U_2,2}, e_{U_3,1})\}$.
The corresponding Interval valued neutrosophic soft multiset is given as:

$$(I, A) = \{(a_1, (\{\frac{\langle[0.2,0.3],[0.5,0.6],[0.8,0.9]\rangle}{s_1}, \frac{\langle[0.7,0.8],[0.2,0.4],[0.1,0.3]\rangle}{s_2}, \frac{\langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle}{s_3}\}, \{\frac{\langle[0.7,0.9],[0.5,0.6],[0.4,0.5]\rangle}{c_1}, \frac{\langle[0.5,0.6],[0.2,0.3],[0.1,0.2]\rangle}{c_2}, \frac{\langle[0.8,0.9],[0.4,0.5],[0.8,0.9]\rangle}{c_3}\}, \{\frac{\langle[0.7,0.9],[0.4,0.6],[0.4,0.5]\rangle}{a_1}, \frac{\langle[0.5,0.7],[0.4,0.7],[0.2,0.3]\rangle}{a_2}\})),$$

$$(a_2, (\{\frac{\langle[0.7,0.8],[0.2,0.4],[0.1,0.3]\rangle}{s_1}, \frac{\langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle}{s_2}, \frac{\langle[0.1,0.2],[0.3,0.4],[0.5,0.6]\rangle}{s_3}\}, \{\frac{\langle[0.5,0.6],[0.2,0.3],[0.1,0.2]\rangle}{c_1}, \frac{\langle[0.8,1.0],[0.2,0.3],[0.1,0.3]\rangle}{c_2}, \frac{\langle[0.4,0.5],[0.9,1.0],[0.1,0.2]\rangle}{c_3}\}, \{\frac{\langle[0.5,0.7],[0.4,0.7],[0.2,0.3]\rangle}{a_1}, \frac{\langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle}{a_2}\})),$$

$$(a_3, (\{\frac{\langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle}{s_1}, \frac{\langle[0.1,0.2],[0.3,0.4],[0.5,0.6]\rangle}{s_2}, \frac{\langle[0.5,0.7],[0.5,0.6],[0.2,0.3]\rangle}{s_3}\}, \{\frac{\langle[0.5,0.6],[0.2,0.3],[0.1,0.2]\rangle}{c_1}, \frac{\langle[0.8,1.0],[0.2,0.3],[0.1,0.3]\rangle}{c_2}, \frac{\langle[0.4,0.5],[0.9,1.0],[0.1,0.2]\rangle}{c_3}\}, \{\frac{\langle[0.7,0.9],[0.4,0.6],[0.4,0.5]\rangle}{a_1}, \frac{\langle[0.5,0.7],[0.4,0.7],[0.2,0.3]\rangle}{a_2}\}))\}.$$

$$[\hat{A}_{lk}] = \left[\begin{array}{c|ccc} & e_{U_1,1} & e_{U_1,2} & e_{U_1,3} \\ u_{(1,1)} & \langle[0.2, 0.3], [0.5, 0.6], [0.8, 0.9]\rangle & \langle[0.7, 0.8], [0.2, 0.4], [0.1, 0.3]\rangle & \langle[0.8, 0.9], [0.2, 0.3], [0.1, 0.2]\rangle \\ u_{(1,2)} & \langle[0.7, 0.8], [0.2, 0.4], [0.1, 0.3]\rangle & \langle[0.8, 0.9], [0.4, 0.5], [0.2, 0.4]\rangle & \langle[0.1, 0.2], [0.3, 0.4], [0.5, 0.6]\rangle \\ u_{(1,3)} & \langle[0.8, 0.9], [0.2, 0.3], [0.1, 0.2]\rangle & \langle[0.1, 0.2], [0.3, 0.4], [0.5, 0.6]\rangle & \langle[0.5, 0.7], [0.5, 0.6], [0.2, 0.3]\rangle \\ u_{(2,1)} & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle \\ u_{(2,2)} & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle \\ u_{(2,3)} & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle \\ u_{(3,1)} & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle \\ u_{(3,2)} & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle \end{array} \right.$$

$$\left. \begin{array}{cc|cc} e_{U_2,1} & e_{U_2,2} & e_{U_3,1} & e_{U_3,2} \\ \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle \\ \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle \\ \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle \\ \langle[0.7, 0.9], [0.5, 0.6], [0.4, 0.5]\rangle & \langle[0.5, 0.6], [0.2, 0.3], [0.1, 0.2]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle \\ \langle[0.5, 0.6], [0.2, 0.3], [0.1, 0.2]\rangle & \langle[0.8, 1], [0.2, 0.3], [0.1, 0.3]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle \\ \langle[0.8, 0.9], [0.4, 0.5], [0.8, 0.9]\rangle & \langle[0.4, 0.5], [0.9, 1], [0.1, 0.2]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle \\ \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.7, 0.9], [0.4, 0.6], [0.4, 0.5]\rangle & \langle[0.5, 0.7], [0.4, 0.7], [0.2, 0.3]\rangle \\ \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]\rangle & \langle[0.5, 0.7], [0.4, 0.7], [0.2, 0.3]\rangle & \langle[0.8, 0.9], [0.2, 0.4], [0.1, 0.3]\rangle \end{array} \right]_{8\times 7}$$

Definition 3.4. If $m = n$, then IVNSMS-matrix $[\hat{A}_{lk}]_{m\times n}$ is a square IVNSMS-matrix. i.e., For every U_i-IVNSMS-part $[\hat{a}^i_{lk}]_{m_i\times n_i}$ of (I_A, E) has equal number of rows and columns.

Definition 3.5. The transpose of an IVNSMS-matrix $[\hat{A}_{lk}]$ of order $m \times n$ is another IVNSMS-matrix $[\hat{A}_{lk}]^T$ of order $n \times m$ by swapping its rows and columns.

Example 3.6. Consider Example 3.3, let $[\hat{A}_{lk}]_{7\times 7}$ is written as:

$$[\hat{A}_{lk}] = \left[\begin{array}{c|ccc|cc|cc} & e_{U_1,1} & e_{U_1,2} & e_{U_1,3} & e_{U_2,1} & e_{U_2,2} & e_{U_3,1} & e_{U_3,2} \\ \hline u_{(1,1)} & \langle[0.2,0.3],[0.5,0.6],[0.8,0.9]\rangle & \langle[0.7,0.8],[0.2,0.4],[0.1,0.3]\rangle & \langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(1,2)} & \langle[0.6,0.7],[0.2,0.4],[0.2,0.3]\rangle & \langle[0.5,0.7],[0.3,0.6],[0.2,0.4]\rangle & \langle[0.1,0.2],[0.3,0.5],[0.6,0.7]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(1,3)} & \langle[0.7,0.9],[0.4,0.5],[0.1,0.2]\rangle & \langle[0.2,0.3],[0.4,0.4],[0.4,0.6]\rangle & \langle[0.5,0.7],[0.5,0.6],[0.2,0.3]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \hline u_{(2,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.8,0.9],[0.4,0.6],[0.5,0.6]\rangle & \langle[0.5,0.6],[0.4,0.8],[0.2,0.4]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(2,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.4,0.6],[0.3,0.5],[0.2,0.3]\rangle & \langle[0.9,1],[0.1,0.3],[0.1,0.3]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \hline u_{(3,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.8,0.9],[0.3,0.4],[0.3,0.5]\rangle & \langle[0.4,0.6],[0.4,0.7],[0.2,0.3]\rangle \\ u_{(3,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.5,0.7],[0.4,0.7],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle \end{array}\right]_{7\times 7}$$

Then the $[\hat{A}_{lk}]^T_{7\times 7}$ is written as:

$$[\hat{A}_{lk}]^T = \left[\begin{array}{c|ccc|cc|cc} & u_{(1,1)} & u_{(1,2)} & u_{(1,3)} & u_{(2,1)} & u_{(2,2)} & u_{(3,1)} & u_{(3,2)} \\ \hline e_{U_1,1} & \langle[0.2,0.3],[0.5,0.6],[0.8,0.9]\rangle & \langle[0.6,0.7],[0.2,0.4],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.4,0.5],[0.1,0.2]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ e_{U_1,2} & \langle[0.7,0.8],[0.2,0.4],[0.1,0.3]\rangle & \langle[0.5,0.7],[0.3,0.6],[0.2,0.4]\rangle & \langle[0.2,0.3],[0.4,0.4],[0.4,0.6]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ e_{U_1,3} & \langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.1,0.2],[0.3,0.5],[0.6,0.7]\rangle & \langle[0.5,0.7],[0.5,0.6],[0.2,0.3]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \hline e_{U_2,1} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.8,0.9],[0.4,0.6],[0.5,0.6]\rangle & \langle[0.4,0.6],[0.3,0.5],[0.2,0.3]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ e_{U_2,2} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.5,0.6],[0.4,0.8],[0.2,0.4]\rangle & \langle[0.9,1],[0.1,0.3],[0.1,0.3]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \hline e_{U_3,1} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.8,0.9],[0.3,0.4],[0.3,0.5]\rangle & \langle[0.5,0.7],[0.4,0.7],[0.2,0.3]\rangle \\ e_{U_3,2} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.4,0.6],[0.4,0.7],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle \end{array}\right]_{7\times 7}$$

Definition 3.7. An IVNSMS-matrix is called a symmetric IVNSMS-matrix, if transpose of $[\hat{A}_{lk}]$ equals $[\hat{A}_{lk}]$. i.e., $[\hat{A}_{lk}]^T = [\hat{A}_{lk}]$.

Definition 3.8. Let $[\hat{A}_{lk}] \in \hat{N}_{m\times n}$. Then $[\hat{A}_{lk}]$ is called a semi zero IVNSMS-matrix $[\hat{A}_{lk\approx\phi_i}]_{m\times n}$, if at least one of the U_i-IVNSMS-part $[\hat{a}^i_{lk}]_{m_i\times n_i}$ of $[\hat{A}_{lk}]$ equals $[\hat{0}_{lk}]_{m_i\times n_i} = [\langle[0,0],[1,1],[1,1]\rangle]$ for every element $u_{(i,j)} \in U_i$.

Definition 3.9. Let $[\hat{A}_{lk}] \in \hat{N}_{m\times n}$. Then $[\hat{A}_{lk}]$ is called a semi universal IVNSMS-matrix $[\hat{A}_{lk\approx W_i}]_{m\times n}$, if at least one of the U_i-IVNSMS-part $[\hat{a}^i_{lk}]_{m_i\times n_i}$ of $[\hat{A}_{lk}]$ equals $[\hat{1}_{lk}]_{m_i\times n_i} = [\langle[1,1],[0,0],[0,0]\rangle]$ for every element $u_{(i,j)} \in U_i$.

Definition 3.10. Let $[\hat{A}_{lk}] \in \hat{N}_{m\times n}$. Then $[\hat{A}_{lk}]$ is called a zero IVNSMS-matrix $[\hat{O}_{lk}]_{m\times n}$, if every U_i-IVNSMS-part $[\hat{a}^i_{lk}]_{m_i\times n_i}$ of $[\hat{A}_{lk}]$ equals $[\hat{0}_{lk}]_{m_i\times n_i} = [\langle[0,0],[1,1],[1,1]\rangle]$ for every element $u_{(i,j)} \in U_i$.

Definition 3.11. Let $[\hat{A}_{lk}] \in \hat{N}_{m\times n}$. Then $[\hat{A}_{lk}]$ is called a universal IVNSMS-matrix $[\hat{W}_{lk}]_{m\times n}$, if every U_i-IVNSMS-part $[\hat{a}^i_{lk}]_{m_i\times n_i}$ of $[\hat{A}_{lk}]$ equals $[\hat{1}_{lk}]_{m_i\times n_i} = [\langle[1,1],[0,0],[0,0]\rangle]$ for every element $u_{(i,j)} \in U_i$.

Example 3.12. Let $U_1 = \{u_{(1,1)}, u_{(1,2)}, u_{(1,3)}\}$, $U_2 = \{u_{(2,1)}, u_{(2,2)}\}$ and $U_3 = \{u_{(3,1)}, u_{(3,2)}\}$ be the universes under consideration. Let $E_{U_1,1} = \{e_{U_1,1}, e_{U_1,2}, e_{U_1,3}\}$, $E_{U_2,1} = \{e_{U_2,1}, e_{U_2,2}\}$ and $E_{U_3,1} = \{e_{U_3,1}, e_{U_3,2}\}$ be the parameter sets that describe the universes U_1, U_2 and U_3 respectively. Let $U = \prod_{i=1}^{3} IVNS(U_i)$, $E = \prod_{i=1}^{3} E_{U_i}$. Then, (i) a semi zero IVNSMS-matrix is given by

$$[\hat{A}_{lk\approx\phi_1}]_{7\times 7} = \left[\begin{array}{c|ccc} & e_{U_1,1} & e_{U_1,2} & e_{U_1,3} \\ \hline u_{(1,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(1,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(1,3)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \hline u_{(2,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(2,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \hline u_{(3,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{array}\right.$$

$$\left.\begin{bmatrix} e_{U_2,1} & e_{U_2,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.7,0.9],[0.5,0.6],[0.4,0.5]\rangle & \langle[0.5,0.6],[0.2,0.3],[0.1,0.2]\rangle \\ \langle[0.5,0.6],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.3]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix} \begin{bmatrix} e_{U_3,1} & e_{U_3,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.7,0.9],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.7],[0.2,0.3]\rangle \\ \langle[0.5,0.7],[0.4,0.7],[0.2,0.3]\rangle & \langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle \end{bmatrix}\right]_{7\times 7}$$

(ii) a semi universal IVNSMS-matrix is,

$$[\hat{A}_{lk\approx W_1}]_{7\times 7} = \left[\begin{bmatrix} & e_{U_1,1} & e_{U_1,2} & e_{U_1,3} \\ u_{(1,1)} & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle \\ u_{(1,2)} & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle \\ u_{(1,3)} & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle \\ u_{(2,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(2,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix}\right.$$

$$\left.\begin{bmatrix} e_{U_2,1} & e_{U_2,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.7,0.9],[0.5,0.6],[0.4,0.5]\rangle & \langle[0.5,0.6],[0.2,0.3],[0.1,0.2]\rangle \\ \langle[0.5,0.6],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.3]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix} \begin{bmatrix} e_{U_3,1} & e_{U_3,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.7,0.9],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.7],[0.2,0.3]\rangle \\ \langle[0.5,0.7],[0.4,0.7],[0.2,0.3]\rangle & \langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle \end{bmatrix}\right]_{7\times 7}$$

(iii) a zero IVNSMS-matrix is

$$[\hat{O}_{lk}]_{7\times 7} = \left[\begin{bmatrix} & e_{U_1,1} & e_{U_1,2} & e_{U_1,3} \\ u_{(1,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(1,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(1,3)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(2,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(2,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix}\right.$$

$$\left.\begin{bmatrix} e_{U_2,1} & e_{U_2,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix} \begin{bmatrix} e_{U_3,1} & e_{U_3,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix}\right]_{7\times 7}$$

(iv) an universal IVNSMS-matrix is given as:

$$[\hat{W}_{lk}]_{7\times 7} = \left[\begin{bmatrix} & e_{U_1,1} & e_{U_1,2} & e_{U_1,3} \\ u_{(1,1)} & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle \\ u_{(1,2)} & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle \\ u_{(1,3)} & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle \\ u_{(2,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(2,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix}\right.$$

$$\left.\begin{bmatrix} e_{U_2,1} & e_{U_2,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle \\ \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix} \begin{bmatrix} e_{U_3,1} & e_{U_3,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle \\ \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle & \langle[1.0,1.0],[0.0,0.0],[0.0,0.0]\rangle \end{bmatrix}\right]_{7\times 7}$$

Definition 3.13. Let $[\hat{A}_{lk}], [\hat{B}_{lk}] \in \hat{N}_{m\times n}$. Let $[\hat{a}^i_{lk}]_{m_i\times n_i}$ and $[\hat{b}^i_{lk}]_{m_i\times n_i}$ be U_i-IVNSMS-parts of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ respectively. Then $[\hat{A}_{lk}]$ is an IVNSM-submatrix of $[\hat{B}_{lk}]$, denoted by $[\hat{A}_{lk}] \subseteq [\hat{B}_{lk}]$, if $[\hat{a}^i_{lk}]_{m_i\times n_i} \subseteq [\hat{b}^i_{lk}]_{m_i\times n_i}$ for all i. i.e. $infT^a_{lk}(u_{(i,j)}) \leq infT^b_{lk}(u_{(i,j)}),\ supT^a_{lk}(u_{(i,j)}) \leq supT^b_{lk}(u_{(i,j)}),\ infI^a_{lk}(u_{(i,j)}) \geq infI^b_{lk}(u_{(i,j)}), supI^a_{lk}(u_{(i,j)}) \geq supI^b_{lk}(u_{(i,j)})$ and $infF^a_{lk}(u_{(i,j)}) \geq infF^b_{lk}(u_{(i,j)}), supF^a_{lk}(u_{(i,j)}) \geq supF^b_{lk}(u_{(i,j)})$.

Definition 3.14. Let $[\hat{A}_{lk}], [\hat{B}_{lk}] \in \hat{N}_{m\times n}$. Let $[\hat{a}^i_{lk}]_{m_i\times n_i}$ and $[\hat{b}^i_{lk}]_{m_i\times n_i}$ be U_i-IVNSMS-parts of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ respectively. Then $[\hat{A}_{lk}]$ is an IVNSM-equal matrix of $[\hat{B}_{lk}]$, denoted by $[\hat{A}_{lk}] = [\hat{B}_{lk}]$, if $[\hat{a}^i_{lk}]_{m_i\times n_i} = [\hat{b}^i_{lk}]_{m_i\times n_i}$ for all i. i.e. $infT^a_{lk}(u_{(i,j)}) = infT^b_{lk}(u_{(i,j)}),\ supT^a_{lk}(u_{(i,j)}) =$

$supT^b_{lk}(u_{(i,j)})$, $infI^a_{lk}(u_{(i,j)}) = infI^b_{lk}(u_{(i,j)})$, $supI^a_{lk}(u_{(i,j)}) = supI^b_{lk}(u_{(i,j)})$ and $infF^a_{lk}(u_{(i,j)}) = infF^b_{lk}(u_{(i,j)})$, $supF^a_{lk}(u_{(i,j)}) = supF^b_{lk}(u_{(i,j)})$.

Definition 3.15. Let $[\hat{A}_{lk}], [\hat{B}_{lk}] \in \hat{N}_{m\times n}$. Let $[\hat{a}^i_{lk}]_{m_i\times n_i}$ and $[\hat{b}^i_{lk}]_{m_i\times n_i}$ be U_i-IVNSMS-parts of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ respectively. Then union of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ is denoted by $[\hat{A}_{lk}]\cup[\hat{B}_{lk}]$ and is defined by $[\hat{a}^i_{lk}]_{m_i\times n_i}\cup[\hat{b}^i_{lk}]_{m_i\times n_i} = [\hat{c}^i_{lk}]_{m_i\times n_i}$ $\forall\, i$ where $[\hat{c}^i_{lk}]_{m_i\times n_i} = [\langle[infT^c_{lk}(u_{(i,j)}), supT^c_{lk}(u_{(i,j)})], [infI^c_{lk}(u_{(i,j)}), supI^c_{lk}(u_{(i,j)})], [infF^c_{lk}(u_{(i,j)}), supF^c_{lk}(u_{(i,j)})]\rangle]$,

$$infT^c_{lk}(u_{(i,j)}) = max\{infT^a_{lk}(u_{(i,j)}), infT^b_{lk}(u_{(i,j)})\},$$
$$supT^c_{lk}(u_{(i,j)}) = max\{supT^a_{lk}(u_{(i,j)}), supT^b_{lk}(u_{(i,j)})\},$$
$$infI^c_{lk}(u_{(i,j)}) = min\{infI^a_{lk}(u_{(i,j)}), infI^b_{lk}(u_{(i,j)})\},$$
$$supI^c_{lk}(u_{(i,j)}) = min\{supI^a_{lk}(u_{(i,j)}), supI^b_{lk}(u_{(i,j)})\},$$
$$infF^c_{lk}(u_{(i,j)}) = min\{infF^a_{lk}(u_{(i,j)}), infF^b_{lk}(u_{(i,j)})\}\ and$$
$$supF^c_{lk}(u_{(i,j)}) = min\{supF^a_{lk}(u_{(i,j)}), supF^b_{lk}(u_{(i,j)})\}.$$

Definition 3.16. Let $[\hat{A}_{lk}], [\hat{B}_{lk}] \in \hat{N}_{m\times n}$. Let $[\hat{a}^i_{lk}]_{m_i\times n_i}$ and $[\hat{b}^i_{lk}]_{m_i\times n_i}$ be U_i-IVNSMS-parts of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ respectively. Then intersection of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ is denoted by $[\hat{A}_{lk}]\cap[\hat{B}_{lk}]$ and is defined by $[\hat{a}^i_{lk}]_{m_i\times n_i}\cap[\hat{b}^i_{lk}]_{m_i\times n_i} = [\hat{c}^i_{lk}]_{m_i\times n_i}$ $\forall\, i$, where $[\hat{c}^i_{lk}]_{m_i\times n_i} = [\langle[infT^c_{lk}(u_{(i,j)}), supT^c_{lk}(u_{(i,j)})], [infI^c_{lk}(u_{(i,j)}), supI^c_{lk}(u_{(i,j)})], [infF^c_{lk}(u_{(i,j)}), supF^c_{lk}(u_{(i,j)})]\rangle]$,

$$infT^c_{lk}(u_{(i,j)}) = min\{infT^a_{lk}(u_{(i,j)}), infT^b_{lk}(u_{(i,j)})\},$$
$$supT^c_{lk}(u_{(i,j)}) = min\{supT^a_{lk}(u_{(i,j)}), supT^b_{lk}(u_{(i,j)})\},$$
$$infI^c_{lk}(u_{(i,j)}) = max\{infI^a_{lk}(u_{(i,j)}), infI^b_{lk}(u_{(i,j)})\},$$
$$supI^c_{lk}(u_{(i,j)}) = max\{supI^a_{lk}(u_{(i,j)}), supI^b_{lk}(u_{(i,j)})\},$$
$$infF^c_{lk}(u_{(i,j)}) = max\{infF^a_{lk}(u_{(i,j)}), infF^b_{lk}(u_{(i,j)})\}\ and$$
$$supF^c_{lk}(u_{(i,j)}) = max\{supF^a_{lk}(u_{(i,j)}), supF^b_{lk}(u_{(i,j)})\}.$$

Definition 3.17. Let $[\hat{A}_{lk}] \in \hat{N}_{m\times n}$. Let $[\hat{a}^i_{lk}]_{m_i\times n_i}$ be U_i-IVNSMS-part of $[\hat{A}_{lk}]$. Then complement of $[\hat{A}_{lk}]$ is denoted by $[\hat{A}_{lk}]^c$ and is defined by $[\hat{a}^i_{lk}]^c_{m_i\times n_i} = [\langle[infF^a_{lk}(u_{(i,j)}), supF^a_{lk}(u_{(i,j)})], [1 - supI^a_{lk}(u_{(i,j)}), 1 - infI^a_{lk}(u_{(i,j)})], [infT^a_{lk}(u_{(i,j)}), supT^a_{lk}(u_{(i,j)})]\rangle]$ for every element $u_{(i,j)} \in U_i$.

Example 3.18. Consider Example 3.12, let $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ be two IVNSMS-matrices.

$[\hat{A}_{lk}] =$ (7×7)

	$e_{U_1,1}$	$e_{U_1,2}$	$e_{U_1,3}$	$e_{U_2,1}$	$e_{U_2,2}$	$e_{U_3,1}$	$e_{U_3,2}$
$u_{(1,1)}$	⟨[0.2, 0.3], [0.5, 0.6], [0.8, 0.9]⟩	⟨[0.7, 0.8], [0.2, 0.4], [0.1, 0.3]⟩	⟨[0.8, 0.9], [0.2, 0.3], [0.1, 0.2]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩
$u_{(1,2)}$	⟨[0.7, 0.8], [0.2, 0.4], [0.1, 0.3]⟩	⟨[0.8, 0.9], [0.4, 0.5], [0.2, 0.4]⟩	⟨[0.1, 0.2], [0.3, 0.4], [0.5, 0.6]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩
$u_{(1,3)}$	⟨[0.8, 0.9], [0.2, 0.3], [0.1, 0.2]⟩	⟨[0.1, 0.2], [0.3, 0.4], [0.5, 0.6]⟩	⟨[0.5, 0.7], [0.5, 0.6], [0.2, 0.3]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩
$u_{(2,1)}$	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.7, 0.9], [0.5, 0.6], [0.4, 0.5]⟩	⟨[0.5, 0.6], [0.2, 0.3], [0.1, 0.2]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩
$u_{(2,2)}$	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.5, 0.6], [0.2, 0.3], [0.1, 0.2]⟩	⟨[0.8, 1], [0.2, 0.3], [0.1, 0.3]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩
$u_{(3,1)}$	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.7, 0.9], [0.4, 0.6], [0.4, 0.5]⟩	⟨[0.5, 0.7], [0.4, 0.7], [0.2, 0.3]⟩
$u_{(3,2)}$	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.5, 0.7], [0.4, 0.7], [0.2, 0.3]⟩	⟨[0.8, 0.9], [0.2, 0.4], [0.1, 0.3]⟩

$[\hat{B}_{lk}] =$

	$e_{U_1,1}$	$e_{U_1,2}$	$e_{U_1,3}$
$u_{(1,1)}$	⟨[0.5, 0.6], [0.2, 0.3], [0.7, 0.8]⟩	⟨[0.8, 0.9], [0.2, 0.4], [0.1, 0.2]⟩	⟨[0.5, 0.7], [0.4, 0.5], [0.2, 0.3]⟩
$u_{(1,2)}$	⟨[0.8, 0.9], [0.1, 0.2], [0.4, 0.5]⟩	⟨[0.7, 0.9], [0.3, 0.5], [0.6, 0.7]⟩	⟨[0.2, 0.4], [0.5, 0.7], [0.2, 0.3]⟩
$u_{(1,3)}$	⟨[0.6, 0.7], [0.2, 0.4], [0.8, 0.9]⟩	⟨[0.4, 0.6], [0.2, 0.5], [0.8, 0.9]⟩	⟨[0.7, 0.8], [0.2, 0.4], [0.1, 0.2]⟩
$u_{(2,1)}$	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩
$u_{(2,2)}$	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩
$u_{(3,1)}$	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩
$u_{(3,2)}$	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩	⟨[0.0, 0.0], [1.0, 1.0], [1.0, 1.0]⟩

$$\left.\begin{bmatrix} e_{U_2,1} & e_{U_2,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.8,0.9],[0.5,0.6],[0.2,0.3]\rangle & \langle[0.5,0.6],[0.4,0.5],[0.3,0.4]\rangle \\ \langle[0.6,0.7],[0.5,0.6],[0.4,0.5]\rangle & \langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix} \begin{bmatrix} e_{U_3,1} & e_{U_3,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle & \langle[0.7,0.8],[0.5,0.6],[0.4,0.7]\rangle \\ \langle[0.5,0.6],[0.4,0.5],[0.7,0.8]\rangle & \langle[0.5,0.6],[0.2,0.4],[0.5,0.6]\rangle \end{bmatrix}\right]_{7\times 7}$$

$$[\hat{A}_{lk}] \cup [\hat{B}_{lk}] = \left[\begin{bmatrix} & e_{U_1,1} & e_{U_1,2} & e_{U_1,3} \\ u_{(1,1)} & \langle[0.5,0.6],[0.2,0.3],[0.7,0.8]\rangle & \langle[0.8,0.9],[0.2,0.4],[0.1,0.2]\rangle & \langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle \\ u_{(1,2)} & \langle[0.8,0.9],[0.1,0.2],[0.1,0.3]\rangle & \langle[0.8,0.9],[0.3,0.5],[0.2,0.4]\rangle & \langle[0.2,0.4],[0.3,0.4],[0.2,0.3]\rangle \\ u_{(1,3)} & \langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.4,0.6],[0.2,0.4],[0.5,0.6]\rangle & \langle[0.7,0.8],[0.2,0.4],[0.1,0.2]\rangle \\ u_{(2,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(2,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix}\right.$$

$$\left.\begin{bmatrix} e_{U_2,1} & e_{U_2,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.8,0.9],[0.5,0.6],[0.2,0.3]\rangle & \langle[0.5,0.6],[0.2,0.3],[0.1,0.2]\rangle \\ \langle[0.6,0.7],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.3]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix} \begin{bmatrix} e_{U_3,1} & e_{U_3,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle & \langle[0.7,0.8],[0.4,0.6],[0.2,0.3]\rangle \\ \langle[0.5,0.7],[0.4,0.5],[0.2,0.3]\rangle & \langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle \end{bmatrix}\right]_{7\times 7}$$

$$[\hat{A}_{lk}] \cap [\hat{B}_{lk}] = \left[\begin{bmatrix} & e_{U_1,1} & e_{U_1,2} & e_{U_1,3} \\ u_{(1,1)} & \langle[0.2,0.3],[0.5,0.6],[0.8,0.9]\rangle & \langle[0.7,0.8],[0.2,0.4],[0.1,0.3]\rangle & \langle[0.5,0.7],[0.4,0.5],[0.2,0.3]\rangle \\ u_{(1,2)} & \langle[0.7,0.8],[0.2,0.4],[0.4,0.5]\rangle & \langle[0.7,0.9],[0.4,0.5],[0.6,0.7]\rangle & \langle[0.1,0.2],[0.5,0.7],[0.5,0.6]\rangle \\ u_{(1,3)} & \langle[0.6,0.7],[0.2,0.4],[0.8,0.9]\rangle & \langle[0.1,0.2],[0.3,0.5],[0.8,0.9]\rangle & \langle[0.5,0.7],[0.5,0.6],[0.2,0.3]\rangle \\ u_{(2,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(2,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix}\right.$$

$$\left.\begin{bmatrix} e_{U_2,1} & e_{U_2,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.7,0.9],[0.5,0.6],[0.4,0.5]\rangle & \langle[0.5,0.6],[0.4,0.5],[0.3,0.4]\rangle \\ \langle[0.5,0.6],[0.5,0.6],[0.4,0.5]\rangle & \langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix} \begin{bmatrix} e_{U_3,1} & e_{U_3,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.7,0.9],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.5,0.7],[0.4,0.7]\rangle \\ \langle[0.5,0.6],[0.4,0.7],[0.7,0.8]\rangle & \langle[0.5,0.6],[0.2,0.4],[0.5,0.6]\rangle \end{bmatrix}\right]_{7\times 7}$$

$$[\hat{A}_{lk}]^c = \left[\begin{bmatrix} & e_{U_1,1} & e_{U_1,2} & e_{U_1,3} \\ u_{(1,1)} & \langle[0.8,0.9],[0.4,0.5],[0.2,0.3]\rangle & \langle[0.1,0.3],[0.6,0.8],[0.7,0.8]\rangle & \langle[0.1,0.2],[0.7,0.8],[0.8,0.9]\rangle \\ u_{(1,2)} & \langle[0.1,0.3],[0.6,0.8],[0.7,0.8]\rangle & \langle[0.2,0.4],[0.5,0.6],[0.8,0.9]\rangle & \langle[0.5,0.6],[0.6,0.7],[0.1,0.2]\rangle \\ u_{(1,3)} & \langle[0.1,0.2],[0.7,0.8],[0.8,0.9]\rangle & \langle[0.5,0.6],[0.6,0.7],[0.1,0.2]\rangle & \langle[0.2,0.3],[0.4,0.5],[0.5,0.7]\rangle \\ u_{(2,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(2,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,1)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ u_{(3,2)} & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix}\right.$$

$$\left.\begin{bmatrix} e_{U_2,1} & e_{U_2,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.4,0.5],[0.4,0.5],[0.7,0.9]\rangle & \langle[0.1,0.2],[0.7,0.8],[0.5,0.6]\rangle \\ \langle[0.1,0.2],[0.7,0.8],[0.5,0.6]\rangle & \langle[0.1,0.3],[0.7,0.8],[0.8,1]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \end{bmatrix} \begin{bmatrix} e_{U_3,1} & e_{U_3,2} \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle & \langle[0.0,0.0],[1.0,1.0],[1.0,1.0]\rangle \\ \langle[0.4,0.5],[0.4,0.6],[0.7,0.9]\rangle & \langle[0.2,0.3],[0.3,0.6],[0.5,0.7]\rangle \\ \langle[0.2,0.3],[0.3,0.6],[0.5,0.7]\rangle & \langle[0.1,0.3],[0.6,0.8],[0.8,0.9]\rangle \end{bmatrix}\right]_{7\times 7}$$

Proposition 3.19. *Let* $[\hat{A}_{lk}] \in \hat{N}_{m\times n}$. *Then*
(1) $[\hat{O}_{lk}]^c = [\hat{W}_{lk}]$.
(2) $([\hat{A}_{lk}]^c)^c = [\hat{A}_{lk}]$.
(3) $[\hat{A}_{lk\approx\phi_i}]^c = [\hat{A}_{lk\approx U_i}]$.
(4) $[\hat{A}_{lk\approx U_i}]^c = [\hat{A}_{lk\approx\phi_i}]$.

Proof. The proof is obvious from the Definition 3.17.

Proposition 3.20. *Let* $[\hat{A}_{lk}],\ [\hat{B}_{lk}] \in \hat{N}_{m\times n}$. *Then*
(1) $[\hat{A}_{lk}] \subseteq [\hat{W}_{lk}]$.
(2) $[\hat{O}_{lk}] \subseteq [\hat{A}_{lk}]$.
(3) $[\hat{A}_{lk}] \subseteq [\hat{B}_{lk}]$ *and* $[\hat{B}_{lk}] \subseteq [\hat{A}_{lk}]$ *if and only if* $[\hat{A}_{lk}] = [\hat{B}_{lk}]$.

Proof. The proof is clear from Definition 3.13.

Proposition 3.21. *Let* $[\hat{A}_{lk}],\ [\hat{B}_{lk}],\ [\hat{C}_{lk}] \in \hat{N}_{m\times n}$. *Then*
(1) $[\hat{A}_{lk}] \subseteq [\hat{B}_{lk}]$ *and* $[\hat{B}_{lk}] \subseteq [\hat{C}_{lk}]$ *then* $[\hat{A}_{lk}] \subseteq [\hat{C}_{lk}]$.
(2) $[\hat{A}_{lk}] = [\hat{B}_{lk}]$ *and* $[\hat{B}_{lk}] = [\hat{C}_{lk}]$ *then* $[\hat{A}_{lk}] = [\hat{C}_{lk}]$.

Proof. The proof is an immediate consequence of Definitions 3.13 and 3.14.

Proposition 3.22. *Let* $[\hat{A}_{lk}],\ [\hat{B}_{lk}],\ [\hat{C}_{lk}] \in \hat{N}_{m\times n}$. *Then*
(1) $[\hat{A}_{lk}] \cup [\hat{A}_{lk}] = [\hat{A}_{lk}]$.
(2) $[\hat{A}_{lk}] \cup [\hat{O}_{lk}] = [\hat{A}_{lk}]$.
(3) $[\hat{A}_{lk}] \cup [\hat{W}_{lk}] = [\hat{W}_{lk}]$.
(4) $[\hat{A}_{lk}] \cup [\hat{B}_{lk}] = [\hat{B}_{lk}] \cup [\hat{A}_{lk}]$.
(5) $([\hat{A}_{lk}] \cup [\hat{B}_{lk}]) \cup [\hat{C}_{lk}] = [\hat{A}_{lk}] \cup ([\hat{B}_{lk}] \cup [\hat{C}_{lk}])$.
(6) If $[\hat{A}_{lk\approx\phi_i}] \subseteq [\hat{A}_{lk}]$ *then* $[\hat{A}_{lk\approx\phi_i}] \cup [\hat{A}_{lk}] = [\hat{A}_{lk}]$.
(7) If $[\hat{A}_{lk}] \subseteq [\hat{A}_{lk\approx U_i}]$ *then* $[\hat{A}_{lk}] \cup [\hat{A}_{lk\approx U_i}] = [\hat{A}_{lk\approx U_i}]$.

Proof. The proof follows from Definition 3.15.

Proposition 3.23. *Let* $[\hat{A}_{lk}],\ [\hat{B}_{lk}],\ [\hat{C}_{lk}] \in \hat{N}_{m\times n}$. *Then*
(1) $[\hat{A}_{lk}] \cap [\hat{A}_{lk}] = [\hat{A}_{lk}]$.
(2) $[\hat{A}_{lk}] \cap [\hat{O}_{lk}] = [\hat{O}_{lk}]$.
(3) $[\hat{A}_{lk}] \cap [\hat{W}_{lk}] = [\hat{A}_{lk}]$.
(4) $[\hat{A}_{lk}] \cap [\hat{B}_{lk}] = [\hat{B}_{lk}] \cap [\hat{A}_{lk}]$.
(5) $([\hat{A}_{lk}] \cap [\hat{B}_{lk}]) \cap [\hat{C}_{lk}] = [\hat{A}_{lk}] \cap ([\hat{B}_{lk}] \cap [\hat{C}_{lk}])$.
(6) If $[\hat{A}_{lk\approx\phi_i}] \subseteq [\hat{A}_{lk}]$ *then* $[\hat{A}_{lk\approx\phi_i}] \cap [\hat{A}_{lk}] = [\hat{A}_{lk\approx\phi_i}]$.
(7) If $[\hat{A}_{lk}] \subseteq [\hat{A}_{lk\approx U_i}]$ *then* $[\hat{A}_{lk}] \cap [\hat{A}_{lk\approx U_i}] = [\hat{A}_{lk}]$.

Proof. The proof is an immediate consequence of Definition 3.16.

Proposition 3.24. *Let* $[\hat{A}_{lk}],\ [\hat{B}_{lk}],\ [\hat{C}_{lk}] \in \hat{N}_{m\times n}$. *Then*
(1) $[\hat{A}_{lk}] \cap ([\hat{B}_{lk}] \cup [\hat{C}_{lk}]) = ([\hat{A}_{lk}] \cap [\hat{B}_{lk}]) \cup ([\hat{A}_{lk}] \cap [\hat{C}_{lk}])$.
(2) $[\hat{A}_{lk}] \cup ([\hat{B}_{lk}] \cap [\hat{C}_{lk}]) = ([\hat{A}_{lk}] \cup [\hat{B}_{lk}]) \cap ([\hat{A}_{lk}] \cup [\hat{C}_{lk}])$.

Proof. (1). Let $[\hat{A}_{lk}],\ [\hat{B}_{lk}]$ and $[\hat{C}_{lk}]$ be three matrices in $\hat{N}_{m\times n}$. Let $[\hat{a}^i_{lk}]_{m_i\times n_i},\ [\hat{b}^i_{lk}]_{m_i\times n_i}$ and $[\hat{c}^i_{lk}]_{m_i\times n_i}$ be U_i-IVNSMS-parts of $[\hat{A}_{lk}],\ [\hat{B}_{lk}]$ and $[\hat{C}_{lk}]$ respectively.

$$\begin{aligned}
[\hat{B}_{lk}] \cup [\hat{C}_{lk}] =& [\hat{b}^i_{lk}]_{m_i\times n_i} \cup [\hat{c}^i_{lk}]_{m_i\times n_i} \text{ for all i.}\\
=& [\langle [infT^b_{lk}(u_{(i,j)}), supT^b_{lk}(u_{(i,j)})], [infI^b_{lk}(u_{(i,j)}), supI^b_{lk}(u_{(i,j)})],\\
& [infF^b_{lk}(u_{(i,j)}), supF^b_{lk}(u_{(i,j)})]\rangle] \cup [\langle [infT^c_{lk}(u_{(i,j)}), supT^c_{lk}(u_{(i,j)})],\\
& [infI^c_{lk}(u_{(i,j)}), supI^c_{lk}(u_{(i,j)})], [infF^c_{lk}(u_{(i,j)}), supF^c_{lk}(u_{(i,j)})]\rangle].\\
=& [\langle [max\{infT^b_{lk}(u_{(i,j)}), infT^c_{lk}(u_{(i,j)})\}, max\{supT^b_{lk}(u_{(i,j)}), supT^c_{lk}(u_{(i,j)})\}],\\
& [min\{infI^b_{lk}(u_{(i,j)}), infI^c_{lk}(u_{(i,j)})\}, min\{supI^b_{lk}(u_{(i,j)}), supI^c_{lk}(u_{(i,j)})\}],\\
& [min\{infF^b_{lk}(u_{(i,j)}), infF^c_{lk}(u_{(i,j)})\}, min\{supF^b_{lk}(u_{(i,j)}), supF^c_{lk}(u_{(i,j)})\}]\rangle].
\end{aligned}$$

$$\begin{aligned}
[\hat{A}_{lk}] \cap ([\hat{B}_{lk}] \cup [\hat{C}_{lk}]) =& [\hat{a}^i_{lk}]_{m_i\times n_i} \cap ([\hat{b}^i_{lk}]_{m_i\times n_i} \cup [\hat{c}^i_{lk}]_{m_i\times n_i}) \text{ for all i.}\\
=& [\langle [infT^a_{lk}(u_{(i,j)}), supT^a_{lk}(u_{(i,j)})], [infI^a_{lk}(u_{(i,j)}), supI^a_{lk}(u_{(i,j)})],\\
& [infF^a_{lk}(u_{(i,j)}), supF^a_{lk}(u_{(i,j)})]\rangle] \cap ([\langle [max\{infT^b_{lk}(u_{(i,j)}),\\
& infT^c_{lk}(u_{(i,j)})\}, max\{supT^b_{lk}(u_{(i,j)}), supT^c_{lk}(u_{(i,j)})\}],\\
& [min\{infI^b_{lk}(u_{(i,j)}), infI^c_{lk}(u_{(i,j)})\}, min\{supI^b_{lk}(u_{(i,j)}),\\
& supI^c_{lk}(u_{(i,j)})\}], [min\{infF^b_{lk}(u_{(i,j)}), infF^c_{lk}(u_{(i,j)})\},\\
& min\{supF^b_{lk}(u_{(i,j)}), supF^c_{lk}(u_{(i,j)})\}]\rangle]).
\end{aligned}$$

$$=[\langle[min\{infT^a_{lk}(u_{(i,j)}), max\{infT^b_{lk}(u_{(i,j)}), infT^c_{lk}(u_{(i,j)})\}\},$$
$$min\{supT^a_{lk}(u_{(i,j)}), max\{supT^b_{lk}(u_{(i,j)}), supT^c_{lk}(u_{(i,j)})\}\}],$$
$$[max\{infI^a_{lk}(u_{(i,j)}), min\{infI^b_{lk}(u_{(i,j)}), infI^c_{lk}(u_{(i,j)})\}\},$$
$$max\{supI^a_{lk}(u_{(i,j)}), min\{supI^b_{lk}(u_{(i,j)}), supI^c_{lk}(u_{(i,j)})\}\}],$$
$$[max\{infF^a_{lk}(u_{(i,j)}), min\{infF^b_{lk}(u_{(i,j)}), infF^c_{lk}(u_{(i,j)})\}\},$$
$$max\{supF^a_{lk}(u_{(i,j)}), min\{supF^b_{lk}(u_{(i,j)}), supF^c_{lk}(u_{(i,j)})\}\}]\rangle].$$
$$=[\langle[max\{min\{infT^a_{lk}(u_{(i,j)}), infT^b_{lk}(u_{(i,j)})\}, min\{infT^a_{lk}(u_{(i,j)}),$$
$$infT^c_{lk}(u_{(i,j)})\}\}, max\{min\{supT^a_{lk}(u_{(i,j)}), supT^b_{lk}(u_{(i,j)})\},$$
$$min\{supT^a_{lk}(u_{(i,j)}), supT^c_{lk}(u_{(i,j)})\}\}], [min\{max\{infI^a_{lk}(u_{(i,j)}),$$
$$infI^b_{lk}(u_{(i,j)})\}, max\{infI^a_{lk}(u_{(i,j)}), infI^c_{lk}(u_{(i,j)})\}\},$$
$$min\{max\{supI^a_{lk}(u_{(i,j)}), supI^b_{lk}(u_{(i,j)})\}, max\{supI^a_{lk}(u_{(i,j)}),$$
$$supI^c_{lk}(u_{(i,j)})\}\}], [min\{max\{infF^a_{lk}(u_{(i,j)}), infF^b_{lk}(u_{(i,j)})\},$$
$$max\{infF^a_{lk}(u_{(i,j)}), infF^c_{lk}(u_{(i,j)})\}\}, min\{max\{supF^a_{lk}(u_{(i,j)}),$$
$$supF^b_{lk}(u_{(i,j)})\}, max\{supF^a_{lk}(u_{(i,j)}), supF^c_{lk}(u_{(i,j)})\}\}]\rangle].$$
$$=[\langle[min\{infT^a_{lk}(u_{(i,j)}), infT^b_{lk}(u_{(i,j)})\}, min\{supT^a_{lk}(u_{(i,j)}), supT^b_{lk}(u_{(i,j)})\}],$$
$$[max\{infI^a_{lk}(u_{(i,j)}), infI^b_{lk}(u_{(i,j)})\}, max\{supI^a_{lk}(u_{(i,j)}), supI^b_{lk}(u_{(i,j)})\}],$$
$$[max\{infF^a_{lk}(u_{(i,j)}), infF^b_{lk}(u_{(i,j)})\}, max\{supF^a_{lk}(u_{(i,j)}), supF^b_{lk}(u_{(i,j)})\}]\rangle]\cup$$
$$[\langle[min\{infT^a_{lk}(u_{(i,j)}), infT^c_{lk}(u_{(i,j)})\}, min\{supT^a_{lk}(u_{(i,j)}), supT^c_{lk}(u_{(i,j)})\}],$$
$$[max\{infI^a_{lk}(u_{(i,j)}), infI^c_{lk}(u_{(i,j)})\}, max\{infI^a_{lk}(u_{(i,j)}), infI^c_{lk}(u_{(i,j)})\}],$$
$$[max\{supF^a_{lk}(u_{(i,j)}), supF^b_{lk}(u_{(i,j)})\}, max\{supF^a_{lk}(u_{(i,j)}), supF^c_{lk}(u_{(i,j)})\}]\rangle].$$
$$=([\hat{a}^i_{lk}]_{m_i\times n_i}\cap[\hat{b}^i_{lk}]_{m_i\times n_i})\cup([\hat{a}^i_{lk}]_{m_i\times n_i}\cap[\hat{c}^i_{lk}]_{m_i\times n_i}) \text{ for all i.}$$
$$=([\hat{A}_{lk}]\cap[\hat{B}_{lk}])\cup([\hat{A}_{lk}]\cap[\hat{C}_{lk}]).$$

This implies $[\hat{A}_{lk}]\cap([\hat{B}_{lk}]\cup[\hat{C}_{lk}])=([\hat{A}_{lk}]\cap[\hat{B}_{lk}])\cup([\hat{A}_{lk}]\cap[\hat{C}_{lk}])$.
Likewise, we can demonstrate (2).

Proposition 3.25. *Let $[\hat{A}_{lk}],\ [\hat{B}_{lk}]\in\hat{N}_{m\times n}$. Then they satisfy the De Morgan's laws.*
(1) $([\hat{A}_{lk}]\cup[\hat{B}_{lk}])^c=[\hat{A}_{lk}]^c\cap[\hat{B}_{lk}]^c$.
(2) $([\hat{A}_{lk}]\cap[\hat{B}_{lk}])^c=[\hat{A}_{lk}]^c\cup[\hat{B}_{lk}]^c$.

Proof. (1). Let $[\hat{A}_{lk}],[\hat{B}_{lk}]\in\hat{N}_{m\times n}$. Let $[\hat{a}^i_{lk}]_{m_i\times n_i}$ and $[\hat{b}^i_{lk}]_{m_i\times n_i}$ be U_i-IVNSMS-parts of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ respectively.

$$[\hat{A}_{lk}]\cup[\hat{B}_{lk}]=[\hat{a}^i_{lk}]_{m_i\times n_i}\cup[\hat{b}^i_{lk}]_{m_i\times n_i} \text{ for all i.}$$
$$=[\langle[infT^a_{lk}(u_{(i,j)}), supT^a_{lk}(u_{(i,j)})], [infI^a_{lk}(u_{(i,j)}), supI^a_{lk}(u_{(i,j)})],$$
$$[infF^a_{lk}(u_{(i,j)}), supF^a_{lk}(u_{(i,j)})]\rangle]\cup[\langle[infT^b_{lk}(u_{(i,j)}), supT^b_{lk}(u_{(i,j)})],$$
$$[infI^b_{lk}(u_{(i,j)}), supI^b_{lk}(u_{(i,j)})], [infF^b_{lk}(u_{(i,j)}), supF^b_{lk}(u_{(i,j)})]\rangle].$$
$$=[\langle[max\{infT^a_{lk}(u_{(i,j)}), infT^b_{lk}(u_{(i,j)})\}, max\{supT^a_{lk}(u_{(i,j)}), supT^b_{lk}(u_{(i,j)})\}],$$
$$[min\{infI^a_{lk}(u_{(i,j)}), infI^b_{lk}(u_{(i,j)})\}, min\{supI^a_{lk}(u_{(i,j)}), supI^b_{lk}(u_{(i,j)})\}],$$
$$[min\{infF^a_{lk}(u_{(i,j)}), infF^b_{lk}(u_{(i,j)})\}, min\{supF^a_{lk}(u_{(i,j)}), supF^b_{lk}(u_{(i,j)})\}]\rangle].$$

$$
\begin{aligned}
([\hat{A}_{lk}] \cup [\hat{B}_{lk}])^c =&[\langle[[min\{infF^a_{lk}(u_{(i,j)}), infF^b_{lk}(u_{(i,j)})\}, min\{supF^a_{lk}(u_{(i,j)}), supF^b_{lk}(u_{(i,j)})\}],\\
&[min\{1-supI^a_{lk}(u_{(i,j)}), 1-supI^b_{lk}(u_{(i,j)})\}, min\{1-infI^a_{lk}(u_{(i,j)}),\\
&1-infI^b_{lk}(u_{(i,j)})\}], [max\{infT^a_{lk}(u_{(i,j)}), infT^b_{lk}(u_{(i,j)})\},\\
&max\{supT^a_{lk}(u_{(i,j)}), supT^b_{lk}(u_{(i,j)})\}]]\rangle].\\
=&[\langle[infF^a_{lk}(u_{(i,j)}), supF^a_{lk}(u_{(i,j)})], [1-supI^a_{lk}(u_{(i,j)}), 1-infI^a_{lk}(u_{(i,j)})],\\
&[infT^a_{lk}(u_{(i,j)}), supT^a_{lk}(u_{(i,j)})]\rangle] \cap [infF^b_{lk}(u_{(i,j)}), supF^b_{lk}(u_{(i,j)})],\\
&[1-supI^b_{lk}(u_{(i,j)}), 1-infI^b_{lk}(u_{(i,j)})], [infT^b_{lk}(u_{(i,j)}), supT^b_{lk}(u_{(i,j)})]\rangle].\\
=&[\hat{a}^i_{lk}]^c_{m_i\times n_i} \cap [\hat{b}^i_{lk}]^c_{m_i\times n_i} \quad \text{for all i.}\\
=&[\hat{A}_{lk}]^c \cap [\hat{B}_{lk}]^c.
\end{aligned}
$$

The proof of (2) is similar to (1).

Definition 3.26. Let $[\hat{A}_{lk}], [\hat{B}_{lk}] \in \hat{N}_{m\times n}$. Let $[\hat{a}^i_{lk}]_{m_i\times n_i}$ and $[\hat{b}^i_{lk}]_{m_i\times n_i}$ be U_i-IVNSMS-parts of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ respectively. Then AND product of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ is defined by

$$[\hat{a}^i_{lk}]_{m_i\times n_i} \wedge [\hat{b}^i_{lk}]_{m_i\times n_i} = [\hat{c}^i_{lp}]_{m_i\times n_i^2} \quad \text{for all i,}$$

where $[\hat{c}^i_{lp}] = [\langle[infT^c_{lp}(u_{(i,j)}), supT^c_{lp}(u_{(i,j)})], [infI^c_{lp}(u_{(i,j)}), supI^c_{lp}(u_{(i,j)})], [infF^c_{lp}(u_{(i,j)}), supF^c_{lp}(u_{(i,j)})]\rangle]$,

$$
\begin{aligned}
infT^c_{lp}(u_{(i,j)}) &= min\{infT^a_{lk}(u_{(i,j)}), infT^b_{kj}(u_{(i,j)})\},\\
supT^c_{lp}(u_{(i,j)}) &= min\{supT^a_{lk}(u_{(i,j)}), supT^b_{kj}(u_{(i,j)})\},\\
infI^c_{lp}(u_{(i,j)}) &= max\{infI^a_{lk}(u_{(i,j)}), infI^b_{kj}(u_{(i,j)})\},\\
supI^c_{lp}(u_{(i,j)}) &= max\{supI^a_{lk}(u_{(i,j)}), supI^b_{kj}(u_{(i,j)})\},\\
infF^c_{lp}(u_{(i,j)}) &= max\{infF^a_{lk}(u_{(i,j)}), infF^b_{kj}(u_{(i,j)})\} \; and\\
supF^c_{lp}(u_{(i,j)}) &= max\{supF^a_{lk}(u_{(i,j)}), supF^b_{kj}(u_{(i,j)})\}
\end{aligned}
$$

such that $p = n_i(k-1)+j$.

$$
[\hat{A}_{lk}]_{m\times n} \wedge [\hat{B}_{lk}]_{m\times n} = \begin{bmatrix}
[\hat{a}^1_{lk}]_{m_1\times n_1} \wedge [\hat{b}^1_{lk}]_{m_1\times n_1} & [\hat{0}_{lk}]_{m_1\times n_2^2} & \cdots & [\hat{0}_{lk}]_{m_1\times n_N^2}\\
[\hat{0}_{lk}]_{m_2\times n_1^2} & [\hat{a}^2_{lk}]_{m_2\times n_2} \wedge [\hat{b}^2_{lk}]_{m_2\times n_2} & \cdots & [\hat{0}_{lk}]_{m_2\times n_N^2}\\
\vdots & \vdots & \cdots & \vdots\\
[\hat{0}_{lk}]_{m_N\times n_1^2} & [\hat{0}_{lk}]_{m_N\times n_2^2} & \cdots & [\hat{a}^N_{lk}]_{m_N\times n_N} \wedge [\hat{b}^N_{lk}]_{m_N\times n_N}
\end{bmatrix}_{m\times ns}
$$

where $m = m_1 + m_2 + \cdots + m_N$ and $ns = n_1^2 + n_2^2 + \cdots + n_N^2$.

Definition 3.27. Let $[\hat{A}_{lk}], [\hat{B}_{lk}] \in \hat{N}_{m\times n}$. Let $[\hat{a}^i_{lk}]_{m_i\times n_i}$ and $[\hat{b}^i_{lk}]_{m_i\times n_i}$ be U_i-IVNSMS-parts of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ respectively. Then OR product of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ is defined by

$$[\hat{a}^i_{lk}]_{m_i\times n_i} \vee [\hat{b}^i_{lk}]_{m_i\times n_i} = [\hat{c}^i_{lp}]_{m_i\times n_i^2} \quad \text{for all i,}$$

where $[\hat{c}^i_{lp}] = [\langle[infT^c_{lp}(u_{(i,j)}), supT^c_{lp}(u_{(i,j)})], [infI^c_{lp}(u_{(i,j)}), supI^c_{lp}(u_{(i,j)})], [infF^c_{lp}(u_{(i,j)}), supF^c_{lp}(u_{(i,j)})]\rangle]$,

$$
\begin{aligned}
infT^c_{lp}(u_{(i,j)}) &= max\{infT^a_{lk}(u_{(i,j)}), infT^b_{kj}(u_{(i,j)})\},\\
supT^c_{lp}(u_{(i,j)}) &= max\{supT^a_{lk}(u_{(i,j)}), supT^b_{kj}(u_{(i,j)})\},\\
infI^c_{lp}(u_{(i,j)}) &= min\{infI^a_{lk}(u_{(i,j)}), infI^b_{kj}(u_{(i,j)})\},\\
supI^c_{lp}(u_{(i,j)}) &= min\{supI^a_{lk}(u_{(i,j)}), supI^b_{kj}(u_{(i,j)})\},\\
infF^c_{lp}(u_{(i,j)}) &= min\{infF^a_{lk}(u_{(i,j)}), infF^b_{kj}(u_{(i,j)})\} \; and\\
supF^c_{lp}(u_{(i,j)}) &= min\{supF^a_{lk}(u_{(i,j)}), supF^b_{kj}(u_{(i,j)})\}
\end{aligned}
$$

$$[\hat{A}_{lk}]_{m\times n}\vee[\hat{B}_{lk}]_{m\times n} = \begin{bmatrix} [\hat{a}^1_{lk}]_{m_1\times n_1}\vee[\hat{b}^1_{lk}]_{m_1\times n_1} & [\hat{0}]_{m_1\times n_2^2} & \cdots & [\hat{0}]_{m_1\times n_N^2} \\ [\hat{0}]_{m_2\times n_1^2} & [\hat{a}^2_{lk}]_{m_2\times n_2}\vee[\hat{b}^2_{lk}]_{m_2\times n_2} & \cdots & [\hat{0}]_{m_2\times n_N^2} \\ \vdots & \vdots & \cdots & \vdots \\ [\hat{0}]_{m_N\times n_1^2} & [\hat{0}]_{m_N\times n_2^2} & \cdots & [\hat{a}^N_{lk}]_{m_N\times n_N}\vee[\hat{b}^N_{lk}]_{m_N\times n_N} \end{bmatrix}_{m\times ns}$$

where $m = m_1 + m_2 + \cdots + m_N$ and $ns = n_1^2 + n_2^2 + \cdots + n_N^2$.

Example 3.28. Let us consider the matrices $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ as in Example 3.18. Then the AND product of $[\hat{a}^1_{lk}]_{3\times 3}$ and $[\hat{b}^1_{lk}]_{3\times 3}$ is

$$[\hat{a}^1_{lk}]_{3\times 3}\wedge[\hat{b}^1_{lk}]_{3\times 3} =$$
$$\left[\begin{array}{llll}
 & (e_{U_1,1},e_{U_1,1}) & (e_{U_1,1},e_{U_1,2}) & (e_{U_1,1},e_{U_1,3}) \\
u_{(1,1)} & \langle[0.2,0.3],[0.5,0.6],[0.8,0.9]\rangle & \langle[0.2,0.3],[0.5,0.6],[0.8,0.9]\rangle & \langle[0.2,0.3],[0.5,0.6],[0.8,0.9]\rangle \\
u_{(1,2)} & \langle[0.7,0.8],[0.2,0.4],[0.4,0.5]\rangle & \langle[0.7,0.8],[0.3,0.5],[0.6,0.7]\rangle & \langle[0.2,0.4],[0.5,0.7],[0.2,0.3]\rangle \\
u_{(1,3)} & \langle[0.6,0.7],[0.2,0.4],[0.8,0.9]\rangle & \langle[0.4,0.6],[0.2,0.5],[0.8,0.9]\rangle & \langle[0.7,0.8],[0.2,0.4],[0.1,0.2]\rangle \\
 & (e_{U_1,2},e_{U_1,1}) & (e_{U_1,2},e_{U_1,2}) & (e_{U_1,2},e_{U_1,3}) \\
u_{(1,1)} & \langle[0.5,0.6],[0.2,0.4],[0.7,0.8]\rangle & \langle[0.7,0.8],[0.2,0.4],[0.1,0.3]\rangle & \langle[0.5,0.7],[0.4,0.5],[0.2,0.3]\rangle \\
u_{(1,2)} & \langle[0.8,0.9],[0.4,0.5],[0.4,0.5]\rangle & \langle[0.7,0.9],[0.4,0.5],[0.6,0.7]\rangle & \langle[0.2,0.4],[0.5,0.7],[0.2,0.4]\rangle \\
u_{(1,3)} & \langle[0.1,0.2],[0.3,0.4],[0.8,0.9]\rangle & \langle[0.1,0.2],[0.3,0.5],[0.8,0.9]\rangle & \langle[0.1,0.2],[0.3,0.4],[0.5,0.6]\rangle \\
 & (e_{U_1,3},e_{U_1,1}) & (e_{U_1,3},e_{U_1,2}) & (e_{U_1,3},e_{U_1,3}) \\
u_{(1,1)} & \langle[0.5,0.6],[0.2,0.3],[0.7,0.8]\rangle & \langle[0.8,0.9],[0.2,0.4],[0.1,0.2]\rangle & \langle[0.5,0.7],[0.4,0.5],[0.2,0.3]\rangle \\
u_{(1,2)} & \langle[0.1,0.2],[0.3,0.4],[0.5,0.6]\rangle & \langle[0.1,0.2],[0.3,0.5],[0.6,0.7]\rangle & \langle[0.1,0.2],[0.5,0.7],[0.5,0.6]\rangle \\
u_{(1,3)} & \langle[0.5,0.7],[0.5,0.6],[0.8,0.9]\rangle & \langle[0.4,0.6],[0.5,0.6],[0.8,0.9]\rangle & \langle[0.5,0.7],[0.5,0.6],[0.2,0.3]\rangle
\end{array}\right]_{3\times 9}.$$

Likewise, we can constract $[\hat{a}^2_{lk}]_{2\times 2}\wedge[\hat{b}^2_{lk}]_{2\times 2}$ and $[\hat{a}^3_{lk}]_{2\times 2}\wedge[\hat{b}^3_{lk}]_{2\times 2}$ for universes U_2 and U_3.

$$[\hat{a}^2_{lk}]_{2\times 2}\wedge[\hat{b}^2_{lk}]_{2\times 2} = \left[\begin{array}{lll}
 & (e_{U_2,1},e_{U_2,1}) & (e_{U_2,1},e_{U_2,2}) \\
u_{(2,1)} & \langle[0.7,0.9],[0.5,0.6],[0.4,0.5]\rangle & \langle[0.5,0.6],[0.5,0.6],[0.4,0.5]\rangle \\
u_{(2,2)} & \langle[0.5,0.6],[0.5,0.6],[0.4,0.5]\rangle & \langle[0.5,0.6],[0.2,0.4],[0.1,0.3]\rangle \\
 & (e_{U_2,2},e_{U_2,1}) & (e_{U_2,2},e_{U_2,2}) \\
u_{(1,1)} & \langle[0.5,0.6],[0.5,0.6],[0.2,0.3]\rangle & \langle[0.5,0.6],[0.4,0.5],[0.3,0.4]\rangle \\
u_{(1,2)} & \langle[0.6,0.7],[0.5,0.6],[0.4,0.5]\rangle & \langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle
\end{array}\right]_{2\times 4}.$$

$$[\hat{a}^3_{lk}]_{2\times 2}\wedge[\hat{b}^3_{lk}]_{2\times 2} = \left[\begin{array}{lll}
 & (e_{U_3,1},e_{U_3,1}) & (e_{U_3,1},e_{U_3,2}) \\
u_{(3,1)} & \langle[0.7,0.9],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.7,0.8],[0.5,0.6],[0.4,0.7]\rangle \\
u_{(3,2)} & \langle[0.5,0.6],[0.4,0.7],[0.7,0.8]\rangle & \langle[0.5,0.6],[0.4,0.7],[0.5,0.6]\rangle \\
 & (e_{U_3,2},e_{U_3,1}) & (e_{U_3,2},e_{U_3,2}) \\
u_{(3,1)} & \langle[0.5,0.7],[0.4,0.7],[0.2,0.3]\rangle & \langle[0.5,0.7],[0.5,0.7],[0.4,0.7]\rangle \\
u_{(3,2)} & \langle[0.5,0.6],[0.4,0.5],[0.7,0.8]\rangle & \langle[0.5,0.6],[0.2,0.4],[0.5,0.6]\rangle
\end{array}\right]_{2\times 4}.$$

Similarly, we can build OR product of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$.

$$[\hat{a}^1_{lk}]_{3\times 3}\vee[\hat{b}^1_{lk}]_{3\times 3} =$$
$$\left[\begin{array}{llll}
 & (e_{U_1,1},e_{U_1,1}) & (e_{U_1,1},e_{U_1,2}) & (e_{U_1,1},e_{U_1,3}) \\
u_{(1,1)} & \langle[0.5,0.6],[0.2,0.3],[0.7,0.8]\rangle & \langle[0.8,0.9],[0.2,0.4],[0.1,0.2]\rangle & \langle[0.5,0.7],[0.4,0.5],[0.2,0.3]\rangle \\
u_{(1,2)} & \langle[0.8,0.9],[0.1,0.2],[0.1,0.3]\rangle & \langle[0.7,0.9],[0.2,0.4],[0.1,0.3]\rangle & \langle[0.7,0.8],[0.2,0.4],[0.1,0.3]\rangle \\
u_{(1,3)} & \langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle \\
 & (e_{U_1,2},e_{U_1,1}) & (e_{U_1,2},e_{U_1,2}) & (e_{U_1,2},e_{U_1,3}) \\
u_{(1,1)} & \langle[0.7,0.8,[0.2,0.3],[0.1,0.3]\rangle & \langle[0.8,0.9],[0.2,0.4],[0.1,0.2]\rangle & \langle[0.7,0.8],[0.2,0.4],[0.1,0.3]\rangle \\
u_{(1,2)} & \langle[0.8,0.9],[0.1,0.2],[0.2,0.4]\rangle & \langle[0.8,0.9],[0.3,0.5],[0.2,0.4]\rangle & \langle[0.8,0.9],[0.4,0.5],[0.2,0.3]\rangle \\
u_{(1,3)} & \langle[0.6,0.7],[0.2,0.4],[0.5,0.6]\rangle & \langle[0.4,0.6],[0.2,0.4],[0.5,0.6]\rangle & \langle[0.7,0.8],[0.2,0.4],[0.1,0.2]\rangle \\
 & (e_{U_1,3},e_{U_1,1}) & (e_{U_1,3},e_{U_1,2}) & (e_{U_1,3},e_{U_1,3}) \\
u_{(1,1)} & \langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle \\
u_{(1,2)} & \langle[0.8,0.9],[0.1,0.2],[0.4,0.5]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.5,0.6]\rangle & \langle[0.2,0.4],[0.3,0.4],[0.2,0.3]\rangle \\
u_{(1,3)} & \langle[0.6,0.7],[0.2,0.4],[0.2,0.3]\rangle & \langle[0.5,0.7],[0.2,0.5],[0.2,0.3]\rangle & \langle[0.7,0.8],[0.2,0.4],[0.1,0.2]\rangle
\end{array}\right]_{3\times 9}.$$

$$[\hat{a}^2_{lk}]_{2\times 2}\vee[\hat{b}^2_{lk}]_{2\times 2} = \left[\begin{array}{lll}
 & (e_{U_2,1},e_{U_2,1}) & (e_{U_2,1},e_{U_2,2}) \\
u_{(2,1)} & \langle[0.8,0.9],[0.5,0.6],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.4,0.5],[0.3,0.4]\rangle \\
u_{(2,2)} & \langle[0.6,0.7],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle \\
 & (e_{U_2,2},e_{U_2,1}) & (e_{U_2,2},e_{U_2,2}) \\
u_{(1,1)} & \langle[0.8,0.9],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.5,0.6],[0.2,0.3],[0.1,0.2]\rangle \\
u_{(1,2)} & \langle[0.8,1],[0.2,0.3],[0.1,0.3]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.3]\rangle
\end{array}\right]_{2\times 4}.$$

$$[\hat{a}^3_{lk}]_{2\times 2}\vee[\hat{b}^3_{lk}]_{2\times 2} = \left[\begin{array}{lll}
 & (e_{U_3,1},e_{U_3,1}) & (e_{U_3,1},e_{U_3,2}) \\
u_{(3,1)} & \langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle & \langle[0.7,0.9],[0.4,0.6],[0.4,0.5]\rangle \\
u_{(3,2)} & \langle[0.5,0.7],[0.4,0.5],[0.2,0.3]\rangle & \langle[0.5,0.7],[0.2,0.4],[0.2,0.3]\rangle
\end{array}\right.$$

$$\begin{matrix} & (e_{U_3,2}, e_{U_3,1}) & (e_{U_3,2}, e_{U_3,2}) \\ u_{(3,1)} & \langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle & \langle[0.7,0.8],[0.4,0.6],[0.2,0.3]\rangle \\ u_{(3,2)} & \langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle & \langle[0.8,0.9],[0.2,0.4],[0.1,0.3]\rangle \end{bmatrix}_{2\times 4}.$$

Definition 3.29. Let $[\hat{C}_{lp}] \in IVNSMM_{m\times n^2}$, $I_k^i = \{p : \langle[infT_{lp}, supT_{lp}], [infI_{lp}, supI_{lp}], [infF_{lp}, supF_{lp}]\rangle \neq 0,\ (k-1)n_i < p \leq kn_i\}$, for all $k \in 1, 2, \cdots n_i,\ i = 1, 2, \cdots N$. Then IVNSMM-max-min decision function, represented by $IVNSMM_{mMM}$, is defined by

$$IVNSMM_{mMM} : IVNSMM_{m\times n^2} \longrightarrow IVNSMM_{m\times 1},$$

$IVNSMM_{mMM} = [\langle[infT_{lp}, supT_{lp}], [infI_{lp}, supI_{lp}], [infF_{lp}, supF_{lp}]\rangle] = [\hat{d}_{l1}^i] = [\langle[\max_k\{inf\acute{T}_{lpk}\}, \max_k\{sup\acute{T}_{lpk}\}], [\min_k\{inf\acute{I}_{lpk}\}, \min_k\{sup\acute{I}_{lpk}\}], [\min_k\{inf\acute{F}_{lpk}\}, \min_k\{sup\acute{F}_{lpk}\}]\rangle]$, where

$$inf\acute{T}_{lpk} = \begin{cases} \min_{p\in I_k^i}\{infT_{lpk}\}, & \text{if } I_k^i \neq \phi, \\ 0, & \text{if } I_k^i = \phi. \end{cases} \qquad sup\acute{T}_{lpk} = \begin{cases} \min_{p\in I_k^i}\{supT_{lpk}\}, & \text{if } I_k^i \neq \phi, \\ 0, & \text{if } I_k^i = \phi. \end{cases}$$

$$inf\acute{I}_{lpk} = \begin{cases} \max_{p\in I_k^i}\{infI_{lpk}\}, & \text{if } I_k^i \neq \phi, \\ 0, & \text{if } I_k^i = \phi. \end{cases} \qquad sup\acute{I}_{lpk} = \begin{cases} \max_{p\in I_k^i}\{supI_{lpk}\}, & \text{if } I_k^i \neq \phi, \\ 0, & \text{if } I_k^i = \phi. \end{cases}$$

$$inf\acute{F}_{lpk} = \begin{cases} \max_{p\in I_k^i}\{infF_{lpk}\}, & \text{if } I_k^i \neq \phi, \\ 0, & \text{if } I_k^i = \phi. \end{cases} \qquad sup\acute{F}_{lpk} = \begin{cases} \max_{p\in I_k^i}\{supF_{lpk}\}, & \text{if } I_k^i \neq \phi, \\ 0, & \text{if } I_k^i = \phi. \end{cases}$$

The max-min decision IVNSMS-matrix is a column IVNSMS-matrix $IVNSMM\ [\hat{C}_{lp}]$.

Definition 3.30. Let $U_i = \{u_{(i,1)}, u_{(i,2)}, \cdots u_{(i,m_N)}\}$ be an U_i-IVNSMS-part and $IVNSMM_{mMM} = [\langle[infT_{lp}, supT_{lp}],$
$[infI_{lp}, supI_{lp}], [infF_{lp}, supF_{lp}]\rangle] = [\hat{d}_{l1}^i]$. Then

$$opt_N[\hat{d}_{l1}^i] = \{u_{(i,j)}/\hat{d}_{l1}^i : u_{(i,j)} \in U_i,\ \hat{d}_{l1}^i \neq 0\},\ d_{l1}^i = max\{s_l^i\},$$

where $s_l^i = (\frac{infT_{lp}+supT_{lp}}{2}) + (1 - \frac{infI_{lp}+supI_{lp}}{2}) + (1 - \frac{infF_{lp}+supF_{lp}}{2})$ is known as an U_i-optimum neutrosophic set.

4 Application

In this section, we present a group decision making algorithm for interval-valued neutrosophic soft multiset matrices, and we employ it to a real world marketing problem to make the best optimal decision. Interval valued neutrosophic soft multisets help to analyse the situation in problems involving complex scenerios.

Algorithm:

Step 1: Find the interval-valued neutrosophic soft multisets (I_A, E) and (J_B, E).

Step 2: Build the interval-valued neutrosophic soft matrix $[\hat{a}_{lk}^i]_{m_i\times n_i}$ of U_i-IVNSMS-part of (I_A, E) and $[\hat{b}_{lk}^i]_{m_i\times n_i}$ of U_i-IVNSMS-part of (J_B, E).

Step 3: Construct the interval-valued neutrosophic soft multiset matrices $[\hat{A}_{lk}]_{m\times n}$ and $[\hat{B}_{lk}]_{m\times n}$ on interval-valued neutrosophic soft multisets (I_A, E) and (J_B, E).

Step 4: Utilize Definition 3.26 and frame the AND-product of $[\hat{A}_{lk}]_{m\times n}$ and $[\hat{B}_{lk}]_{m\times n}$ and denote it by $[\hat{C}_{lp}]_{m\times n^2}$.

Step 5: Employ the IVNSMM-max-min decision function and find the max-min decision IVNSMS-matrix for $[\hat{C}_{lp}]_{m\times n^2}$.

Step 6: Find an optimum interval-valued neutrosophic set of U_i-IVNSMS-part of $[\hat{C}_{lp}]_{m\times n^2}$.

Step 7: Choose the alternative that has the maximum score value as the best optimal decision.

Our suggested algorithm provides a great mechanism to find the best alternative for complex, real world problems in a more accurate manner. Better and more reliable decisions can be made in complex and uncertain environments involving multiple universes at once by incorporating interval-valued neutrosophic soft multisets into real world problems.

Example 4.1. Suppose that a home appliance manufacturing company X needs to select an effective Television channel, Newspaper/magazine type and a forceful digital marketing cite for their ad campaign. Let (I_A, E) be an interval-valued neutrosophic soft multiset which describes "different types of TV channels", "types of Newspapers/magazines for making ad posters"and "effective digital marketing cites"respectively that Company X is considering. Let $U_1 = \{u_{(1,1)} = News\ channels,\ u_{(1,2)} = Sports\ channels,\ u_{(1,3)} = Entertainment\ \&\ life\ style\ channels,\ u_{(1,4)} = Science\ \&\ discovery\ channels, u_{(1,5)} = Kids\ channels\}$ be the universe for different types of channels capable for advertisement, $U_2 = \{u_{(2,1)} = Broadsheet\ newspapers, u_{(2,2)} = Regional\ newspapers,\ u_{(2,3)} = National\ newspapers,\ u_{(2,4)} = Tabloids,\ u_{(2,5)} = Business\ newspapers\}$ be the universe for different types of newspapers and $U_3 = \{u_{(3,1)} = Social\ media,\ u_{(3,2)} = Google\ ad,\ u_{(3,3)} = Amazon\}$ be the universe for effective digital marketing cites. Let $\{E_{U_1}, E_{U_2}, E_{U_3}\}$ be a collection of parameters which describes above universes, where
$E_{U_1} = \{e_{U_1,1} = content\ relevance,\ e_{U_1,2} = sponsered\ programs,\ e_{U_1,3} = frequent\ high\ trp\ programs,\ e_{U_1,4} = past\ histories\}$,
$E_{U_2} = \{e_{U_2,1} = geographic\ coverage,\ e_{U_2,2} = good\ effectiveness,\ e_{U_2,3} = extradinary\ benefits\ in\ online\ edition\}$ and
$E_{U_3} = \{e_{U_3,1} = product\ awarness\ globally,\ e_{U_3,2} = driving\ direct\ sale,\ e_{U_3,3} = platform\ popularity\ \&\ trend\}$.
Let $U = \prod_{i=1}^{3} IVNS(U_i)$, $E = \prod_{i=1}^{3} E_{U_i}$. Assume that E_1 and E_2 are two strategists of company X, whose guidance and strategies enable the company to work successfully. Now, the company X wants to choose a combination of TV channel type, Newspaper type and Digital marketing cite for ad campaign.
Step 1: Strategists E_1 and E_2 have to define interval-valued neutrosophic number as linguistic values in Table 1.

The decision makers E_1 and E_2 assign linguistic variables to alternatives to define interval-

Table 1. Linguistic variables

S.No.	Code	Linguistic Variable	Interval Valued Neutrosophic Number
1.	VHR	Very High Reach	$\langle[0.8, 1], [0.2, 0.3], [0.1, 0.2]\rangle$
2.	HR	High Reach	$\langle[0.7, 0.9], [0.3, 0.4], [0.2, 0.3]\rangle$
3.	MR	Moderate Reach	$\langle[0.5, 0.7], [0.4, 0.6], [0.4, 0.5]\rangle$
4.	LR	Low Reach	$\langle[0.3, 0.5], [0.7, 0.8], [0.7, 0.8]\rangle$
5.	VLR	Very Low Reach	$\langle[0.2, 0.3], [0.7, 0.9], [0.8, 1]\rangle$

valued neutrosophic soft multisets (I_A, E) and (J_B, E). The set of choice parameters A and B are given as:
$A = \{a_1 = (e_{U_1,1}, e_{U_2,1}, e_{U_3,1}),\ a_2 = (e_{U_1,2,}, e_{U_2,3}, e_{U_3,2}),\ a_3 = (e_{U_1,4}, e_{U_2,2}, e_{U_3,1}),$
$a_4 = (e_{U_1,3}, e_{U_2,3}, e_{U_3,3})\}$.
$B = \{b_1 = (e_{U_1,1}, e_{U_2,1}, e_{U_3,1}),\ b_2 = (e_{U_1,2,}, e_{U_2,3}, e_{U_3,2}),\ b_3 = (e_{U_1,3}, e_{U_2,1}, e_{U_3,3}),$
$b_4 = (e_{U_1,4}, e_{U_2,2}, e_{U_3,3})\}$.
Then interval-valued neutrosophic soft multisets for experts E_1 and E_2 are (I_A, E) and (J_B, E) respectively which are now given by
$(I_A, E) =$
$\{(a_1, (\{\frac{HR}{u_{(1,1)}}, \frac{MR}{u_{(1,2)}}, \frac{VHR}{u_{(1,3)}}, \frac{MR}{u_{(1,4)}}, \frac{VLR}{u_{(1,5)}}\}, \{\frac{MR}{u_{(2,1)}}, \frac{MR}{u_{(2,2)}}, \frac{VHR}{u_{(2,3)}}, \frac{LR}{u_{(2,4)}}, \frac{VHR}{u_{(2,5)}}\}, \{\frac{VHR}{u_{(3,1)}}, \frac{HR}{u_{(3,2)}}, \frac{LR}{u_{(3,3)}}\})),$
$(a_2, (\{\frac{VHR}{u_{(1,1)}}, \frac{VLR}{u_{(1,2)}}, \frac{VHR}{u_{(1,3)}}, \frac{VLR}{u_{(1,4)}}, \frac{VLR}{u_{(1,5)}}\}, \{\frac{MR}{u_{(2,1)}}, \frac{LR}{u_{(2,2)}}, \frac{LR}{u_{(2,3)}}, \frac{LR}{u_{(2,4)}}, \frac{HR}{u_{(2,5)}}\}, \{\frac{HR}{u_{(3,1)}}, \frac{VHR}{u_{(3,2)}}, \frac{VHR}{u_{(3,3)}}\})),$
$(a_3, (\{\frac{MR}{u_{(1,1)}}, \frac{VHR}{u_{(1,2)}}, \frac{HR}{u_{(1,3)}}, \frac{LR}{u_{(1,4)}}, \frac{VLR}{u_{(1,5)}}\}, \{\frac{VHR}{u_{(2,1)}}, \frac{LR}{u_{(2,2)}}, \frac{LR}{u_{(2,3)}}, \frac{VLR}{u_{(2,4)}}, \frac{VHR}{u_{(2,5)}}\}, \{\frac{VHR}{u_{(3,1)}}, \frac{HR}{u_{(3,2)}}, \frac{LR}{u_{(3,3)}}\})),$
$(a_4, (\{\frac{HR}{u_{(1,1)}}, \frac{MR}{u_{(1,2)}}, \frac{VHR}{u_{(1,3)}}, \frac{VLR}{u_{(1,4)}}, \frac{VLR}{u_{(1,5)}}\}, \{\frac{MR}{u_{(2,1)}}, \frac{LR}{u_{(2,2)}}, \frac{LR}{u_{(2,3)}}, \frac{LR}{u_{(2,4)}}, \frac{HR}{u_{(2,5)}}\}, \{\frac{VHR}{u_{(3,1)}}, \frac{VHR}{u_{(3,2)}}, \frac{MR}{u_{(3,3)}}\}))\}.$

$(J_B, E) =$
$\{(b_1, (\{\frac{HR}{u_{(1,1)}}, \frac{MR}{u_{(1,2)}}, \frac{VHR}{u_{(1,3)}}, \frac{MR}{u_{(1,4)}}, \frac{VLR}{u_{(1,5)}}\}, \{\frac{VHR}{u_{(2,1)}}, \frac{VLR}{u_{(2,2)}}, \frac{VHR}{u_{(2,3)}}, \frac{LR}{u_{(2,4)}}, \frac{HR}{u_{(2,5)}}\}, \{\frac{HR}{u_{(3,1)}}, \frac{VHR}{u_{(3,2)}}, \frac{MR}{u_{(3,3)}}\})),$
$(b_2, (\{\frac{HR}{u_{(1,1)}}, \frac{LR}{u_{(1,2)}}, \frac{VHR}{u_{(1,3)}}, \frac{MR}{u_{(1,4)}}, \frac{LR}{u_{(1,5)}}\}, \{\frac{HR}{u_{(2,1)}}, \frac{MR}{u_{(2,2)}}, \frac{MR}{u_{(2,3)}}, \frac{LR}{u_{(2,4)}}, \frac{VHR}{u_{(2,5)}}\}, \{\frac{VHR}{u_{(3,1)}}, \frac{MR}{u_{(3,2)}}, \frac{MR}{u_{(3,3)}}\})),$
$(b_3, (\{\frac{MR}{u_{(1,1)}}, \frac{HR}{u_{(1,2)}}, \frac{HR}{u_{(1,3)}}, \frac{LR}{u_{(1,4)}}, \frac{VLR}{u_{(1,5)}}\}, \{\frac{VHR}{u_{(2,1)}}, \frac{VLR}{u_{(2,2)}}, \frac{VHR}{u_{(2,3)}}, \frac{LR}{u_{(2,4)}}, \frac{HR}{u_{(2,5)}}\}, \{\frac{VHR}{u_{(3,1)}}, \frac{VHR}{u_{(3,2)}}, \frac{LR}{u_{(3,3)}}\})),$

$(b_4, (\{\frac{HR}{u_{(1,1)}}, \frac{VHR}{u_{(1,2)}}, \frac{VHR}{u_{(1,3)}}, \frac{VLR}{u_{(1,4)}}, \frac{LR}{u_{(1,5)}}\}, \{\frac{HR}{u_{(2,1)}}, \frac{LR}{u_{(2,2)}}, \frac{HR}{u_{(2,3)}}, \frac{LR}{u_{(2,4)}}, \frac{MR}{u_{(2,5)}}\}, \{\frac{VHR}{u_{(3,1)}}, \frac{VHR}{u_{(3,2)}}, \frac{LR}{u_{(3,3)}}\}))\}$.

Step 2: Now, let us build interval-valued neutrosophic soft matrices $[\hat{a}^1_{lk}]$ and $[\hat{b}^1_{lk}]$ for U_1-IVNSMS-part of (I_A, E) and (J_B, E).

$$[\hat{a}^1_{lk}] = \begin{bmatrix} HR & VHR & HR & MR \\ MR & VLR & MR & VHR \\ VHR & VHR & VHR & HR \\ MR & VLR & VLR & LR \\ VLR & VLR & VLR & VLR \end{bmatrix} \qquad [\hat{b}^1_{lk}] = \begin{bmatrix} HR & HR & MR & HR \\ MR & LR & HR & VHR \\ VHR & VHR & HR & VHR \\ MR & MR & LR & VLR \\ VLR & LR & VLR & LR \end{bmatrix}$$

Similarly, we construct the other U_i-IVNSMS-part of IVNSM-sets (I_A, E) and (J_B, E).

$$[\hat{a}^2_{lk}] = \begin{bmatrix} MR & VHR & MR \\ MR & LR & LR \\ VHR & LR & LR \\ LR & VLR & LR \\ VHR & VHR & HR \end{bmatrix} \qquad [\hat{b}^2_{lk}] = \begin{bmatrix} VHR & HR & HR \\ VLR & LR & MR \\ VHR & HR & MR \\ LR & LR & LR \\ HR & MR & VHR \end{bmatrix}$$

$$[\hat{a}^3_{lk}] = \begin{bmatrix} VHR & HR & VHR \\ HR & VHR & VHR \\ LR & VHR & MR \end{bmatrix} \qquad [\hat{b}^3_{lk}] = \begin{bmatrix} HR & VHR & VHR \\ VHR & MR & VHR \\ MR & MR & LR \end{bmatrix}$$

Step 3: Then we construct the interval-valued neutrosophic soft multiset matrix $[\hat{A}_{lk}]_{m\times n}$ for (I_A, E) and interval-valued neutrosophic soft multiset matrix $[\hat{B}_{lk}]_{m\times n}$ for (J_B, E).

$$[\hat{A}_{lk}] = \begin{bmatrix} \begin{bmatrix} HR & VHR & HR & MR \\ MR & VLR & MR & VHR \\ VHR & VHR & VHR & HR \\ MR & VLR & VLR & LR \\ VLR & VLR & VLR & VLR \end{bmatrix} & \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{bmatrix} & \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{bmatrix} \\ \begin{bmatrix} \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \end{bmatrix} & \begin{bmatrix} MR & VHR & MR \\ MR & LR & LR \\ VHR & LR & LR \\ LR & VLR & LR \\ VHR & VHR & HR \end{bmatrix} & \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{bmatrix} \\ \begin{bmatrix} \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \end{bmatrix} & \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{bmatrix} & \begin{bmatrix} VHR & HR & VHR \\ HR & VHR & VHR \\ LR & VHR & MR \end{bmatrix} \end{bmatrix}_{13\times 10}$$

$$[\hat{B}_{lk}] = \begin{bmatrix} \begin{bmatrix} HR & HR & MR & HR \\ MR & LR & HR & VHR \\ VHR & VHR & HR & VHR \\ MR & MR & LR & VLR \\ VLR & LR & VLR & LR \end{bmatrix} & \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{bmatrix} & \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{bmatrix} \\ \begin{bmatrix} \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \end{bmatrix} & \begin{bmatrix} VHR & HR & HR \\ VLR & LR & MR \\ VHR & HR & MR \\ LR & LR & LR \\ HR & MR & VHR \end{bmatrix} & \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{bmatrix} \\ \begin{bmatrix} \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \\ \phi & \phi & \phi & \phi \end{bmatrix} & \begin{bmatrix} \phi & \phi & \phi \\ \phi & \phi & \phi \\ \phi & \phi & \phi \end{bmatrix} & \begin{bmatrix} HR & VHR & VHR \\ VHR & MR & VHR \\ MR & MR & LR \end{bmatrix} \end{bmatrix}_{13\times 10}$$

Step 4: The AND product of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$ is constructed and named as $[\hat{C}_{lp}]$.

Now, consider $[\hat{a}^1_{lk}]$ and $[\hat{b}^1_{lk}]$ of U_1-IVNSMS-part of (I_A, E) and (J_B, E). The AND product $[\hat{c}^1_{lp}]$ is determined as follows:

$$[\hat{c}^1_{lp}] = \begin{bmatrix} \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle \\ \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle \\ \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle \\ \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle \\ \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle \end{bmatrix}$$

$$\begin{matrix}
\langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle \\
\langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle \\
\langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle \\
\langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle \\
\langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle
\end{matrix}$$

$$\begin{matrix}
\langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle \\
\langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle \\
\langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle \\
\langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle \\
\langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle
\end{matrix}$$

$$\left.\begin{matrix}
\langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle \\
\langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle \\
\langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle \\
\langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle \\
\langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle
\end{matrix}\right]_{5\times16}$$

Similarly, we construct the AND product of other U_i-IVNSMS-parts of $[\hat{A}_{lk}]$ and $[\hat{B}_{lk}]$.

$$[\hat{c}^2_{lp}] = \left[\begin{matrix}
\langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle \\
\langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle \\
\langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle \\
\langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle \\
\langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle
\end{matrix}\right.$$

$$\begin{matrix}
\langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle \\
\langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle \\
\langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle \\
\langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle \\
\langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle
\end{matrix}$$

$$\left.\begin{matrix}
\langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle \\
\langle[0.2,0.3],[0.7,0.9],[0.8,1]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle \\
\langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle \\
\langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle \\
\langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle
\end{matrix}\right]_{5\times9}$$

$$[\hat{c}^3_{lp}] = \left[\begin{matrix}
\langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle \\
\langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle \\
\langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle
\end{matrix}\right.$$

$$\begin{matrix}
\langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle \\
\langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle \\
\langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle
\end{matrix}$$

$$\left.\begin{matrix}
\langle[0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle \\
\langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.8,1],[0.2,0.3],[0.1,0.2]\rangle \\
\langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle & \langle[0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle
\end{matrix}\right]_{3\times9}$$

Step 5: Now, we apply IVNSMM-max-min decision function to each universe $U_i,\ i = 1, 2, 3$ in order to create the column IVNSMS-matrix $IVNSMM[\hat{C}_{lp}]$.

$$IVNSMM[\hat{C}_{lp}] = \begin{bmatrix} [\hat{d}^1_{l1}] \\ [\hat{d}^2_{l1}] \\ [\hat{d}^3_{l1}] \end{bmatrix}, \text{ where}$$

$$[\hat{d}^1_{l1}] = \begin{bmatrix}
\langle[infT_{11}, supT_{11}], [infI_{11}, supI_{11}], [infF_{11}, supF_{11}]\rangle \\
\langle[infT_{21}, supT_{21}], [infI_{21}, supI_{21}], [infF_{21}, supF_{21}]\rangle \\
\langle[infT_{31}, supT_{31}], [infI_{31}, supI_{31}], [infF_{31}, supF_{31}]\rangle \\
\langle[infT_{41}, supT_{41}], [infI_{41}, supI_{41}], [infF_{41}, supF_{41}]\rangle \\
\langle[infT_{51}, supT_{51}], [infI_{51}, supI_{51}], [infF_{51}, supF_{51}]\rangle
\end{bmatrix},$$

$$[\hat{d}_{l1}^{2}] = \begin{bmatrix} \langle [infT_{11}, supT_{11}], [infI_{11}, supI_{11}], [infF_{11}, supF_{11}] \rangle \\ \langle [infT_{21}, supT_{21}], [infI_{21}, supI_{21}], [infF_{21}, supF_{21}] \rangle \\ \langle [infT_{31}, supT_{31}], [infI_{31}, supI_{31}], [infF_{31}, supF_{31}] \rangle \\ \langle [infT_{41}, supT_{41}], [infI_{41}, supI_{41}], [infF_{41}, supF_{41}] \rangle \\ \langle [infT_{51}, supT_{51}], [infI_{51}, supI_{51}], [infF_{51}, supF_{51}] \rangle \end{bmatrix} \text{ and}$$

$$[\hat{d}_{l1}^{3}] = \begin{bmatrix} \langle [infT_{11}, supT_{11}], [infI_{11}, supI_{11}], [infF_{11}, supF_{11}] \rangle \\ \langle [infT_{21}, supT_{21}], [infI_{21}, supI_{21}], [infF_{21}, supF_{21}] \rangle \\ \langle [infT_{31}, supT_{31}], [infI_{31}, supI_{31}], [infF_{31}, supF_{31}] \rangle \end{bmatrix}.$$

Let us discover $\hat{d}_{11}^{1}$ for $l = 1$. Since, $l = 1$ and $k = \{1, 2, 3, 4\}$ we have $\hat{d}_{11}^{1} = \langle [inf\acute{T}_{11}, sup\acute{T}_{11}], [inf\acute{I}_{11}, sup\acute{I}_{11}], [inf\acute{F}_{11}, sup\acute{F}_{11}] \rangle$.
Let $x_{1k}^{1} = \{x_{11}^{1}, x_{12}^{1}, x_{13}^{1}, x_{14}^{1}\}$, where $x_{1k}^{1} = \langle [infT_{1p}, supT_{1p}], [infI_{1p}, supI_{1p}], [infF_{1p}, supF_{1p}] \rangle$. We have to find x_{1k}^{1} for all $k = \{1, 2, 3, 4\}$. Firstly, let us consider x_{11}^{1}, $I_{1}^{1} = \{p : 0 < p \leq 4\}$ for $k = 1$ and $n_1 = 4$.
Now,

$$\begin{aligned} x_{11}^{1} =& \langle [min\{infT_{11}^{1}, infT_{12}^{1}, infT_{13}^{1}, infT_{14}^{1}\}, min\{supT_{11}^{1}, supT_{12}^{1}, supT_{13}^{1}, supT_{14}^{1}\}], \\ & [max\{infI_{11}^{1}, infI_{12}^{1}, infI_{13}^{1}, infI_{14}^{1}\}, max\{supI_{11}^{1}, supI_{12}^{1}, supI_{13}^{1}, supI_{14}^{1}\}], \\ & [max\{infF_{11}^{1}, infF_{12}^{1}, infF_{13}^{1}, infF_{14}^{1}\}, max\{supF_{11}^{1}, supF_{12}^{1}, supF_{13}^{1}, supF_{14}^{1}\}] \rangle. \\ x_{11}^{1} =& \langle [min\{0.7, 0.7, 0.5, 0.7\}, min\{0.9, 0.9, 0.7, 0.9\}], [max\{0.3, 0.3, 0.4, 0.3\}, \\ & max\{0.4, 0.4, 0.6, 0.4\}], [max\{0.2, 0.2, 0.4, 0.2\}, max\{0.3, 0.3, 0.5, 0.3\}] \rangle. \\ =& \langle [0.5, 0.7], [0.4, 0.6], [0.4, 0.5] \rangle. \end{aligned}$$

Similarly, for $k = 2$ and $n_1 = 4$, we can find x_{12}^{1}, $I_{2}^{1} = \{p : 4 < p \leq 8\}$.

$$\begin{aligned} x_{12}^{1} =& \langle [min\{0.7, 0.7, 0.5, 0.7\}, min\{0.9, 0.9, 0.7, 0.9\}], [max\{0.3, 0.3, 0.4, 0.3\}, \\ & max\{0.4, 0.4, 0.6, 0.4\}], [max\{0.2, 0.2, 0.4, 0.2\}, max\{0.3, 0.3, 0.5, 0.3\}] \rangle. \\ =& \langle [0.5, 0.7], [0.4, 0.6], [0.4, 0.5] \rangle. \end{aligned}$$

For $k = 3$ and $n_1 = 4$, we have to find x_{13}^{1}, $I_{3}^{1} = \{p : 8 < p \leq 12\}$.

$$\begin{aligned} x_{13}^{1} =& \langle [min\{0.7, 0.7, 0.5, 0.7\}, min\{0.9, 0.9, 0.7, 0.9\}], [max\{0.3, 0.3, 0.4, 0.3\}, \\ & max\{0.4, 0.4, 0.6, 0.4\}], [max\{0.2, 0.2, 0.4, 0.2\}, max\{0.3, 0.3, 0.5, 0.3\}] \rangle. \\ =& \langle [0.5, 0.7], [0.4, 0.6], [0.4, 0.5] \rangle. \end{aligned}$$

Finally, for $k = 4$ and $n_1 = 4$ we have to find x_{14}^{1}, $I_{4}^{1} = \{p : 12 < p \leq 16\}$.

$$\begin{aligned} x_{14}^{1} =& \langle [min\{0.5, 0.5, 0.5, 0.5\}, min\{0.7, 0.7, 0.7, 0.7\}], [max\{0.4, 0.4, 0.4, 0.4\}, \\ & max\{0.6, 0.6, 0.6, 0.6\}], [max\{0.4, 0.4, 0.4, 0.4\}, max\{0.5, 0.5, 0.5, 0.5\}] \rangle. \\ =& \langle [0.5, 0.7], [0.4, 0.6], [0.4, 0.5] \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} d_{11}^{1} =& \langle [infT_{11}, supT_{11}], [infI_{11}, supI_{11}], [infF_{11}, supF_{11}] \rangle \\ =& \langle [max\{inf\acute{T}_{111}^{1}\}, max\{sup\acute{T}_{111}^{1}\}], [min\{inf\acute{I}_{111}^{1}\}, min\{sup\acute{I}_{111}^{1}\}], \\ & [min\{inf\acute{F}_{111}^{1}\}, min\{sup\acute{F}_{111}^{1}\}] \rangle. \\ =& \langle [0.5, 0.7], [0.4, 0.6], [0.4, 0.5] \rangle. \end{aligned}$$

Hence, we obtain

$$[\hat{d}^1_{l1}] = \begin{bmatrix} \langle [0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle \\ \langle [0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle \\ \langle [0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle \\ \langle [0.2,0.3],[0.7,0.9],[0.8,1]\rangle \\ \langle [0.2,0.3],[0.7,0.9],[0.8,1]\rangle \end{bmatrix}.$$

Similarly, we can obtain

$$[\hat{d}^2_{l1}] = \begin{bmatrix} \langle [0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle \\ \langle [0.2,0.3],[0.7,0.9],[0.8,1]\rangle \\ \langle [0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle \\ \langle [0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle \\ \langle [0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle \end{bmatrix} \text{ and}$$

$$[\hat{d}^3_{l1}] = \begin{bmatrix} \langle [0.7,0.9],[0.3,0.4],[0.2,0.3]\rangle \\ \langle [0.5,0.7],[0.4,0.6],[0.4,0.5]\rangle \\ \langle [0.3,0.5],[0.7,0.8],[0.7,0.8]\rangle \end{bmatrix}.$$

Step 6: Now, we compute the score of elements in each universe by using the score function s^i_l. Hence we get

$$[s_1]^1 = \begin{bmatrix} 1.65 \\ 0.9 \\ 2.2 \\ 0.55 \\ 0.55 \end{bmatrix}, \quad [s_1]^2 = \begin{bmatrix} 2.2 \\ 0.55 \\ 1.65 \\ 0.9 \\ 1.65 \end{bmatrix}, \quad [s_1]^3 = \begin{bmatrix} 2.2 \\ 1.65 \\ 0.9 \end{bmatrix}.$$

Finally, we obtain the optimum neutrosophic set for every universe U_i. Thus,
$opt_N[d^1_{l1}] = \{u_{(1,1)}/1.65, u_{(1,2)}/0.9, u_{(1,3)}/2.2, u_{(1,4)}/0.55, u_{(1,5)}/0.55\}$,
$opt_N[d^2_{l1}] = \{u_{(2,1)}/2.2, u_{(2,2)}/0.55, u_{(2,3)}/1.65, u_{(2,4)}/0.9, u_{(2,5)}/1.65\}$,
$opt_N[d^3_{l1}] = \{u_{(3,1)}/2.2, u_{(3,2)}/1.65, u_{(3,3)}/0.9\}$.

Step 7: According to the score values, we obtain $\{u_{(1,3)}, u_{(2,1)}, u_{(3,1)}\}$ as the best optimal decision. As a result, strategists E_1 and E_2 would recommend the company X to spend more money on Entertainment & life style channels, Broadsheet newspapers and Social Medias like facebook, instagram and youtube advertisements in order to boost product sales and brand recognition among their target audience.

MATLAB code for the suggested algorithm:

MATLAB is quite useful for matrix operations and mathematical calculations. To make the AND operations and computations in our technique easier, we provide a MATLAB code and run our algorithm and get the results directly.

The result for each universe got by executing the code, is given below.

```
Enter the number of universes:3
Enter the A and B IVNS-Matrices for universe 1
Enter the a.ml values in a vector:
[.7 .8 .7 .5;.5 .2 .5 .8;.8 .8 .8 .7;.5 .2 .2 .3;.2 .2 .2 .2]
Enter the a.mu values in a vector:
[.9 01 .9 .7;.7 .3 .7 01;01 01 01 .9;.7 .3 .3 .5;.3 .3 .3 .3]
Enter the a.il values in a vector:
[.3 .2 .3 .4;.4 .7 .4 .2;.2 .2 .2 .3;.4 .7 .7 .7;.7 .7 .7 .7]
Enter the a.iu values in a vector:
[.4 .3 .4 .6;.6 .9 .6 .3;.3 .3 .3 .4;.6 .9 .9 .9;.9 .9 .9 .9]
Enter the a.nl values in a vector:
[.2 .1 .2 .4;.4 .8 .4 .1;.1 .1 .1 .2;.4 .8 .8 .7;.8 .8 .8 .8]
```

```
Enter the a.nu values in a vector:
[.3 .2 .3 .5;.5 01 .5 .2;.2 .2 .2 .3;.5 01 01 .8;01 01 01 01]
Enter the b.ml values in a vector:
[.7 .7 .5 .7;.5 .3 .7 .8;.8 .8 .7 .8;.5 .5 .3 .2;.2 .3 .2 .3]
Enter the b.mu values in a vector:
[.9 .9 .7 .9;.7 .5 .9 01;01 01 .9 01;.7 .7 .5 .3;.3 .5 .3 .5]
Enter the b.il values in a vector:
[.3 .3 .4 .3;.4 .7 .3 .2;.2 .2 .3 .2;.4 .4 .7 .7;.7 .7 .7 .7]
Enter the b.iu values in a vector:
[.4 .4 .6 .4;.6 .8 .4 .3;.3 .3 .4 .3;.6 .6 .8 .9;.9 .9 .9 .9]
Enter the b.nl values in a vector:
[.2 .2 .4 .2;.4 .7 .2 .1;.1 .1 .2 .1;.4 .4 .7 .8;.8 .7 .8 .7]
Enter the b.nu values in a vector:
[.3 .3 .5 .3;.5 .8 .3 .2;.2 .2 .3 .2;.4 .4 .8 01;01 .8 01 .8]

The score matrix for the universe 1 is
u =

    1.6500
    0.9000
    2.2000
    0.5500
    0.5500

The best optimal decision for universe 1 is u_1_3

Enter the A and B IVNS-Matrices for universe 2
Enter the a.ml values in a vector:
[.5 .8 .5;.5 .3 .3;.8 .3 .3;.3 .2 .3;.8 .8 .7]
Enter the a.mu values in a vector:
[.7 01 .7;.7 .5 .5;01 .5 .5;.5 .3 .5;01 01 .9]
Enter the a.il values in a vector:
[.4 .2 .4;.4 .7 .7;.2 .7 .7;.7 .7 .7;.2 .2 .3]
Enter the a.iu values in a vector:
[.6 .3 .6;.6 .8 .8;.3 .8 .8;.8 .9 .8;.3 .3 .4]
Enter the a.nl values in a vector:
[.4 .1 .4;.4 .7 .7;.1 .7 .7;.7 .8 .7;.1 .1 .2]
Enter the a.nu values in a vector:
[.5 .2 .5;.5 .8 .8;.2 .8 .8;.8 01 .8;.2 .2 .3]
Enter the b.ml values in a vector:
[.8 .7 .7;.2 .3 .5;.8 .7 .5;.3 .3 .3;.7 .5 .8]
Enter the b.mu values in a vector:
[01 .9 .9;.3 .5 .7;01 .9 .7;.5 .5 .5;.9 .7 01]
Enter the b.il values in a vector:
[.2 .3 .3;.7 .7 .4;.2 .3 .4;.7 .7 .7;.3 .4 .2]
Enter the b.iu values in a vector:
[.3 .4 .4;.9 .8 .6;.3 .4 .6;.8 .8 .8;.4 .6 .3]
Enter the b.nl values in a vector:
[.1 .2 .2;.8 .7 .4;.1 .2 .4;.7 .7 .7;.2 .7 .1]
Enter the b.nu values in a vector:
[.2 .3 .3;01 .8 .5;.2 .3 .5;.8 .8 .8;.3 .8 .2]

The score matrix for the universe 2 is
u =

    2.2000
    0.5500
```

```
    1.6500
    0.9000
    1.6500

The best optimal decision for universe 2 is u_2_1

Enter the A and B IVNS−Matrices for universe 3
Enter the a.ml values in a vector:[.8 .7 .8;.7 .8 .8;.3 .8 .5]
Enter the a.mu values in a vector:[01 .9 01;.9 01 01;.5 01 .7]
Enter the a.il values in a vector:[.2 .3 .2;.3 .2 .2;.7 .2 .4]
Enter the a.iu values in a vector:[.3 .4 .3;.4 .3 .3;.8 .3 .6]
Enter the a.nl values in a vector:[.1 .2 .1;.2 .1 .1;.7 .1 .4]
Enter the a.nu values in a vector:[.2 .3 .2;.3 .2 .2;.8 .2 .5]
Enter the b.ml values in a vector:[.7 .8 .8;.8 .5 .8;.5 .5 .3]
Enter the b.mu values in a vector:[.9 01 01;01 .7 01;.7 .7 .5]
Enter the b.il values in a vector:[.3 .2 .2;.2 .4 .2;.4 .4 .7]
Enter the b.iu values in a vector:[.4 .3 .3;.3 .6 .3;.6 .6 .8]
Enter the b.nl values in a vector:[.2 .1 .1;.1 .4 .1;.4 .4 .7]
Enter the b.nu values in a vector:[.3 .2 .2;.2 .5 .2;.5 .5 .8]

The score matrix for the universe 3 is
u =

    2.2000
    1.6500
    0.9000
The best optimal decision for universe 3 is u_3_1
```

5 Conclusion

In this paper, we defined the notion for interval-valued neutrosophic soft multiset matrices. Additionally, we discussed the properties and operators of interval-valued neutrosophic soft multiset matrices with examples. Later, we proposed a decision making algorithm for interval-valued neutrosophic soft multisets. Finally, a real world marketing problem is used to illustrate the operation of our proposed algorithm.

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