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# Weak absolute summability theorems and related generalized vector variational inequality problems

P. K. Das, P. Baliarsingh and L. Nayak

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**Corresponding Author: L. Nayak**

**Abstract**. In the present paper, we define the idea of a generalized vector variational operator of order $r, (r \in \mathbb{N})$ via difference sequences and it is being used to study some weak absolute summability theorems. Certain results on newly defined vector variational operator are discussed. Finally, the existence theorem of a generalized vector variational inequality problem is studied to show the existence of minimum eigenvalue problems.

## 1 Introduction

The theory of variational inequality problem (VIP) was defined and studied by Stampacchia ([33],1964) to represent the Signorini contact problem in a particular form. Later on, this theory became very interesting for its unified model as the researchers found that the equality and inequality problems can be reformed as a variational inequality problem. Now a days, many crucial problems arising in different areas of applied mathematics, physics, engineering, medical science, finance, etc. can be studied with the help of variational inequalities. Some of such important problems include the networking problems, image processing problems, contact problems, obstacle problems, viscocity problems, equilibrium problems, constrained or unconstrained minimization problems, and many more. The structure classical variational inequality problems was based on a real linear or nonlinear functional defined either over a Banach space or a topological space or a manifold. In the year 2000, the concept of the variational inequality problem was extended and studied by Giannessi [19] in the setting of the finite dimensional Euclidean spaces. Then several developments along with relevant results for the vector variational inequality and vector complementarity problems have been implemented. For this perspective, we may refer to Behera and Panda [7, 8], Gianessi [20]. Chen [9], Chen and Yang [10], Daniilidis and Hadjisavvas [17], Behera and Das [6] and the references there in.

### 1.1 Absolute almost convergence and weak absolute almost convergence

The notion of absolute almost convergence and weak absolute almost convergence are introduced by Das et al. [12] and later, the ideas have been studied by many researchers (for details, we may refer to [13, 14, 27]). Let $\ell_\infty$ denote the Banach space of all real bounded sequences $x = (x_n)$ normed as usual by $\|x\| = \sup_n |x_n|$.

A linear functional $\mathcal{L}$ on $\ell_\infty$ is said to be a Banach limit (see, [1], p. 32) if it has the following properties:

(i) $\mathcal{L}(x) \geq 0$, if $x \geq 0$ (i.e., $x_n \geq 0$ for all $n \in \mathbb{N}$),

(ii) $\mathcal{L}(e) = 1$, where $e = (1, 1, 1, \dots)$,

(iii) $\mathcal{L}(Dx) = \mathcal{L}(x)$, where $(Dx)_n = x_{n+1}$.

According to Lorentz [23], a sequence $x \in \ell_\infty$ is said to be *almost convergent* to a number $s$, if $\mathcal{L}(x) = s$, for every Banach limit $\mathcal{L}$. He has proved that $x$ is almost convergent to $s$ if and only if

$$\frac{1}{m+1}\sum_{i=0}^{m} x_{m+i} \to s \text{ as } m \to \infty \text{ uniformly in } n.$$

Due to Maddox [25], a sequence $x \in \ell_\infty$ is to be strongly almost convergent to a number $s$ if

$$\frac{1}{m+1}\sum_{i=0}^{m} \|x_{n+i} - s\| \to 0 \text{ as } m \to \infty \text{ uniformly in } \ n.$$

To introduce the concept of absolutely almost convergent of the sequence $x \in \ell_\infty$, Das, Kuttner and Nanda [12] have developed the sequence $\varphi_{m,n}$ as follows:
For $m, n \geq 0$, the sequence $(\varphi_{m,n})$ is defined by

$$\varphi_{m,n} = t_{m,n} - t_{m-1,n}, \tag{1.1a}$$

where

$$t_{m,n} = \frac{1}{m+1}\sum_{i=0}^{m} x_{m+i} \text{ with } t_{-1,n} = x_{n-1}. \tag{1.1b}$$

**Definition 1.1.** [12] A sequence $x \in \ell_\infty$ is said to be *absolutely almost convergent* if

$$\sum_m \|\varphi_{m,n}\| \text{ coverges unformly in } n, \tag{1.2}$$

where summation without limits runs from 0 to $\infty$.

The results on summable sequences have been discussed by various researchers. For references, we refer King [21], Das et al. [11, 12, 13, 14], Maddox [24, 25], Nanda et al. [30, 31] and the references therein.
The idea of difference sequence space via the difference operator $\Delta$ of order one has been defined by Kızmaz [22]

$$(\Delta x)_k = x_k - x_{k+1}, (k \in \mathbb{N} = \{0, 1, 2, 3, \dots\}).$$

Later on, it was generalized to the case of integer order $r$ by Et and Colak [18] using the difference operator $\Delta^r$, where

$$(\Delta^r x)_k = \sum_{i=0}^{m}(-1)^i \binom{r}{i} x_{k+i}, (k \in \mathbb{N}).$$

Recently, the idea has been extended and used in various applied and pure fields of mathematics such as linear algebra, fractional calculus, approximation theory (see, for details [2, 3, 4, 32, 5, 28, 29, 16]).

## 1.2 Vector variational operator

In order to develop the VV-operator $\Delta T_{m,n}$, the weak absolutely almost convergence of the sequence $x \in \ell_\infty$, the concept of difference operator $\Delta$ is used to a weak sequence $T_{m,n}$ that is related with $\varphi_{m,n}$ defined in (1.1a). We summarize the idea of weak absolutely almost summability and related VV-operator. Assume that $T : K \subset X \to C(X, Y)$ is any mapping, $(z_n)$ is any sequences in $K$ and $(T_{m,n})$ is the functional sequence of $\varphi_{m,n}$. The difference sequence of $T_{m,n}$ is $(\Delta T_{m,n})$ (called VV-operator) defined in $Y \subset \ell_\infty$ is developed as follows:

(i) The sequence $(z_n)$ in $X$ is defined by $z_n = x_{n_k}$, $k \in \mathbb{N}$.

(ii) For $m, n \geq 0$, $(T_{m,n})$ is the sequence in $Y \subset \ell_\infty$, where

$$T_{m,n} = \frac{1}{m+1}\sum_{i=0}^{m}\langle T(z_n), x_{m+i}\rangle = \langle T(z_n, t_{m,n})\rangle \tag{1.3}$$

and

$$T_{-1,n} = \langle T(z_n), t_{-1,n}\rangle = \langle T(z_n), x_{n-1}\rangle. \tag{1.4}$$

(iii) For $m, n \geq 0$, the vector variational (VV) operator is defined by

$$\Delta T_{m,n} = T_{m,n} - T_{m-1,n} = \langle T(z_n), \varphi_{m,n} \rangle.$$

The weak absolutely almost convergence of a sequence is defined (Das [15]) as follows:

**Definition 1.2.** Let $x \in X \subset \ell_\infty$ be any sequence. Then

(i) The sequence $x$ is said to be *weak almost convergent* to a number $\mathcal{S}$, if $\mathcal{L}(x) = \xi$ for every weak topological limit $\mathcal{L}$ (the immediate neighborhood of $\mathcal{L}$ associated with $U \in \tau$).

(ii) The sequence $x$ is said to be *weak absolutely almost convergent* if

$$\sum_m \|\Delta T_{m,n}\| \text{ converges uniformly in } n.$$

Das [15] has extended some summability theorems studied by Das et. al [12] in variational prospective with the help of the VV-operator which are weaker in the sense of functional summability and its convergence.

## 2 The $r^{\text{th}}$-difference VV operator and weak absolute almost summability theorems

Let $X, Y \subset \ell_\infty$ and $(X, P_X)$ be an ordered topological spaces equipped with closed convex pointed cone $P_X$ with nonempty interior. Also, let $(Y, P_Y)$ be an ordered topological spaces with topology $\sigma$ equipped with closed convex pointed cone $P_Y$ with nonempty interior. Then the topology $\tau$ of $X$, i.e., $\tau(X)$ consists of open contractible subsets of $X$ and topology $\sigma$ of $Y$, i.e., $\sigma(Y)$ consists of open contractible subsets of $Y$. Every continuous map $f : X \to Y$ induces a topology $\sigma$ consists of open contractible subset $V$ of $Y$, and $f(U)$ is open contractible and finitely representable in $V \in \sigma$ for each $U \in \tau$, i.e., if there are $U_1, U_2, \ldots, U_n \in \tau_X$ such that $U \subset \overset{n}{\underset{i=1}{\cup}} U_i$ for each $U \subset \tau$, then there exists a $V \in \sigma$ such that $V \subset f(U) = \overset{n}{\underset{i=1}{\cup}} f(U_i)$. Therefore for $x_i \in U_i$, $i = 1, 2, \ldots, n$; the finite set $A = \{x_1, x_2, \ldots, x_n\}$ satisfies $\Gamma_A \subseteq \overset{n}{\underset{i=1}{\cup}} U_i$ where $\Gamma_A$ is the contractible set indexed by finite set $A \subset X$. In general, one can choose $U = int(\Gamma_A)$, the interior of $\Gamma_A$. Now if $\Gamma_A$ is open, then $U = \Gamma_A$.

Now, we denote the following sets:

(i) $C(X, Y)$ is the family of all continuous maps from $X$ to $Y$,

(ii) $L(X, Y)$ is the family of all linear maps from $X$ to $Y$,

(iii) $BL(X, Y)$ is the family of all bounded linear maps from $X$ to $Y$,

(iv) $LC(X, Y)$ is the family of all linear and continuous maps from $X$ to $Y$,

(v) $LU(X, Y)$ is the family of all linear uniformly continuous functionals from $X$ to $Y$ implying $f \in LC(X, Y)$ if and only if the linear continuous map $f \in C(X, Y)$ induces a topology $\sigma$ consists of open contractible subset $V$ of $Y$, and $f(U)$ is open contractible and finitely representable in $V \in \sigma$ for each $U \in \tau$.

Let $T : K \subset X \to LU(X, Y)$ be any map (may continuous or noncontinuous). Here the VV-operator is extended to $r^{\text{th}}$-difference VV Operator (or difference VV-operator of order $r$) as follows:
For $m, n \geq 0, r \geq 1$, the sequences $(t_{m,n})$, $(\varphi^r_{m,n})$ and $(\Phi^r_{m,n})$ are defined as follows:

$$t_{m,n} = \frac{1}{m+1} \sum_{i=0}^{m} x_{m+i} \text{ with } t_{-1,n} = x_{n-1} \tag{2.1a}$$

$$\varphi^r_{m,n} = \Delta^r t_{m,n} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} t_{m-i,n} \tag{2.1b}$$

$$\Phi^r_{m,n} = \Delta^r \varphi_{m,n} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \varphi_{m-i,n}. \tag{2.1c}$$

Note that the sequences defined above are in $X \subset \ell_\infty$. For our main investigation, we need to define the followings:

**Definition 2.1.** A sequence $x \in \ell_\infty$ is said to be

(i) *absolutely almost convergent* of order $r \geq 1$ if

$$\sum_m \|\varphi^r_{m,n}\| \text{ converges unformly in } n. \tag{2.2}$$

(ii) *weak absolutely almost convergent* of order $r \geq 1$ if

$$\sum_m \|\Phi^r_{m,n}\| \text{ converges unformly in } n. \tag{2.3}$$

**Theorem 2.2.** For $r, m, n \in \mathbb{N}$, we have

(i) $\varphi^r_{mn} - \varphi^r_{m-1,n} = \Phi^r_{mn}$,

(ii) $(m-i+1)t_{m-i,n} - (m-i-1)t_{m-i-1} = x_{2m-2i} + x_{2m-2i-1} - x_{m-i-1}$.

*Proof.* (i) From the definition, for $r, m, n \in \mathbb{N}$, we have

$$\varphi^r_{m,n} = \Delta^r t_{m,n} = \sum_{i=0}^{r}(-1)^i \binom{r}{i} t_{m-i,n},$$

Then,

$$\begin{aligned}
\varphi^r_{mn} - \varphi^r_{m-1,n} &= \sum_{i=0}^{r}(-1)^i \binom{r}{i} t_{m-i,n} - \sum_{i=0}^{r}(-1)^i \binom{r}{i} t_{m-i-1,n} \\
&= \sum_{i=0}^{r}(-1)^i \binom{r}{i} [t_{m-i,n} - t_{m-i-1,n}] \\
&= \sum_{i=0}^{r}(-1)^i \binom{r}{i} \varphi_{m-i,n} = \Phi^r_{mn}.
\end{aligned}$$

(ii) Also, we write

$$t_{m,n} = \frac{1}{m+1}\sum_{i=0}^{m} x_{m+i} \text{ with } t_{-1,n} = x_{n-1},$$

Thus, we easily calculate that

$$t_{m-i,n} = \frac{1}{m-i+1}\sum_{j=0}^{m-i} x_{m-i+j}$$

and

$$t_{m-i-1,n} = \frac{1}{m-i}\sum_{j=0}^{m-i-1} x_{m-i-1+j}.$$

Finally, we have

$$\begin{aligned}
(m-i+1)t_{m-i,n} - (m-i-1)t_{m-i-1} &= \sum_{j=0}^{m-i} x_{m-i+j,n} - \sum_{j=0}^{m-i-1} x_{m-i-1+j,n} \\
&= x_{2m-2i} + x_{2m-2i-1} - x_{m-i-1}. \quad \square
\end{aligned}$$

### 2.1 The $r^{\text{th}}$-VV operator

For $m, n \geq 0$, the concept of vector variational operation of $T_{m,n}$ of order $r \in \mathbb{N}$ (in short, $r^{\text{th}}$-**VV operation**) of $T_{m,n}$) is defined as follows:

$$\Delta^r T_{m,n} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} T_{m-i,n}. \tag{2.4}$$

**Remark 2.3.** For $r = 1$, the equation (2.4) is reduced to first vector variational operation of $\Delta T_{m,n}$ (see Das [15]).

Now, for more detail, we can calculate

$$\begin{aligned}\Delta^r T_{m,n} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} T_{m-i,n} &= \sum_{i=0}^{r} (-1)^i \binom{r}{i} \langle T(z_n), t_{m-i,n} \rangle \\ &= \left\langle T(z_n), \sum_{i=0}^{r} (-1)^i \binom{r}{i} t_{m-i,n} \right\rangle \\ &= \langle T(z_n), \Delta^r t_{m,n} \rangle \\ &= \langle T(z_n), \varphi^r_{m,n} \rangle \end{aligned} \tag{2.5}$$

**Theorem 2.4.** For $r, m, n \in \mathbb{N}$, we have

(i)
$$T^r_{mn} - T^r_{m-1,n} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \langle T(z_n), \varphi_{m-i,n} \rangle,$$

(ii)
$$\begin{aligned}&(m-i+1)\Delta^r T_{m,n} - (m-i-1)\Delta^r T_{m-1,n} - \Delta^r T_{m-1,n} \\ &\qquad = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \langle T(z_n), x_{2m-2i} - x_{m-i-1} \rangle.\end{aligned}$$

*Proof.* (i) Suppose $r, m, n \in \mathbb{N}$. Then, we have

$$\Delta^r T_{m,n} = \sum_{i=0}^{r} (-1)^i \binom{r}{i} T_{m-i,n} = \langle T(z_n), \varphi^r_{mn} \rangle.$$

As a consequence,

$$\begin{aligned}\Delta^r T_{mn} - \Delta^r T_{m-1,n} &= \langle T(z_n), \varphi^r_{m,n} \rangle - \langle T(z_n), \varphi^r_{m-1,n} \rangle \\ &= \sum_{i=0}^{r} (-1)^i \binom{r}{i} \langle T(z_n), \varphi_{m-i,n} \rangle.\end{aligned}$$

(ii) Now, we take the left hand side of (ii) and find

$$\begin{aligned}&(m-i+1)\Delta^r T_{m,n} - (m-i-1)\Delta^r T_{m-1,n} - \Delta^r T_{m-1,n} \\ &\qquad = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left\langle T(z_n), \sum_{j=0}^{m-i} x_{m-i+j} - \sum_{j=0}^{m-i-1} x_{m-i-1+j} \right\rangle \\ &\qquad\qquad - \sum_{i=0}^{r} (-1)^i \binom{r}{i} \langle T(z_n), x_{2m-2i-1} \rangle \\ &\qquad = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \langle T(z_n), x_{2m-2i} - x_{m-i-1} \rangle.\end{aligned}$$

□

Now, we define the weak absolute convergence of a sequence $x \in X \subset \ell_\infty$ as follows.

A sequence $x \in X \subset \ell_\infty$ is said to be *weak almost absolute convergent* to a number $\mathcal{S}$, if $\mathcal{L}(x) = \xi$ for every weak absolute topological limit $\mathcal{L}$ (the immediate neighborhood of $\mathcal{L}$ associated with $U \in \tau$).

A sequence $x \in X \subset \ell_\infty$ is said to be weak absolute convergent of order $r \geq 1$ if

$$\sum_m \|\Delta^r T_{m,n}\| \text{ converges uniformly in } n \text{ for all } r \geq 1.$$

Assume that for $i \leq m$,

$$A(\varphi^r_{mn}) = (m-i+1)t_{m-i,n} - (m-i-1)t_{m-i-1} - x_{2m-2i-1}.$$

## 2.2 Results on weak absolute almost summability theorems

The following theorem states that the absolutely almost convergence implying weak absolutely almost convergence of the sequence $x \in K \in X$.

**Theorem 2.5.** Let $T : K \to BL(X, Y)$ be any map and the sequence $x \in K$ be absolutely almost convergent of order $r \geq 1$. Then, $x$ is weak absolutely almost convergent with respect to difference operator $\Delta^r$, $r \geq 1$.

*Proof.* Since $T : K \to BL(X, Y)$, there exists a number $R > 0$ such that $\|T(x)\| \leq R$ for all $x \in K$. Thus

$$\begin{aligned} \sum_m \|\Delta^r T_{m,n}\| &= \sum_m \|\langle T(z_n), \varphi^r_{m,n}\rangle\| \\ &\leq \|T(z_n)\| \sum_m \|\varphi^r_{m,n}\| \end{aligned}$$

converges uniformly in $n$ for all $r \geq 1$, since $x \in K$ is absolutely almost convergent, we have $\sum\limits_m \|\varphi^r_{m,n}\|$ converges uniformly in $n$ for all $r \geq 1$. Hence $x \in K$ is weak absolutely almost convergent of order $r \geq 1$. □

Throughout of this section, $T$ is considered as a bounded linear continuous functional from $K$ to $BL(X, Y)$, i.e., $T : K \to BL(X, Y)$.
Consider

$$\Delta^r T_{m,n}(Ax) = \langle T(z_n) - \lambda\xi, \varphi^r_{m,n}(A(x))\rangle$$

which implies

$$\begin{aligned} \Delta^r T_{m,n}(Ax) &= \langle T(z_n) - \lambda\xi, \varphi^r_{m,n}(A(x))\rangle \\ &= \langle T(z_n) - \lambda\xi, \sum_k a(n,k,m;r)x_k\rangle \\ &= \sum_k a(n,k,m;r)\langle T(z_n) - \lambda\xi, x_k\rangle \end{aligned}$$

where for $m \geq 1$, $r \geq 1$,

$$a(n,k,m;r) = \frac{1}{m+1}\sum_{k=0}^{\infty}\sum_{j=1}^{m}\sum_{i=0}^{r}(-1)^i\binom{r}{i} i\, a_{n-i+j,k};$$

and

$$a(n,k,0;r) = \sum_{i=0}^{r}(-1)^i\binom{r}{i}a_{n-i,k}.$$

Let $(p_n)$ be a sequence of real numbers such that $p_n \geq 0$ and $\sup p_n < \infty$. We define the following classes:

$$[TA, p] = \left\{ x : \sum_n \|TA_n(x)\|^{p_n} < \infty \right\},$$

$$[T\hat{A}, p] = \left\{ x : \sum_m \|T_{m,n}(Ax)\|^{p_m} \text{ converges uniformly in } n \right\},$$

$$\left[T\hat{\hat{A}}, p; r\right] = \left\{ x : \sup_n \sum_m \|\Delta^r T_{m,n}(Ax)\|^{p_m} < \infty \right\}.$$

If $p_n = p$ for all $n$, we write $[A]_p$ for $[A, p]$. If $p = 1$, we omit the suffix $p$ write $[A$ for $[A_p]$. Note that $[TA$ denotes the set of all weak absolute summable sequences. Similarly if $p_m = p$ for all $m$, we write $[T\hat{A}]_p$ and $[T\hat{\hat{A}}]_p$ for $[T\hat{A}, p$ and $[T\hat{\hat{A}}, p]$ respectively. If $p = 1$ we write $[T\hat{A}]$ and $[T\hat{\hat{A}}]$ for $[T\hat{A}]_p$ and $[T\hat{\hat{A}}]_p$ respectively. We have

**Theorem 2.6.** $[T\hat{A}, p] \subset \left[T\hat{\hat{A}}, p\right]$.

*Proof.* Let $x \in [T\hat{A}, p]$. Then there is an integer $M > 0$ such that

$$\sum_{m \geq K} \|\Delta^r T_{m,n}(Ax)\|^{p_m} \leq 1. \tag{2.6}$$

Hence it is enough to show that for fixed $m$, $\Delta^r T_{m,n}(Ax)$ is bounded. It follows from (2.6) that

$$\|\Delta^r T_{m,n}(Ax)\| \leq 1 \text{ for } m \geq K \text{ and all } n.$$

If $m \geq 1$, then

$$(m+1)\Delta^r T_{m,n}(Ax) - (m-1)\Delta^r T_{m-1,n}(Ax) = \sum_k a_{n+m,k} \sum_{i=0}^{r} (-1)^i \binom{r}{i} \langle T(z_n) - \lambda\xi, x_k \rangle.$$

Since, $T(z_n) - \lambda\xi \in BL(X, Y)$, so for any fixed $m \geq M$,

$$\sum_k a_{n+m,k} \langle T(z_n) - \lambda\xi, x_k \rangle = \left\langle T(z_n) - \lambda\xi, \sum_k a_{n+m,k} x_k \right\rangle$$

is bounded in $Y$, implying $\Delta T_{mn}(Ax)$ is bounded in $Y$ for all $m, n$. This proves that $[T\hat{A}, p] \subset [T\hat{\hat{A}}, p]$. This completes the proof of the theorem. □

**Theorem 2.7.** $[TA, p]$ is linear matric space paranormed by the maps $\nu : K \to \mathbb{R}$ (the set of all real numbers) and $w : K \to \mathbb{R}$ defined by

$$v(x) = \left\{ \sum_n \|TA_n(x)\|^{p_n} \right\}^{\frac{1}{M}}$$

where $M = \max\{1, \sup p_n\}$. The space $[T\hat{A}, p]$ is paranormed by

$$w(x) = \sup_n \left\{ \sum_m \|\Delta^r T_{m,n}(Ax)\|^{p_m} \right\}^{\frac{1}{M}}. \tag{2.7}$$

If $\inf p_m > 0$, then the space $\left[T\hat{\hat{A}}, p\right]$ is paranormed by (2.7).

*Proof.* Because of Theorem 2.6, (2.7) is meaningful, for $x \in [TA, p]$. The proof is a routine verification and uses standard techniques, therefore we omit the details. But it may be noted that there is an essential difference between the proof of $[T\hat{A}, p]$ to be paranormed and that of $[T\hat{\hat{A}}, p]$. As one step in the proof we have to show that $\lambda x \to 0$ as $\lambda \to 0$ for fixed $x$. If $x \in [T\hat{A}, p]$, then for given $\epsilon > 0$, there is a $K > 0$ such that for all $n$,

$$\sum_{m \geq K} \|\Delta^r T_{m,n}(Ax)\|^{p_m} < \epsilon. \tag{2.8}$$

If $|\lambda| \leq 1$,

$$\sum_{m \geq K} \|\Delta^r T_{m,n}(\lambda Ax)\|^{p_m} \leq \sum_{m \geq K} \|\Delta^r T_{m,n}(Ax)\|^{p_m} < \epsilon$$

and since, for fixed $K$,

$$\sum_{m=0}^{K-1} \|\Delta T_{m,n}(\lambda Ax)\|^{p_m} \to 0 \text{ as } \lambda \to 0,$$

this proves the result. But if we are given that $x \in [T\hat{\hat{A}}, p]$, then (2.8) need not be true. Now if $\inf p_m > 0$, then there is some constant $\theta > 0$ such that $p_m \geq \theta$ for all $\theta$. Hence for $|\lambda| \leq 1$, $|\lambda|^{p_m} \leq |\lambda|^\theta$, so that

$$w(\lambda x) \leq |\lambda|^\theta w(x).$$

Thus $\left[T\hat{\hat{A}}, p\right]$ is paranormed by (2.7). This completes the proof of the theorem. □

In order to prove next Theorem we recall the following lemma studied by Maddox([24], p. 168).

**Lemma 2.8.** [24] Suppose that

(i) $\sum_m \|a_{mn}\|$ converges for each $n$,

(ii) $\sum_m \|a_{mn}\| \to 0$ as $n \to \infty$.

Then $\sum_m \|a_{mn}\|$ converges uniformly in $n$.

**Theorem 2.9.** *If $p \geq 1$, then $[TA, p] \subset [T\hat{A}, p]$.*

*Proof.* Suppose that $x \in [TA, p]$ and $m \geq 1$. Then, it is trivial for $p = 1$ and for $p > 1$,

$$\begin{aligned}
&\left\| \frac{1}{m(m+1)} \sum_{i=0}^{r} \sum_{j=1}^{m} \sum_{k} (-1)^i \binom{r}{i} j \; a_{n-i+j,k} \left\langle T(z_n) - \lambda\xi, x_k \right\rangle \right\|^p \\
&\leq \frac{1}{m^p(m+1)^p} \sum_{i=0}^{r} \sum_{j=1}^{m} (-1)^i \binom{r}{i} j^p \sum_{k} \|a_{n-i+j,k} \left\langle T(z_n) - \lambda\xi, x_k \right\rangle\|^p
\end{aligned}$$

This is due to the Holder's inequality. Hence,

$$\begin{aligned}
&\sum_{m} \|\Delta^r T_{m,n}(Ax)\|^p \\
&\leq \sum_{m} \frac{1}{m^p(m+1)^p} \sum_{i=0}^{r} \sum_{j=1}^{m} (-1)^i \binom{r}{i} j^p \sum_{k} \|a_{n-i+j,k} \left\langle T(z_n) - \lambda\xi, x_k \right\rangle\|^p \\
&= \sum_{i=0}^{r} \sum_{j} (-1)^i \binom{r}{i} j^p \sum_{k} \|a_{n-i+j,k} \left\langle T(z_n) - \lambda\xi, x_k \right\rangle\|^p \sum_{m=1}^{\infty} \frac{1}{m^p(m+1)^p} \\
&= \sum_{i=0}^{r} \sum_{j} (-1)^i \binom{r}{i} \left\| \sum_{k} a_{n-i+j,k} \left\langle T(z_n) - \lambda\xi, x_k \right\rangle \right\|^p.
\end{aligned}$$

Since $a(n,k,0)=a_{nk}$, we have

$$\sum_m \|\triangle^r T_{m,n}(Ax)\|^p \le \sum_{i=n+1}^{\infty} \left\| \sum_k a_{ik} \left\langle T(z_n)-\lambda\xi, x_k\right\rangle \right\|^p .$$

Therefore the hypothesis of Lemma 2.8 is satisfied with $a_{mn} = \|\triangle^r T_{m,n}(Ax)\|^p$. This completes the proof of the theorem. □

## 3 Applications

Let $\mathcal{B}$ be a Banach space (reflexive or non-reflexive) and $X=(X,\tau)$ be a quasi-reflexive topological vector space modeled in $\mathcal{B}$ with topology $\tau$ consists of open contractible subsets of $X$. Let $E = ran(T)$ the range of $T$ be nonempty closed convex totally bounded subset $E \subset L(X,Y)$ and the inverse map $T^{-1}: E \to X$ satisfies

(i) $T^{-1}(E)$ is finite dimensional, or

(ii) $T^{-1}(E)$ is finitely representable in $U \cap K \in \tau$ for every finitely representable set in $T(K) \subset L(X,Y)$.

Assume that the pairing $\langle f,x\rangle$ represents the value of $f \in L(X,Y)$ at $x \in K$ and the pairing $\langle f,x\rangle$ represents

$$\langle f,x\rangle = \begin{cases} \langle f,x\rangle, & \text{if } X \text{ and } Y \text{ are of same dimension,}\\ \langle f,x\rangle'\langle f,x\rangle, & \text{if } X \text{ and } Y \text{ are of different dimensions.}\end{cases}$$

Let $y'$ denotes the transpose of $y \in Y$ and $\Omega \subset Y$ be a nonempty subspace in $Y$ satisfies the following property: if $\alpha A'C + \beta C'B \ge 0$ for all $\alpha,\beta > 0$ and $A,B \in \Omega_+ = \{y \in \Omega : y \ge 0\}$, then $C \in \Omega_+$. The following lemma is shown to form the generalized vector variational inequality problem with an eigenvalue.

**Lemma 3.1.** Let $w(y), z(y) \in \Omega_+$ for each $y \in X$, $\langle w\rangle = w'w$ for each $w \in Y$ and

$$\lambda_1 = \min_{y\in K} \frac{\langle w(y)\rangle}{\langle z(y)\rangle}.$$

Then for all $\lambda > 0$ one has $w(y) \ge \lambda z(y)$ for each $y \in X$.

*Proof.* Since $\lambda_1 = \min_{y\in K} \frac{\langle w(y)\rangle}{\langle z(y)\rangle}$ for all $\lambda > 0$, it follows that

$$\begin{aligned} &\langle w(y)\rangle - \lambda\langle z(y)\rangle \ge 0\\ \Rightarrow\quad & w'(w-\lambda z) + \lambda(w'-z')z \ge 0. \end{aligned}$$

Hence we have $w - \lambda z \ge 0$ for all $\lambda > 0$, i.e., for each $y \in X$, $w(y) \ge \lambda z(y)$ for all $\lambda > 0$. This completes the proof. □

Based on the result of Lemma 3.1, the following generalized vector variational inequality problem (GVVIP) is defined. Assume that for $\xi \in LU(X,Y)$ and $T: K \to E \subset LC(X,Y)$ such that $\langle T(y),x\rangle, \langle \xi,x\rangle \in \Omega_+$ for all $x,y \in K$, where $K \subset X$ is a closed convex pointed cone in $X$. Suppose there exists a vector valued function $\eta : K\times K \to X$ such that $\langle \xi, \eta(y,y)\rangle > 0$ for each $y \in K$. The problem to find $\lambda_1 \in \mathbb{R}$ such that

$$\lambda_1 = \min_{y\in K\setminus\{0\}} \frac{\langle T(y), \eta(y,y)\rangle}{\langle \xi, \eta(y,y)\rangle},$$

then $\lambda_1$ is the lowest eigenvalue of the generalized vector variational inequality problem is to find

$$y \in K \text{ such that} \langle T(y), \eta(x,y)\rangle \ge_P \lambda\langle \xi, \eta(x,y)\rangle \text{ for all } x \in K. \qquad \text{(GVVIP)}$$

**Remark 3.2.** In particular, if $\eta(x,y)=y$, then

$$\lambda_1=\min_{y\in K\setminus\{0\}}\frac{\langle T(y),y\rangle}{\langle \xi,y\rangle}$$

is the lowest eigenvalue of the generalized vector variational inequality problems to find

$$y\in K \text{ such that}\langle T(y),x-y\rangle \geq_P \lambda\langle \xi,x-y\rangle \text{ for all } x\in K \qquad \text{(VVIP)}$$

which coincides with the problem studies by Miersmann [26] if $Y=\mathbb{R}^n$.

For $x\neq y\in K\subset X$, let $\widetilde{xy}$ and $\overline{xy}$ be two paths connecting $x$ and $y$ which are defined as follows

$$\begin{aligned}\widetilde{xy}=&\{u\in K: u=y+t\eta(x,y)\in K,\ x,y\in K, t\in[0,1]\}\\ \overline{xy}=&\{u\in K: u=y+t(x-y)\in K,\ x,y\in K,\ t\in[0,1]\}.\end{aligned}$$

Let $S(T)$ and $S(\xi)$ be two solution sets defined by

$$\begin{aligned}S(T)=&\{y\in K:\langle T(u),\eta(x,y)\rangle\geq\langle T(u),x-y\rangle \text{ for all } x\in K, u\in\widetilde{xy}\}\\ S(\xi)=&\{y\in K:\langle \xi(u),\eta(x,y)\rangle\leq\langle \xi(u),x-y\rangle \text{ for all } x\in K, u\in\widetilde{xy}\}.\end{aligned}$$

**Theorem 3.3.** *If $x_0\in K$ solves the problem (VVIP) on $x\in S(T)\cap S(\xi)$, then $x_0\in K$ solves the problem (GVVIP) on $K$.*

*Proof.* Given $x_0\in K$ solves the problem (VVIP) on $x\in S(T)\cap S(\xi)$, implying

$$\begin{aligned}\langle T(u),\eta(x,y)\rangle\geq\langle T(u),x-y\rangle &\geq \lambda\langle\xi(u),x-y\rangle\\ &\geq \lambda\langle\xi(u),\eta(x,y)\rangle.\end{aligned}$$

for all $x\in K$, $u\in\widetilde{xy}$ and $\lambda>0$. Hence $x_0\in K$ solves the problem (GVVIP) on $K$. This completes the proof. □

To study the existence theorem of the vector variational inequalities, the concept of sequentially consistent and stability for a vector inequality problem is introduced (see, for details Das [15]).

(i) The vector inequality problem $G(y,x)\geq_P 0$ for all $x\in X$ and $y\in K$ is sequentially consistent and stable if the following condition holds:

$$\begin{aligned}&\sup_n\left\{\left\|\sum_m\frac{1}{m+1}\sum_{j=1}^m j^p\sum_{i=0}^r(-1)^i\binom{r}{i}\left(\sum_k a_{n+j,k}G(z_n,x_k-y_k)\right)\right\|_Y^{p_m}\right\}^{\frac{1}{M}}\\ \leq\ & 2\sup_n\left\{\sum_m\frac{1}{m+1}\sum_{j=1}^m j^p\sum_{i=0}^r(-1)^i\binom{r}{i}\sum_k a_{n+j,k}\|G(z_n,x_k)\|^{p_m}\right\}^{\frac{1}{M}}\end{aligned}$$

for all $x_k\in K$ and $y_k=\lambda x_k\in K$, $|\lambda|\leq 1$.

(ii) The vector inequality problem $G(y,x)\geq_P 0$ for all $x,y\in K$ is solvable on $K$ if it is sequentially consistent and stable on $K$.

The vector function $G: K\times X\to Y$ is sequentially consistent and dominated with respect to $\eta$ in a convex set if the following condition holds:

$$\begin{aligned}&\sup_n\left\{\left\|\sum_m\frac{1}{m+1}\sum_{i=1}^m j^p\sum_{i=0}^r(-1)^i\binom{r}{i}\left(\sum_k a_{n-i+j,k}G(z_n,\eta(x_k,y_k))\right)\right\|_Y^{p_m}\right\}^{\frac{1}{M}}\\ \leq\ & 2\sup_n\left\{\sum_m\frac{1}{m+1}\sum_{j=1}^m j^p\sum_{i=0}^r(-1)^i\binom{r}{i}\sum_k a_{n-i+j,k}\|G(y_n,x_k-y_k)\|^{p_m}\right\}^{\frac{1}{M}}\end{aligned}$$

for all $x_k$, $y_k = \lambda x_k \in K$, $|\lambda| \leq 1$, $z_n \in N(y_k) = \{x_k : \|x_k - y_k\| < \epsilon, \epsilon > 0\}$.

The following result establishes the solvability theorem of the problem (VVIP) via sequentially consistency and stability of the problem (VVIP).

**Theorem 3.4.** *If all the conditions of Theorem 2.7 are satisfied, then the problem* (VVIP) *is sequentially consistent and stable in $K$, implying* (VVIP) *is solvable on $K$.*

*Proof.* Since all the conditions of Theorem 2.7 are satisfied, $\left[T\hat{\hat{A}}, p\right]$ is paranormed by (2.7), i.e.,

$$w(\lambda x) \leq |\lambda|^{\theta} w(x)$$

which implies

$$w(y) \leq |\lambda|^{\theta} w(x) \leq w(x)$$

for all $x \in K$ and $y = \lambda x \in K$, $|\lambda| \leq 1$, i.e.,

$$w(y) \leq w(x) \Rightarrow \sup_n \left\{ \sum_m \|\Delta^r T_{m,n}(Ay)\|^{p_m} \right\}^{\frac{1}{M}} \leq \sup_n \left\{ \sum_m \|\triangle^r T_{m,n}(Ax)\|^{p_m} \right\}^{\frac{1}{M}}$$

for all $x \in K$ and $y = \lambda x \in K$, $|\lambda| \leq 1$, i.e.,

$$\sup_n \left\{ \sum_m \left\| \sum_k \tfrac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \tbinom{r}{i} a_{n-i+j,k} \langle T(z_n) - \lambda\xi, y_k \rangle \right\|^{p_m} \right\}^{\frac{1}{M}}$$
$$\leq \sup_n \left\{ \sum_m \left\| \sum_k \tfrac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \tbinom{r}{i} a_{n-i+j,k} \langle T(z_n) - \lambda\xi, x_k \rangle \right\|^{p_m} \right\}^{\frac{1}{M}},$$

for all $x_k \in K$ and $y_k = \lambda x_k \in K$, $|\lambda| \leq 1$, implying

$$\sup_n \left\{ \left\| \sum_m \sum_k \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} a_{n-i+j,k} \langle T(z_n) - \lambda\xi, y_k \rangle \right\|^{p_m} \right\}^{\frac{1}{M}}$$
$$\leq \sup_n \left\{ \sum_m \left\| \sum_k \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} a_{n-i+j,k} \langle T(z_n) - \lambda\xi, y_k \rangle \right\|^{p_m} \right\}^{\frac{1}{M}}$$
$$\leq \sup_n \left\{ \sum_m \left\| \sum_k \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} a_{n-i+j,k} \langle T(z_n) - \lambda\xi, x_k \rangle \right\|^{p_m} \right\}^{\frac{1}{M}}$$
$$\leq \sup_n \left\{ \sum_m \sum_k \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} a_{n-i+j,k} \, \|\langle T(z_n) - \lambda\xi, y_k \rangle\|^{p_m} \right\}^{\frac{1}{M}},$$

This implies

$$\sup_n \left\{ \left\| \sum_m \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \langle T(z_n) - \lambda\xi, y_k \rangle \right) \right\|^{p_m} \right\}^{\frac{1}{M}}$$
$$\leq \sup_n \left\{ \sum_m \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \, \|\langle T(z_n) - \lambda\xi, y_k \rangle\|^{p_m} \right) \right\}^{\frac{1}{M}}$$

for all $x_k \in K$ and $y_k = \lambda x_k \in K$, $|\lambda| \leq 1$. Thus

$$\begin{aligned}
&\sup_n \left\{ \left\| \sum_m \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \langle T(z_n) - \lambda\xi, x_k - y_k \rangle \right) \right\|^{p_m} \right\}^{\frac{1}{M}} \\
&\leq \sup_n \left\{ \left\| \sum_m \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \langle T(z_n) - \lambda\xi, x_k \rangle \right) \right\|^{p_m} \right\}^{\frac{1}{M}} \\
&+ \sup_n \left\{ \left\| \sum_m \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \langle T(z_n) - \lambda\xi, y_k \rangle \right) \right\|^{p_m} \right\}^{\frac{1}{M}} \\
&\leq 2 \sup_n \left\{ \sum_m \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \left\| \langle T(z_n) - \lambda\xi, y_k \rangle \right\|^{p_m} \right) \right\}^{\frac{1}{M}}
\end{aligned}$$

for all $x_k \in K$ and $y_k = \lambda x_k \in K$, $|\lambda| \leq 1$. Hence the problem (VVIP) is sequentially consistent and stable in $K$. By the definition, (VVIP) is solvable on $K$. □

To establish the solvability of the problem (GVVIP), the concept of sequentially consistent and dominated with respect to $\eta$ in a convex set is defined as follows.

The mapping $T : K \to L(X, Y)$ is sequentially consistent and dominated associated with $\xi$ with respect to $\eta$ in a convex set if the following condition holds:

$$\begin{aligned}
&\sup_n \left\{ \left\| \sum_m \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \langle T(z_n) - \lambda\xi, \eta(x_k, y_k) \rangle \right) \right\|_Y^{p_m} \right\}^{\frac{1}{M}} \\
\leq \quad &2 \sup_n \left\{ \sum_m \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \left\| \langle T(z_n) - \lambda\xi, x_k - y_k \rangle \right\|_Y^{p_m} \right) \right\}^{\frac{1}{M}}
\end{aligned}$$

for all $x_k \in K$ and $y_k = \lambda x_k \in K$, $|\lambda| \leq 1$.

**Theorem 3.5.** Let the mapping $T : K \to L(X, Y)$ be sequentially consistent and dominated associated with $\xi$ with respect to $\eta$ in a convex set $K$. If all the conditions of Theorem 3.4 are satisfied, then the problem (GVVIP) is sequentially consistent and stable in $K$, i.e., (GVVIP) is solvable on $K$.

*Proof.* Since the mapping $T : K \to L(X, Y)$ is sequentially consistent and dominated associated with $\xi$ with respect to $\eta$ in the convex set $K$, i.e.,

$$\begin{aligned}
&\sup_n \left\{ \left\| \sum_m \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \langle T(z_n) - \lambda\xi, \eta(x_k, y_k) \rangle \right) \right\|_Y^{p_m} \right\}^{\frac{1}{M}} \\
\leq \quad &2 \sup_n \left\{ \sum_m \frac{1}{m+1} \sum_{j=1}^m j^p \sum_{i=0}^r (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \left\| \langle T(z_n) - \lambda\xi, x_k - y_k \rangle \right\|_Y^{p_m} \right) \right\}^{\frac{1}{M}}
\end{aligned} \tag{3.1}$$

for all $x_k \in K$ and $y_k = \lambda x_k \in K$, $|\lambda| \leq 1$. Since all the conditions of Theorem 3.4 are satisfied, the problem (GVVIP) is sequentially consistent and stable in $K$. Hence for every $\epsilon > 0$, we get

$$\sup_n \left\{ \left\| \sum_m \frac{1}{m+1} \sum_{j=1}^{m} j^p \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \langle T(z_n) - \lambda\xi, \eta(x_k, y_k) \rangle \right) \right\|^{p_m} \right\}^{\frac{1}{M}}$$

$$\leq 2\epsilon^{-1} \sup_n \left\{ \left\| \sum_m \frac{1}{m+1} \sum_{j=1}^{m} j^p \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \langle T(z_n) - \lambda\xi, x_k - y_k \rangle \right) \right\|^{p_m} \right\}^{\frac{1}{M}}$$

$$\leq 2\epsilon^{-1} \sup_n \left\{ \sum_m \frac{1}{m+1} \sum_{j=1}^{m} j^p \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \left\| \langle T(z_n) - \lambda\xi, x_k \rangle \right\|^{p_m} \right) \right\}^{\frac{1}{M}}$$

$$\leq 2\epsilon^{-1} \sup_n \left\{ \sum_m \frac{1}{m+1} \sum_{j=1}^{m} j^p \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \left\| \langle T(z_n), x_k \rangle \right\|^{p_m} \right) \right\}^{\frac{1}{M}}$$

$$+ 2\epsilon^{-1} \sup_n \left\{ \sum_m \frac{1}{m+1} \sum_{j=1}^{m} j^p \sum_{i=0}^{r} (-1)^i \binom{r}{i} \left( \sum_k a_{n-i+j,k} \left\| \langle \lambda\xi, x_k \rangle \right\|^{p_m} \right) \right\}^{\frac{1}{M}}$$

for all $x_k \in K$ and $y_k = \lambda x_k \in K$, $|\lambda| \leq 1$. Therefore the problem (GVVIP) is sequentially consistent and stable in $K$, implying (GVVIP) is solvable on $K$, which completes the proof. □

## Conclusion

The concept of vector variational difference operator of order $r$ is newly defined and is used to study the existence of the solution of the vector variational inequality problems in the presence of weak almost absolute summability theorems. Later the existence of solution of the generalized vector variational inequality problems is studied with the help of the concept of sequentially consistent and dominated operator. The work can be further extended to difference vector variational operator of fractional order.

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## Author information


P. K. Das, Department of Mathematics, School of Applied Sciences, KIIT Deemed to be University, Bhubaneswar, India.
E-mail: `dasprasantkumar@yahoo.co.in`

P. Baliarsingh, Department of Mathematics, Institute of Mathematics and Applications, Bhubeneswar, India.
E-mail: `pb.math10@gmail.com`

L. Nayak, Department of Mathematics, School of Applied Sciences, KIIT Deemed to be University, Bhubaneswar, India.
E-mail: `laxmipriyamath@gmail.com`