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A NOTE ON THE PAPERS OF W. S. MARTINDALE III AND Y. WANG

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Abstract In this paper, we revisit the results of Martindale III and Wang on the additivity of n-multiplicative isomorphisms. Motivated by their results on this class of mappings, we generalize the concept of n-multiplicative isomorphism introduced by Wang and establish its additivity under the conditions proposed by Martindale III. As a consequence, we apply the obtained result to the class of standard operator algebras.

1 Introduction

Let $n \geq 2$ be a positive integer. We denote by S_n the set of all permutations of the set $\{1, 2, \cdots, n\}$.

Let $\mathfrak{R}$ and $\mathfrak{S}$ be arbitrary associative rings (where $\mathfrak{R}$ need not have an identity element) and σ an element of S_n. A mapping $\varphi : \mathfrak{R} \to \mathfrak{S}$ is called a *multiplicative homomorphism* (resp., *multiplicative anti-homomorphism*) if $\varphi(a_1a_2) = \varphi(a_1)\varphi(a_2)$ (resp., $\varphi(a_1a_2) = \varphi(a_2)\varphi(a_1)$), for all elements $a_1, a_2 \in \mathfrak{R}$, and a *multiplicative isomorphism* (resp., *multiplicative anti-isomorphism*) if in addition φ is bijective. A mapping $\varphi : \mathfrak{R} \to \mathfrak{S}$ is called a *n-multiplicative homomorphism* (resp., *n-multiplicative anti-homomorphism*) if $\varphi(\prod_{i=1}^n a_i) = \prod_{i=1}^n \varphi(a_i)$ (resp. $\varphi(\prod_{i=1}^n a_i) = \prod_{i=1}^n \varphi(a_{n-i+1})$), for all elements $a_1, \cdots, a_n \in \mathfrak{R}$, and a *$n$-multiplicative isomorphism* (resp., *n-multiplicative anti-isomorphism*) if in addition φ is bijective. A mapping $\varphi : \mathfrak{R} \to \mathfrak{S}$ is called a *n-multiplicative σ-homomorphism* if $\varphi(\prod_{i=1}^n a_i) = \prod_{i=1}^n \varphi(a_{\sigma(i)})$, for all elements $a_1, \cdots, a_n \in \mathfrak{R}$, and a *$n$-multiplicative σ-isomorphism* if in addition φ is bijective. A mapping $\varphi : \mathfrak{R} \to \mathfrak{S}$ is called *additive* if $\varphi(a_1 + a_2) = \varphi(a_1) + \varphi(a_2)$ for all elements $a_1, a_2 \in \mathfrak{R}$.

The study of the question of when n-multiplicative isomorphisms are additive in the area of associative rings has attracted the interest of several researchers. The first result in this direction is due to Martindale III [3] who obtained a pioneer result in 1969, which in his condition requires that the ring possess idempotents. He proved the following result:

Theorem 1.1. *[3, Main Theorem, pp. 695] Let $\mathfrak{R}$ be an associative ring such that $\mathfrak{R}$ contains a family $\{e_\alpha | \alpha \in \Lambda\}$ of nontrivial idempotents satisfying the following properties:*

(i) If $x \in \mathfrak{R}$ is such that $x\mathfrak{R} = 0$, then $x = 0$;

(ii) If $x \in \mathfrak{R}$ is such that $e_\alpha \mathfrak{R} x = 0$ for all $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$);

(iii) For each $\alpha \in \Lambda$ and $x \in \mathfrak{R}$, if $e_\alpha x e_\alpha \mathfrak{R}(1_{\mathfrak{R}} - e_\alpha) = 0$ then $e_\alpha x e_\alpha = 0$.

Then any multiplicative isomorphism φ of $\mathfrak{R}$ onto an arbitrary associative ring $\mathfrak{S}$ is additive.

An interesting consequence of Theorem 1.1 is the following Corollary.

Corollary 1.2. *[3, Corollary, pp. 697] If $\mathfrak{R}$ satisfies the conditions of the Theorem 1.1, then any multiplicative anti-isomorphism φ of $\mathfrak{R}$ onto an arbitrary associative ring $\mathfrak{S}$ is additive.*

The Theorem 1.1 was later generalized by Wang [4]. He proved the following result:

Theorem 1.3. *[4, Corollary 3.1] Let $\mathfrak{R}$ be an associative ring such that $\mathfrak{R}$ contains a family $\{e_\alpha|\alpha \in \Lambda\}$ of nontrivial idempotents satisfying the following properties:*

(i) If $x \in \mathfrak{R}$ is such that $x\mathfrak{R} = 0$, then $x = 0$;

(ii) If $x \in \mathfrak{R}$ is such that $e_\alpha \mathfrak{R} x = 0$ for all $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$);

(iii) For each $\alpha \in \Lambda$ and $x \in \mathfrak{R}$, if $e_\alpha x e_\alpha \mathfrak{R}(1_{\mathfrak{R}} - e_\alpha) = 0$ then $e_\alpha x e_\alpha = 0$.

Then any n-multiplicative isomorphism φ of $\mathfrak{R}$ onto an arbitrary associative ring $\mathfrak{S}$ is additive.

Inspired by these results, in this paper we unify the results of Theorem 1.1, Corollary 1.2, and Theorem 1.3 for the class of n-multiplicative σ-isomorphisms. In addition, we apply the obtained result to the class of standard operator algebras.

We conclude this section by presenting the following well-known result, which will be used throughout this paper.

Let e_1 be any nontrivial idempotent of $\mathfrak{R}$ and formally set $e_2 = 1_{\mathfrak{R}} - e_1$. Then $\mathfrak{R}$ has a *Peirce decomposition* $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$, where $\mathfrak{R}_{ij} = e_i \mathfrak{R} e_j$ $(1 \leq i, j \leq 2)$, satisfying the following multiplicative relations $\mathfrak{R}_{ij}\mathfrak{R}_{kl} \subseteq \delta_{jk}\mathfrak{R}_{il}$, where δ_{jk} is the *Kronecker delta function*. Throughout this article, z_{ij} denotes an arbitrary element of $\mathfrak{R}_{ij}$, so that any $z \in \mathfrak{R}$ admits a unique decomposition $z = z_{11} + z_{12} + z_{21} + z_{22}$, where each z_{ij} is the *(i, j)-Peirce component of z*.

2 The main result

Let us state our main theorem.

Theorem 2.1. *Let $\mathfrak{R}$ be an associative ring such that $\mathfrak{R}$ contains a family $\{e_\alpha|\alpha \in \Lambda\}$ of nontrivial idempotents satisfying the following properties:*

(i) If $x \in \mathfrak{R}$ is such that $x\mathfrak{R} = 0$, then $x = 0$;

(ii) If $x \in \mathfrak{R}$ is such that $e_\alpha \mathfrak{R} x = 0$ for all $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{R}x = 0$ implies $x = 0$);

(iii) For each $\alpha \in \Lambda$ and $x \in \mathfrak{R}$, if $e_\alpha x e_\alpha \mathfrak{R}(1_{\mathfrak{R}} - e_\alpha) = 0$ then $e_\alpha x e_\alpha = 0$.

Let σ be any permutation in S_n. Then any n-multiplicative σ-isomorphism φ of $\mathfrak{R}$ onto an arbitrary associative ring $\mathfrak{S}$ is additive.

Based on the techniques presented by Martindale III [3] we organize the proof of Theorem 2.1 in a series of Lemmas.

Let σ be any permutation in S_n, $\varphi : \mathfrak{R} \to \mathfrak{S}$ a n-multiplicative σ-isomorphism and e_1 an idempotent of the family $\{e_\alpha|\alpha \in \Lambda\}$. From Theorem 1.1 and Corollary 1.2 we can assume that $n \geq 3$. Therefore we begin with the following.

Lemma 2.2. $\varphi(0) = 0$.

Proof. Since φ is surjective we can choose an element $x \in \mathfrak{R}$ such that $\varphi(x) = 0$. It follows that

$$\varphi(0) = \varphi(\underbrace{x0\cdots 0}_{n \text{ terms}}) = \underbrace{\varphi(0)\cdots\varphi(0)\varphi(x)}_{\sigma^{-1}(1) \text{ terms}}\varphi(0)\cdots\varphi(0) = \underbrace{\varphi(0)\cdots\varphi(0)0}_{\sigma^{-1}(1) \text{ terms}}\varphi(0)\cdots\varphi(0) = 0.$$

□

The following lemma will play a fundamental role throughout this article in the study of additivity. For any elements $x, y \in \mathfrak{R}$, consider z to be the preimage of the element $\varphi(x)+\varphi(y) \in \mathfrak{S}$, that is, $z \in \mathfrak{R}$ satisfying $\varphi(z) = \varphi(x) + \varphi(y)$. Based on how the mapping φ acts on products of n elements of $\mathfrak{R}$ in which z appears as one of the factors, this lemma will serve as a tool for investigating the nature of the (i, j)-Peirce components of the element z, when combined with the properties of the Peirce decomposition and conditions (i)–(iii) of Theorem 2.1.

Lemma 2.3. *Let x, y, z be arbitrary elements of $\mathfrak{R}$ such that $\varphi(z) = \varphi(x) + \varphi(y)$. Then the following hold:*

(i) $\varphi(r_1 \cdots r_{n-2}za) = \varphi(r_1 \cdots r_{n-2}xa) + \varphi(r_1 \cdots r_{n-2}ya)$,

(ii) $\varphi(r_1 \cdots r_{n-2}az) = \varphi(r_1 \cdots r_{n-2}ax) + \varphi(r_1 \cdots r_{n-2}ay)$,

(iii) $\varphi(azr_3 \cdots r_n) = \varphi(axr_3 \cdots r_n) + \varphi(ayr_3 \cdots r_n)$,

(iv) $\varphi(zar_3 \cdots r_n) = \varphi(xar_3 \cdots r_n) + \varphi(yar_3 \cdots r_n)$,

(v) $\varphi(zr_2 \cdots r_{n-1}a) = \varphi(xr_2 \cdots r_{n-1}a) + \varphi(yr_2 \cdots r_{n-1}a)$,

(vi) $\varphi(ar_2 \cdots r_{n-1}z) = \varphi(ar_2 \cdots r_{n-1}x) + \varphi(ar_2 \cdots r_{n-1}y)$,

for all elements $a, r_1, \cdots, r_n \in \mathfrak{R}$.

Proof. (i) Let i, j $(1 \leq i, j \leq n)$ be indexes such that $\sigma(i) = n-1$ and $\sigma(j) = n$. Two cases are considered. First case: $1 \leq \sigma^{-1}(n-1) < \sigma^{-1}(n) \leq n$. We split the first case into the following subcases: $1 < \sigma^{-1}(n-1) < \sigma^{-1}(n) < n$, $1 < \sigma^{-1}(n-1) < \sigma^{-1}(n) = n$, $1 = \sigma^{-1}(n-1) < \sigma^{-1}(n) < n$ and $1 = \sigma^{-1}(n-1) < \sigma^{-1}(n) = n$. In this first subcase, we have

$$
\begin{aligned}
&\varphi(r_1 \cdots r_{n-2}za)\\
&= \underbrace{\overbrace{\varphi(r_{\sigma(1)}) \cdots \varphi(r_{\sigma(i-1)})\varphi(z)}^{\sigma^{-1}(n-1) \text{ terms}} \varphi(r_{\sigma(i+1)}) \cdots \varphi(r_{\sigma(j-1)})\varphi(a)}_{\sigma^{-1}(n) \text{ terms}} \varphi(r_{\sigma(j+1)}) \cdots \varphi(r_{\sigma(n)})\\
&= \underbrace{\overbrace{\varphi(r_{\sigma(1)}) \cdots \varphi(r_{\sigma(i-1)})\big(\varphi(x) + \varphi(y)\big)}^{\sigma^{-1}(n-1) \text{ terms}} \varphi(r_{\sigma(i+1)}) \cdots \varphi(r_{\sigma(j-1)})\varphi(a)}_{\sigma^{-1}(n) \text{ terms}}\\
&\qquad \varphi(r_{\sigma(j+1)}) \cdots \varphi(r_{\sigma(n)})\\
&= \underbrace{\overbrace{\varphi(r_{\sigma(1)}) \cdots \varphi(r_{\sigma(i-1)})\varphi(x)}^{\sigma^{-1}(n-1) \text{ terms}} \varphi(r_{\sigma(i+1)}) \cdots \varphi(r_{\sigma(j-1)})\varphi(a)}_{\sigma^{-1}(n) \text{ terms}} \varphi(r_{\sigma(j+1)}) \cdots \varphi(r_{\sigma(n)})\\
&\quad + \underbrace{\overbrace{\varphi(r_{\sigma(1)}) \cdots \varphi(r_{\sigma(i-1)})\varphi(y)}^{\sigma^{-1}(n-1) \text{ terms}} \varphi(r_{\sigma(i+1)}) \cdots \varphi(r_{\sigma(j-1)})\varphi(a)}_{\sigma^{-1}(n) \text{ terms}} \varphi(r_{\sigma(j+1)}) \cdots \varphi(r_{\sigma(n)})\\
&= \varphi(r_1 \cdots r_{n-2}xa) + \varphi(r_1 \cdots r_{n-2}ya).
\end{aligned}
$$

The proofs of the remaining subcases are similar to the proof of first subcase and are therefore omitted. Second case: $1 \leq \sigma^{-1}(n) < \sigma^{-1}(n-1) \leq n$. The proof is entirely analogous to that of the first case and is therefore omitted.

(ii), (iii) and (iv) The proofs are entirely similar to case (i), so we omit them.

(v) Let i, j $(1 \leq i, j \leq n)$ be indexes such that $\sigma(i) = 1$ and $\sigma(j) = n$. Two cases are considered. First case: $1 \leq \sigma^{-1}(1) < \sigma^{-1}(n) \leq n$. We split the first case into the following subcases: $1 < \sigma^{-1}(1) < \sigma^{-1}(n) < n$, $1 < \sigma^{-1}(1) < \sigma^{-1}(n) = n$, $1 = \sigma^{-1}(1) < \sigma^{-1}(n) < n$ and $1 = \sigma^{-1}(1) < \sigma^{-1}(n) = n$. In this first subcase, we have

$$
\begin{aligned}
&\varphi(zr_2 \cdots r_{n-1}a)\\
&= \underbrace{\overbrace{\varphi(r_{\sigma(1)}) \cdots \varphi(r_{\sigma(i-1)})\varphi(z)}^{\sigma^{-1}(1) \text{ terms}} \varphi(r_{\sigma(i+1)}) \cdots \varphi(r_{\sigma(j-1)})\varphi(a)}_{\sigma^{-1}(n) \text{ terms}} \varphi(r_{\sigma(j+1)}) \cdots \varphi(r_{\sigma(n)})
\end{aligned}
$$

$$= \overbrace{\underbrace{\varphi(r_{\sigma(1)})\cdots\varphi(r_{\sigma(i-1)})\big(\varphi(x)+\varphi(y)\big)}_{}}^{\sigma^{-1}(1)\text{ terms}} \underbrace{\varphi(r_{\sigma(i+1)})\cdots\varphi(r_{\sigma(j-1)})\varphi(a)}_{\sigma^{-1}(n)\text{ terms}}$$

$$\varphi(r_{\sigma(j+1)})\cdots\varphi(r_{\sigma(n)})$$

$$= \overbrace{\varphi(r_{\sigma(1)})\cdots\varphi(r_{\sigma(i-1)})\varphi(x)}^{\sigma^{-1}(1)\text{ terms}}\underbrace{\varphi(r_{\sigma(i+1)})\cdots\varphi(r_{\sigma(j-1)})\varphi(a)}_{\sigma^{-1}(n)\text{ terms}}\varphi(r_{\sigma(j+1)})\cdots\varphi(r_{\sigma(n)})$$

$$+ \overbrace{\varphi(r_{\sigma(1)})\cdots\varphi(r_{\sigma(i-1)})\varphi(y)}^{\sigma^{-1}(1)\text{ terms}}\underbrace{\varphi(r_{\sigma(i+1)})\cdots\varphi(r_{\sigma(j-1)})\varphi(a)}_{\sigma^{-1}(n)\text{ terms}}\varphi(r_{\sigma(j+1)})\cdots\varphi(r_{\sigma(n)})$$

$$=\varphi(xr_2\cdots r_{n-1}a)+\varphi(yr_2\cdots r_{n-1}a).$$

The proofs of the remaining subcases are similar to the proof of first subcase and are therefore omitted. Second case: $1 \le \sigma^{-1}(n) < \sigma^{-1}(1) \le n$. The proof is entirely analogous to that of the first case and is therefore omitted.
(vi) The proof is entirely analogous to that of case (v) and is therefore omitted. □

Lemma 2.4. $\varphi(x_{ii}+x_{jk}) = \varphi(x_{ii}) + \varphi(x_{jk})$, $j \neq k$.

Proof. First, assume that $i = j = 1$ and $k = 2$. Since φ is surjective, let z be an element of $\mathfrak{R}$ such that $\varphi(z) = \varphi(x_{11}) + \varphi(x_{12})$. By Lemma 2.3(iv), for an arbitrary element $a_{11} \in \mathfrak{R}_{11}$ we have

$$\varphi(za_{11}) = \varphi(za_{11}\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}) = \varphi(x_{11}a_{11}\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}) + \varphi(x_{12}a_{11}\underbrace{e_1\cdots e_1}_{n-2\text{ terms}})$$
$$= \varphi\big((x_{11}+x_{12})a_{11}\big).$$

This implies that $za_{11} = (x_{11}+x_{12})a_{11}$ which results in

$$\big(z-(x_{11}+x_{12})\big)a_{11} = 0. \tag{2.1}$$

Next, for an arbitrary element $a_{21} \in \mathfrak{R}_{21}$ we have

$$\varphi(za_{21}) = \varphi(za_{21}\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}) = \varphi(x_{11}a_{21}\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}) + \varphi(x_{12}a_{21}\underbrace{e_1\cdots e_1}_{n-2\text{ terms}})$$
$$= \varphi\big((x_{11}+x_{12})a_{21}\big)$$

which implies that $za_{21} = (x_{11}+x_{12})a_{21}$. Thus

$$\big(z-(x_{11}+x_{12})\big)a_{21} = 0. \tag{2.2}$$

Also, for arbitrary elements $a_{22} \in \mathfrak{R}_{22}$ and $r_1, \cdots, r_{n-2} \in \mathfrak{R}$ we have, by Lemma 2.3(i), that

$$\varphi(r_1\cdots r_{n-2}za_{22}) = \varphi(r_1\cdots r_{n-2}x_{11}a_{22}) + \varphi(r_1\cdots r_{n-2}x_{12}a_{22})$$
$$= \varphi\big(r_1\cdots r_{n-2}(x_{11}+x_{12})a_{22}\big).$$

This yields $r_1\cdots r_{n-2}\big(z-(x_{11}+x_{12})\big)a_{22} = 0$ which implies

$$\big(z-(x_{11}+x_{12})\big)a_{22} = 0, \tag{2.3}$$

by Theorem 2.1(ii). Yet, for arbitrary elements $a_{12} \in \mathfrak{R}_{12}$ and $r_1, \cdots, r_{n-2} \in \mathfrak{R}$ we have

$$\varphi(r_1 \cdots r_{n-2} z a_{12}) = \varphi(r_1 \cdots r_{n-2} x_{11} a_{12}) + \varphi(r_1 \cdots r_{n-2} x_{12} a_{12}) \\ = \varphi\big(r_1 \cdots r_{n-2}(x_{11} + x_{12}) a_{12}\big)$$

which implies that $r_1 \cdots r_{n-2}\big(z a_{12} - (x_{11} + x_{12}) a_{12}\big) = 0$. This results in

$$\big(z - (x_{11} + x_{12})\big) a_{12} = 0. \tag{2.4}$$

From identities (2.1)-(2.4) we conclude that

$$\big(z - (x_{11} + x_{12})\big)\mathfrak{R} = 0$$

which yields $z = x_{11} + x_{12}$, by Theorem 2.1(i). Now assume that $i = k = 1$ and $j = 2$. Again we may find an element z of $\mathfrak{R}$ such that $\varphi(z) = \varphi(x_{11}) + \varphi(x_{21})$. By Lemma 2.3(ii), for an arbitrary element $a_{11} \in \mathfrak{R}_{11}$ we have

$$\varphi(a_{11} z) = \varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}} a_{11} z) = \varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}} a_{11} x_{11}) + \varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}} a_{11} x_{21}) \\ = \varphi\big(a_{11}(x_{11} + x_{21})\big).$$

Hence $a_{11} z = a_{11}(x_{11} + x_{21})$, that is,

$$a_{11}\big(z - (x_{11} + x_{21})\big) = 0. \tag{2.5}$$

Next, for an arbitrary element $a_{12} \in \mathfrak{R}_{12}$ we have

$$\varphi(a_{12} z) = \varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}} a_{12} z) = \varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}} a_{12} x_{11}) + \varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}} a_{12} x_{21}) \\ = \varphi\big(a_{12}(x_{11} + x_{21})\big)$$

which implies that $a_{12} z = a_{12}(x_{11} + x_{21})$. This results in

$$a_{12}\big(z - (x_{11} + x_{21})\big) = 0. \tag{2.6}$$

Also, for arbitrary elements $a_{22} \in \mathfrak{R}_{22}$ and $r_1, \cdots, r_{n-2} \in \mathfrak{R}$, we have

$$\varphi(a_{22} z r_1 \cdots r_{n-2}) = \varphi(a_{22} x_{11} r_1 \cdots r_{n-2}) + \varphi(a_{22} x_{21} r_1 \cdots r_{n-2}) \\ = \varphi\big(a_{22}(x_{11} + x_{21}) r_1 \cdots r_{n-2}\big).$$

Hence $\big(a_{22}\big(z - (x_{11} + x_{21})\big)\big) r_1 \cdots r_{n-2} = 0$ which implies that

$$a_{22}\big(z - (x_{11} + x_{21})\big) = 0. \tag{2.7}$$

Now observe that, for arbitrary elements $a_{21} \in \mathfrak{R}_{21}$ and $r_1, \cdots, r_{n-2} \in \mathfrak{R}$, we have

$$\varphi(a_{21} z r_1 \cdots r_{n-2}) = \varphi(a_{21} x_{11} r_1 \cdots r_{n-2}) + \varphi(a_{21} x_{21} r_1 \cdots r_{n-2}) \\ = \varphi\big(a_{21}(x_{11} + x_{21}) r_1 \cdots r_{n-2}\big).$$

This results in $\big(a_{21}\big(z - (x_{11} + x_{21})\big)\big) r_1 \cdots r_{n-2} = 0$ which implies that

$$a_{21}\big(z - (x_{11} + x_{21})\big) = 0. \tag{2.8}$$

From identities (2.5)-(2.8) we obtain

$$\mathfrak{R}\big(z - (x_{11} + x_{21})\big) = 0.$$

By Theorem 2.1(ii) we conclude that $z = x_{11} + x_{21}$. Similarly, we prove the case $i = j = 2$ and $k = 1$ and its complement $i = k = 2$ and $j = 1$. □

In what follows, we are interested in proving that the mapping φ is additive on the spaces $\mathfrak{R}_{12}$ and $\mathfrak{R}_{11}$, as auxiliary steps in the proof of Theorem 2.1. Since φ acts on products of n elements of $\mathfrak{R}$, the results will be established through a succession of intermediate additivity results, obtained separately on $\mathfrak{R}_{12}$ and $\mathfrak{R}_{11}$, by means of a progressive reduction in the order of such products. The final results of the additivity of φ on each of these spaces will be given in the last step of each of these successive processes.

Lemma 2.5. $\varphi(x_{12} + y_{12}t_{22}) = \varphi(x_{12}) + \varphi(x_{12}t_{22})$.

Proof. First, note that the following identity holds

$$\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}}(e_1 + y_{12})(x_{12} + t_{22}) = x_{12} + y_{12}t_{22}.$$

Let i, j $(1 \leq i, j \leq n)$ be indexes such that $\sigma(i) = n-1$ and $\sigma(j) = n$. Two cases are considered. First case: $1 \leq \sigma^{-1}(n-1) < \sigma^{-1}(n) \leq n$. We split the first case into the following subcases: $1 < \sigma^{-1}(n-1) < \sigma^{-1}(n) < n$, $1 < \sigma^{-1}(n-1) < \sigma^{-1}(n) = n$, $1 = \sigma^{-1}(n-1) < \sigma^{-1}(n) < n$ and $1 = \sigma^{-1}(n-1) < \sigma^{-1}(n) = n$. In this first subcase, we have by Lemmas 2.2 and 2.4 that

$$
\begin{aligned}
&\varphi(x_{12} + y_{12}t_{22}) \\
=&\varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}}(e_1 + y_{12})(x_{12} + t_{22})) \\
=&\underbrace{\overbrace{\varphi(e_1) \cdots \varphi(e_1)\varphi(e_1 + y_{12})}^{\sigma^{-1}(n-1) \text{ terms}} \varphi(e_1) \cdots \varphi(e_1)\varphi(x_{12} + t_{22})}_{\sigma^{-1}(n) \text{ terms}} \varphi(e_1) \cdots \varphi(e_1) \\
=&\underbrace{\overbrace{\varphi(e_1) \cdots \varphi(e_1)\big(\varphi(e_1) + \varphi(y_{12})\big)}^{\sigma^{-1}(n-1) \text{ terms}} \varphi(e_1) \cdots \varphi(e_1)\big(\varphi(x_{12}) + \varphi(t_{22})\big)}_{\sigma^{-1}(n) \text{ terms}} \varphi(e_1) \cdots \varphi(e_1) \\
=&\underbrace{\overbrace{\varphi(e_1) \cdots \varphi(e_1)\varphi(e_1)}^{\sigma^{-1}(n-1) \text{ terms}} \varphi(e_1) \cdots \varphi(e_1)\varphi(x_{12})}_{\sigma^{-1}(n) \text{ terms}} \varphi(e_1) \cdots \varphi(e_1) \\
&+ \underbrace{\overbrace{\varphi(e_1) \cdots \varphi(e_1)\varphi(y_{12})}^{\sigma^{-1}(n-1) \text{ terms}} \varphi(e_1) \cdots \varphi(e_1)\varphi(x_{12})}_{\sigma^{-1}(n) \text{ terms}} \varphi(e_1) \cdots \varphi(e_1) \\
&+ \underbrace{\overbrace{\varphi(e_1) \cdots \varphi(e_1)\varphi(e_1)}^{\sigma^{-1}(n-1) \text{ terms}} \varphi(e_1) \cdots \varphi(e_1)\varphi(t_{22})}_{\sigma^{-1}(n) \text{ terms}} \varphi(e_1) \cdots \varphi(e_1) \\
&+ \underbrace{\overbrace{\varphi(e_1) \cdots \varphi(e_1)\varphi(y_{12})}^{\sigma^{-1}(n-1) \text{ terms}} \varphi(e_1) \cdots \varphi(e_1)\varphi(t_{22})}_{\sigma^{-1}(n) \text{ terms}} \varphi(e_1) \cdots \varphi(e_1) \\
=&\varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}} e_1 x_{12}) + \varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}} y_{12}x_{12}) + \varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}} e_1 t_{22}) + \varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}} y_{12}t_{22}) \\
=&\varphi(x_{12}) + \varphi(y_{12}t_{22}).
\end{aligned}
$$

The proofs of the remaining subcases are similar to the proof of first subcase and are therefore omitted. Second case: $1 \leq \sigma^{-1}(n) < \sigma^{-1}(n-1) \leq n$. The proof is entirely similar to that of the first case and is therefore omitted. Therefore, we have $\varphi(x_{12} + y_{12}t_{22}) = \varphi(x_{12}) + \varphi(x_{12}t_{22})$. □

Lemma 2.6. $\varphi(x_{11} + y_{12}t_{21}) = \varphi(x_{11}) + \varphi(x_{21}t_{21})$.

Proof. First, note that the following identity holds

$$\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}}(x_{11} + y_{12})(e_1 + t_{21}) = x_{11} + y_{12}t_{21}.$$

Let i, j $(1 \leq i, j \leq n)$ be indexes such that $\sigma(i) = n-1$ and $\sigma(j) = n$. Two cases are considered. First case: $1 \leq \sigma^{-1}(n-1) < \sigma^{-1}(n) \leq n$. We split the first case into the following subcases: $1 < \sigma^{-1}(n-1) < \sigma^{-1}(n) < n$, $1 < \sigma^{-1}(n-1) < \sigma^{-1}(n) = n$, $1 = \sigma^{-1}(n-1) < \sigma^{-1}(n) < n$ and $1 = \sigma^{-1}(n-1) < \sigma^{-1}(n) = n$. In this first subcase, we have by Lemmas 2.2 and 2.4 again that

$$\begin{aligned}
&\varphi(x_{11} + y_{12}t_{21}) \\
=&\varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}}(x_{11} + y_{12})(e_1 + t_{21})) \\
=&\underbrace{\overbrace{\varphi(e_1)\cdots\varphi(e_1)\varphi(x_{11} + y_{12})}^{\sigma^{-1}(n-1) \text{ terms}}\varphi(e_1)\cdots\varphi(e_1)\varphi(e_1 + t_{21})}_{\sigma^{-1}(n) \text{ terms}}\varphi(e_1)\cdots\varphi(e_1) \\
=&\underbrace{\overbrace{\varphi(e_1)\cdots\varphi(e_1)\big(\varphi(x_{11}) + \varphi(y_{12})\big)}^{\sigma^{-1}(n-1) \text{ terms}}\varphi(e_1)\cdots\varphi(e_1)\big(\varphi(e_1) + \varphi(t_{21})\big)}_{\sigma^{-1}(n) \text{ terms}}\varphi(e_1)\cdots\varphi(e_1) \\
=&\underbrace{\overbrace{\varphi(e_1)\cdots\varphi(e_1)\varphi(x_{11})}^{\sigma^{-1}(n-1) \text{ terms}}\varphi(e_1)\cdots\varphi(e_1)\varphi(e_1)}_{\sigma^{-1}(n) \text{ terms}}\varphi(e_1)\cdots\varphi(e_1) \\
&+\underbrace{\overbrace{\varphi(e_1)\cdots\varphi(e_1)\varphi(y_{12})}^{\sigma^{-1}(n-1) \text{ terms}}\varphi(e_1)\cdots\varphi(e_1)\varphi(e_1)}_{\sigma^{-1}(n) \text{ terms}}\varphi(e_1)\cdots\varphi(e_1) \\
&+\underbrace{\overbrace{\varphi(e_1)\cdots\varphi(e_1)\varphi(x_{11})}^{\sigma^{-1}(n-1) \text{ terms}}\varphi(e_1)\cdots\varphi(e_1)\varphi(t_{21})}_{\sigma^{-1}(n) \text{ terms}}\varphi(e_1)\cdots\varphi(e_1) \\
&+\underbrace{\overbrace{\varphi(e_1)\cdots\varphi(e_1)\varphi(y_{12})}^{\sigma^{-1}(n-1) \text{ terms}}\varphi(e_1)\cdots\varphi(e_1)\varphi(t_{21})}_{\sigma^{-1}(n) \text{ terms}}\varphi(e_1)\cdots\varphi(e_1) \\
=&\varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}}x_{11}e_1) + \varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}}y_{12}e_1) + \varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}}x_{11}t_{21}) + \varphi(\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}}y_{12}t_{21}) \\
=&\varphi(x_{11}) + \varphi(y_{12}t_{21}).
\end{aligned}$$

The proofs of the remaining subcases are similar to the proof of first subcase and are therefore omitted. Second case: $1 \leq \sigma^{-1}(n) < \sigma^{-1}(n-1) \leq n$. The proof is entirely similar to that of the first case and is therefore omitted. Therefore, we have $\varphi(x_{11} + y_{12}t_{21}) = \varphi(x_{11}) + \varphi(x_{21}t_{21})$. □

Lemma 2.7. *φ is additive on $\mathfrak{R}_{12}$.*

Proof. Choose an element $z \in \mathfrak{R}$ such that $\varphi(z) = \varphi(x_{12}) + \varphi(y_{12})$. Hence, for an arbitrary element $a_{11} \in \mathfrak{R}_{11}$ we have

$$\varphi(za_{11}) = \varphi(za_{11}\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}}) = \varphi(x_{12}a_{11}\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}}) + \varphi(y_{12}a_{11}\underbrace{e_1 \cdots e_1}_{n-2 \text{ terms}}) = 0.$$

This implies that $za_{11} = 0 = (x_{12} + x_{12})a_{11}$ which results in

$$\big(z - (x_{12} + x_{12})\big)a_{11} = 0. \tag{2.9}$$

Next, for an arbitrary element $a_{12} \in \mathfrak{R}_{12}$ we have

$$\varphi(za_{12}) = \varphi(z\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}a_{12}) = \varphi(x_{12}\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}a_{12}) + \varphi(y_{12}\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}a_{12})$$
$$= 0 = \varphi\big((x_{12}+y_{12})a_{12}\big),$$

by Lemma 2.3(v). This results in $za_{12} = (x_{12}+y_{12})a_{12}$ which implies that

$$\big(z-(x_{12}+y_{12})\big)a_{12} = 0. \tag{2.10}$$

Also, for an arbitrary element $a_{21} \in \mathfrak{R}_{21}$ we have

$$\varphi(za_{21}) = \varphi(za_{21}\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}) = \varphi(x_{12}a_{21}\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}) + \varphi(y_{12}a_{21}\underbrace{e_1\cdots e_1}_{n-2\text{ terms}})$$
$$= \varphi(x_{12}a_{21}+y_{12}a_{21}) = \varphi\big((x_{12}+y_{12})a_{21}\big),$$

by Lemma 2.6. Hence $za_{21} = (x_{12}+y_{12})a_{21}$ which implies that

$$\big(z-(x_{12}+y_{12})\big)a_{21} = 0. \tag{2.11}$$

Now, observe that $z = e_1 z$. Hence, for an arbitrary element $a_{22} \in \mathfrak{R}_{22}$ we have

$$\varphi(za_{22}) = \varphi(\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}za_{22}) = \varphi(\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}x_{12}a_{22}) + \varphi(\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}y_{12}a_{22})$$
$$= \varphi(x_{12}a_{22}) + \varphi(y_{12}a_{22}) = \varphi(x_{12}a_{22}+y_{12}a_{22}) = \varphi\big((x_{12}+y_{12})a_{22}\big),$$

by Lemma 2.5. It follows that $za_{22} = (y_{12}+y_{12})a_{22}$ which yields

$$\big(z-(y_{12}+y_{12})\big)a_{22} = 0. \tag{2.12}$$

From identities (2.9)-(2.12) we obtain

$$\big(z-(x_{12}+y_{12})\big)\mathfrak{R} = 0$$

which implies that $z = x_{12}+y_{12}$, by Theorem 2.1(i). □

Lemma 2.8. *φ is additive on $\mathfrak{R}_{11}$.*

Proof. Let $x_{11}, y_{11} \in \mathfrak{R}_{11}$ and $z \in \mathfrak{R}$ such that $\varphi(z) = \varphi(x_{11}) + \varphi(y_{11})$. Note that

$$\varphi(z_{11}) = \varphi(\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}ze_1) = \varphi(\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}x_{11}e_1) + \varphi(\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}y_{11}e_1) = \varphi(x_{11}) + \varphi(y_{11}).$$

This shows that $z = z_{11}$. Hence, for an arbitrary element $a_{12} \in \mathfrak{R}_{12}$ we have

$$\varphi(za_{12}) = \varphi(\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}za_{12}) = \varphi(\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}x_{11}a_{12}) + \varphi(\underbrace{e_1\cdots e_1}_{n-2\text{ terms}}y_{11}a_{12})$$
$$= \varphi(x_{11}a_{12}) + \varphi(y_{11}a_{12}) = \varphi(x_{11}a_{12}+y_{11}a_{12}) = \varphi\big((x_{11}+y_{11})a_{12}\big),$$

by Lemma 2.7. It follows that $za_{12} = (x_{11}+y_{11})a_{12}$ which implies that

$$e_1\big(z-(x_{11}+y_{11})\big)e_1\mathfrak{R}(1_{\mathfrak{R}}-e_1) = 0.$$

Therefore, $z = x_{11}+y_{11}$ by Theorem 2.1(iii). □

Lemma 2.9. *φ is additive on $e_1\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12}$.*

Proof. Let $x_{11}, y_{11} \in \mathfrak{R}_{11}$ and $x_{12}, y_{12} \in \mathfrak{R}_{12}$. By Lemmas 2.4, 2.7 and 2.8 we have $\varphi\big((x_{11} + x_{12}) + (y_{11} + y_{12})\big) = \varphi\big((x_{11} + y_{11}) + (x_{12} + y_{12})\big) = \varphi(x_{11} + y_{11}) + \varphi(x_{12} + y_{12}) = \varphi(x_{11}) + \varphi(y_{11}) + \varphi(x_{12}) + \varphi(y_{12}) = \varphi(x_{11} + x_{12}) + \varphi(y_{11} + y_{12})$. □

We are ready for proving our main theorem.

Proof of Theorem 2.1. Let x, y be arbitrary elements of $\mathfrak{R}$ and choose $z \in \mathfrak{R}$ such that $\varphi(z) = \varphi(x) + \varphi(y)$. For arbitrary elements e_α $(\alpha \in \Lambda)$ and $r \in \mathfrak{R}$, by Lemma 2.9 we see that

$$\varphi(e_\alpha rz) = \varphi(\underbrace{e_\alpha \cdots e_\alpha}_{n-2 \text{ terms}} rz) = \varphi(\underbrace{e_\alpha \cdots e_\alpha}_{n-2 \text{ terms}} rx) + \varphi(\underbrace{e_\alpha \cdots e_\alpha}_{n-2 \text{ terms}} ry)$$
$$= \varphi(e_\alpha rx) + \varphi(e_\alpha ry) = \varphi(e_\alpha rx + e_\alpha ry) = \varphi(e_\alpha r(x + y))$$

which implies that $e_\alpha rz = e_\alpha r(x + y)$. It therefore follows that

$$e_\alpha \mathfrak{R}\big(z - (x + y)\big) = 0,$$

for all $\alpha \in \Lambda$. By Theorem 2.1(ii) we conclude that $z = x + y$. □

Remark 2.10. The repeated use of conditions (i) and (ii) of Theorem 2.1, together with Lemma 2.3, to conclude that $\big(z - (x + y)\big)\mathfrak{R} = 0$ or $\mathfrak{R}\big(z - (x + y)\big) = 0$ implies $z = x + y$ (and possibly making use of condition (iii)), suggests that these techniques may be adapted to other types of mappings aimed at the study of additivity, which may indicate that these procedures are not an exclusive consequence of the n-multiplicative σ-isomorphisms.

Corollary 2.11. *Let $\mathfrak{R}$ be a prime associative ring containing a nontrivial idempotent. Then for each permutation σ in S_n the n-multiplicative σ-isomorphism φ of $\mathfrak{R}$ onto an arbitrary associative ring $\mathfrak{S}$ is additive.*

Corollary 2.12. *Let $\mathfrak{R}$ be a prime associative ring containing a nontrivial idempotent. Then any n-multiplicative anti-isomorphism φ of $\mathfrak{R}$ onto an arbitrary associative ring $\mathfrak{S}$ is additive.*

Let $\mathfrak{X}$ be a Banach space. By $\mathfrak{B}(\mathfrak{X})$ we denote the algebra of all bounded linear operators on $\mathfrak{X}$. A subalgebra of $\mathfrak{B}(\mathfrak{X})$ is called a standard operator algebra if it contains all finite rank operators. It is well known that every standard operator algebra is prime. If $\dim \mathfrak{X} \geq 2$, then clearly there exists a nontrivial idempotent operator of rank one in $\mathfrak{B}(\mathfrak{X})$. Therefore, the following corollaries are immediate consequences of Corollaries 2.11 and 2.12.

Corollary 2.13. *Let $\mathfrak{X}$ be a Banach space with* $\dim \mathfrak{X} \geq 2$. *Let $\mathfrak{A}$ be a standard operator algebra on $\mathfrak{X}$. Then for each permutation σ in S_n the n-multiplicative σ-isomorphism φ of $\mathfrak{A}$ onto an arbitrary associative ring $\mathfrak{S}$ is additive.*

Corollary 2.14. *Let $\mathfrak{X}$ be a Banach space with* $\dim \mathfrak{X} \geq 2$. *Let $\mathfrak{A}$ be a standard operator algebra on $\mathfrak{X}$. Then any n-multiplicative anti-isomorphism φ of $\mathfrak{A}$ onto an arbitrary associative ring $\mathfrak{S}$ is additive.*

3 Conclusion remarks

In this paper we presented a study of additivity for a class of mappings that is broader than the class of n-multiplicative isomorphisms, introduced by Wang [4], defined on associative rings satisfying the conditions of Martindale III [3] with values in arbitrary associative rings. This raises several relevant questions concerning possible extensions, such as: is it possible to extend this study by weakening the bijectivity hypothesis, or by altering the nature of the mapping (see, for example, [1] and [2])? Or by considering products more general than those of the monomial type of degree n presented by Wang? Alternatively, could one consider other hypotheses than those proposed by Martindale III?

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