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Trajectory Controllability of Non-Instantaneous Impulsive Dynamical Systems

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Abstract. This paper investigates the trajectory controllability (T-controllability) of evolution equations with non-instantaneous impulses of integer order. Using the theory of C_0-semigroups and Gronwall's inequality, sufficient conditions are derived to guarantee the existence of steering trajectories in a Banach space. The analysis considers both classical initial conditions and nonlocal conditions, providing a unified framework for systems subject to external influences acting over finite time intervals. The theoretical results are further supported by numerical examples illustrating the applicability of the controllability criteria under both local and nonlocal constraints.

1 Introduction

Impulsive differential equations play an important role in modeling dynamical systems that exhibit abrupt changes over time. Such systems arise naturally in physical, biological, and engineering processes. When the changes occur at isolated time instants, the system is described by instantaneous impulsive differential equations, whereas impulses acting over finite time intervals lead to non-instantaneous impulsive differential equations. Recent developments in impulsive evolution systems and their qualitative analysis can be found in [1, 2, 3, 4]. These studies highlight the importance of existence, uniqueness, and stability analysis for evolution equations with both classical and nonlocal conditions.

Controllability is a fundamental qualitative property in control theory. It concerns the ability to steer a system from a given initial state to a desired final state using suitable control inputs. In infinite-dimensional dynamical systems, controllability has been widely studied using semigroup theory and functional analytic methods. Classical developments and trajectory-based approaches can be found in [7, 8, 9, 10, 11]. More recently, controllability of semilinear generalized impulsive systems on finite-dimensional spaces [13] and exact controllability of fractional impulsive systems [14] have been investigated, highlighting the ongoing development in practical and fractional impulsive systems. Related studies by Ramos et al. [15] and Tamilarasi et al. [16] have also examined controllability for fractional impulsive integro-differential systems and approximate controllability of fractional semi-linear delay differential systems with random impulses, demonstrating broader interest in impulsive systems under diverse frameworks. However, exact controllability may be difficult to achieve or may require excessive control effort in many applications.

Trajectory controllability provides a flexible alternative by allowing the system to follow a prescribed trajectory instead of reaching an exact final state. This concept was introduced for nonlinear systems by George [8] and later extended to infinite-dimensional systems using semigroup methods and fixed point techniques [9]. The trajectory controllability of semilinear systems with non-instantaneous impulses under classical and nonlocal conditions has also been investigated in [12].

Motivated by these developments, this paper studies the trajectory controllability of the following non-instantaneous impulsive evolution system:

$$x'(\tau) = A(\tau)x(\tau) + F\left(\tau, x(\tau), \int_0^\tau h(s, x(s))\, ds\right) + W_1(\tau), \qquad \tau \in [s_k, t_{k+1}), \quad k = 0, 1, \ldots, p,$$
$$x'(\tau) = A(\tau)x(\tau) + G(\tau, x(\tau)) + W_2(\tau), \qquad \tau \in [t_k, s_k), \quad k = 1, 2, \ldots, p, \tag{1.1}$$

subject to the classical initial condition $x(0) = x_0$ and the nonlocal condition $x(0) = x_0 - h(x)$.

The remainder of the paper is organized as follows:

- Section 2 presents the necessary preliminaries and definitions related to controllability.
- Section 3 establishes trajectory controllability results for systems with classical initial conditions.
- Section 4 discusses trajectory controllability under nonlocal conditions.
- Section 5 provides concluding remarks.

2 Preliminaries

This section presents basic definitions and preliminary concepts required to study the trajectory controllability of non-instantaneous impulsive evolution systems with both classical and nonlocal conditions. The standard concepts of controllability used here are adapted from the control theory literature; see [7, 8, 9, 18, 19].

Definition 2.1. The system (1.1) is said to be *completely controllable* on the interval $\mathcal{J} = [0, T_0]$ if for any initial state $x_0 \in \mathcal{X}$ and any final state $x_1 \in \mathcal{X}$, there exists a control function $W(\cdot) \in \mathcal{U}$ such that the corresponding state $x(\tau)$ satisfies

$$x(0) = x_0 \quad \text{and} \quad x(T_0) = x_1.$$

Complete controllability guarantees that the system can be transferred between arbitrary states, but it does not specify the path followed by the state trajectory. In many practical situations, following a prescribed trajectory is more desirable because it may reduce control effort or improve system performance. This leads to the concept of trajectory controllability.

Definition 2.2. Let $\mathcal{C}_\mathcal{T}$ denote the set of admissible trajectories connecting x_0 to x_1 over the interval $\mathcal{J}$. The system (1.1) is said to be *trajectory controllable* if for any prescribed trajectory $z(\cdot) \in \mathcal{C}_\mathcal{T}$, there exists a control function $W(\cdot) \in \mathcal{U}$ such that the corresponding state $x(\tau)$ satisfies

$$x(\tau) = z(\tau) \quad \text{a.e. on } \mathcal{J}.$$

3 T-controllability with classical conditions

In this section, we study the trajectory controllability of a non-instantaneous impulsive evolution system with classical initial conditions. Consider the following system:

$$\begin{aligned} x'(\tau) &= A(\tau)x(\tau) + F\left(\tau, x(\tau), \int_0^\tau h(s, x(s))\, ds\right) + W_1(\tau), \quad \tau \in [s_k, t_{k+1}), \\ x'(\tau) &= A(\tau)x(\tau) + G(\tau, x(\tau)) + W_2(\tau), \quad \tau \in [t_k, s_k), \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

defined on the interval $[0, T_0]$.

Here, $x(\tau)$ denotes the state of the system at time $\tau \in [0, T_0]$ taking values in a Banach space $\mathcal{X}$. The family of linear operators $A(\tau)$ acts on $\mathcal{X}$, while F, G, and h represent nonlinear mappings defined on appropriate Banach spaces. The control functions $W_1(\cdot)$ and $W_2(\cdot)$ belong to the admissible control space.

To establish trajectory controllability for the system (3.1), we present the following main result, which is developed using semigroup techniques and standard controllability arguments; see [8, 9, 12].

Theorem 3.1. *If,*

(A1) The linear operator A in the system (3.1) is the infinitesimal generator of a C_0-semigroup.

(A2) The nonlinear map $F : [0, T_0] \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is continuous. There exist non-decreasing functions $l_F, l_h : \mathbb{R}_+ \to \mathbb{R}_+$ and a positive real number r_0 such that

$$\|F(\tau, x_1, y_1) - F(\tau, x_2, y_2)\| \leq l_F(r)\|x_1 - x_2\| + l_h(r)\|y_1 - y_2\|,$$

for all $\tau \in [0, T_0]$, $x_1, x_2, y_1, y_2 \in B_r(\mathcal{X})$ and $r \leq r_0$.

(A3) The nonlinear function h is continuous with respect to τ and there exists a non-decreasing function $m_h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|h(\tau, x_1) - h(\tau, x_2)\| \leq m_h(r)\|x_1 - x_2\|,$$

for all $\tau \in [0, T_0]$, $x_1, x_2 \in B_r(\mathcal{X})$ and $r \leq r_0$.

(A4) The nonlinear map $G : [0, T_0] \times \mathcal{X} \to \mathcal{X}$ is continuous and there exists a non-decreasing function $l_G : \mathbb{R}_+ \to \mathbb{R}_+$ and a positive real number r_0 such that

$$\|G(\tau, x_1) - G(\tau, x_2)\| \leq l_G(r)\|x_1 - x_2\|,$$

for all $\tau \in [0, T_0]$, $x_1, x_2 \in B_r(\mathcal{X})$ and $r \leq r_0$.

Then, the system (3.1) is trajectory controllable over the interval $[0, T_0]$.

Proof. Let $u(\tau)$ be any trajectory in $\mathcal{C}_\mathcal{T}$ satisfying $x(t_k^+) = u(t_k^+)$ along which the system (3.1) is steered from the initial state x_0 at $\tau = 0$ to the desired final state x_1 at $\tau = T_0$.

Over the interval $[0, t_1)$, the system (3.1) becomes

$$\begin{aligned} x'(\tau) &= A(\tau)x(\tau) + F\left(\tau, x(\tau), \int_0^\tau h(s, x(s))\, ds\right) + W_1(\tau), \\ x(0) &= x_0. \end{aligned} \tag{3.2}$$

Consider

$$W_1(\tau) = u'(\tau) - A(\tau)u(\tau) - F\left(\tau, u(\tau), \int_0^\tau h(s, u(s))\, ds\right)$$

over $[0, t_1)$. Substituting in (3.2) gives

$$x'(\tau) = A(\tau)x(\tau) + F\left(\tau, x(\tau), \int_0^\tau h(s, x(s))\, ds\right) + u'(\tau) - A(\tau)u(\tau) - F\left(\tau, u(\tau), \int_0^\tau h(s, u(s))\, ds\right),$$

with $x(0) - u(0) = 0$.

Let $z(\tau) = x(\tau) - u(\tau)$. Then

$$\begin{aligned} z'(\tau) &= A(\tau)z(\tau) + F\left(\tau, x(\tau), \int_0^\tau h(s, x(s))\, ds\right) - F\left(\tau, u(\tau), \int_0^\tau h(s, u(s))\, ds\right), \\ z(0) &= 0. \end{aligned} \tag{3.3}$$

The mild solution is

$$z(\tau) = \int_0^\tau \mathcal{T}(\tau - \zeta)\left[F\left(\zeta, x(\zeta), \int_0^\zeta h(s, x(s))\, ds\right) - F\left(\zeta, u(\zeta), \int_0^\zeta h(s, u(s))\, ds\right)\right] d\zeta, \tag{3.4}$$

where $\mathcal{T}(\tau)$ is the C_0-semigroup generated by A with $\|\mathcal{T}(\tau)\| \leq M$.

Therefore,

$$\begin{aligned} \|z(\tau)\| &\leq M \int_0^\tau \Big[l_F(r)\|x(\zeta) - u(\zeta)\| + l_h(r)m_h(r)\tau\|x(\zeta) - u(\zeta)\|\Big] d\zeta \\ &\leq M \int_0^\tau (l_F(r) + l_h(r)m_h(r)\tau)\|z(\zeta)\| d\zeta. \end{aligned}$$

By Gronwall's inequality, $z(\tau) = 0$ on $[0, t_1)$; hence $x(\tau) = u(\tau)$.

Over $[t_1, s_1)$,

$$\begin{aligned} x'(\tau) &= A(\tau)x(\tau) + G(\tau, x(\tau)) + W_2(\tau), \\ x(t_1) &= u(t_1). \end{aligned} \tag{3.5}$$

Choose

$$W_2(\tau) = u'(\tau) - A(\tau)u(\tau) - G(\tau, u(\tau)).$$

Let $z(\tau) = x(\tau) - u(\tau)$; then

$$z'(\tau) = A(\tau)z(\tau) + G(\tau, x(\tau)) - G(\tau, u(\tau)).$$

Thus

$$\|z(\tau)\| \leq M \int_{t_1}^{\tau} l_G(r)\|z(\zeta)\|d\zeta.$$

Using Gronwall's inequality, $z(\tau) = 0$ on $[t_1, s_1)$.

Continuing over $[s_k, t_{k+1})$,

$$\begin{aligned} x'(\tau) &= A(\tau)x(\tau) + F\left(\tau, x(\tau), \int_0^{\tau} h(s, x(s))\, ds\right) + W_1(\tau), \\ x(s_k) &= u(s_k). \end{aligned} \tag{3.6}$$

Take

$$W_1(\tau) = u'(\tau) - A(\tau)u(\tau) - F\left(\tau, u(\tau), \int_0^{\tau} h(s, u(s))\, ds\right).$$

Let $z(\tau) = x(\tau) - u(\tau)$; then

$$\begin{aligned} z'(\tau) &= A(\tau)z(\tau) + F\left(\tau, x(\tau), \int_0^{\tau} h(s, x(s))\, ds\right) - F\left(\tau, u(\tau), \int_0^{\tau} h(s, u(s))\, ds\right), \\ z(s_k) &= 0. \end{aligned} \tag{3.7}$$

Hence

$$\|z(\tau)\| \leq M \int_{s_k}^{\tau} (l_F(r) + l_h(r)m_h(r)\tau)\|z(\zeta)\|d\zeta,$$

and by Gronwall's inequality $z(\tau) = 0$ on $[s_k, t_{k+1})$.

Over $[t_k, s_k)$,

$$\begin{aligned} x'(\tau) &= A(\tau)x(\tau) + G(\tau, x(\tau)) + W_2(\tau), \\ x(t_k) &= u(t_k). \end{aligned} \tag{3.8}$$

Choose

$$W_2(\tau) = u'(\tau) - A(\tau)u(\tau) - G(\tau, u(\tau)).$$

Then

$$z'(\tau) = A(\tau)z(\tau) + G(\tau, x(\tau)) - G(\tau, u(\tau)),$$

and

$$\|z(\tau)\| \leq M \int_{t_k}^{\tau} l_G(r)\|z(\zeta)\|d\zeta.$$

Using Gronwall's inequality gives $z(\tau) = 0$ on $[t_k, s_k)$.

Since the system is trajectory controllable on all subintervals, it is trajectory controllable on $[0, T_0]$. This completes the proof. □

Example 3.2. Let $\mathcal{X} = L^2([0, \pi], \mathbb{R})$ and consider the system governed by a non-instantaneous impulsive evolution equation:

$$\begin{aligned} \frac{\partial H(\tau, \psi)}{\partial \tau} &= \partial_{\psi}^2 H(\tau, \psi) + F(\tau, H(\tau, \psi)) + w_1(\tau, \psi), && \tau \in [0, 1/3) \cup [2/3, 1], \\ \frac{\partial H(\tau, \psi)}{\partial \tau} &= \partial_{\psi}^2 H(\tau, \psi) + G(\tau, H(\tau, \psi)) + w_2(\tau, \psi), && \tau \in [1/3, 2/3), \\ H(\tau, 0) &= 0, \quad H(\tau, \pi) = 0, && \tau > 0, \\ H(0, \psi) &= H_0(\psi), && 0 < \psi < \pi, \end{aligned} \tag{3.9}$$

over the interval $[0, 1]$.

Define the operator on the space $\mathcal{X}$ by $A = \partial_\psi^2$ with domain $D(A) = \{z \in H^2(0, \pi) \cap H_0^1(0, \pi)\}$. Then A is the infinitesimal generator of a C_0-semigroup $\mathcal{T}(\tau)$. The representation of $\mathcal{T}(\tau)$ is

$$\mathcal{T}(\tau)z = \sum_{m=1}^{\infty} e^{\mu_m \tau} \langle z, \phi_m \rangle \phi_m,$$

where $\phi_m(\psi) = \sqrt{2}\sin(m\psi)$ for all $m = 1, 2, \cdots$ forms an orthonormal basis corresponding to eigenvalues $\mu_m = -m^2$ of the operator A.

Using this formulation, the equation (3.9) can be rewritten as an abstract equation on the space $\mathcal{X}$:

$$\begin{aligned} x'(\tau) &= Ax(\tau) + F(\tau, x(\tau)) + W_1(\tau), \quad \tau \in [0, 1/3) \cup [2/3, 1], \\ x'(\tau) &= Ax(\tau) + G(\tau, x(\tau)) + W_2(\tau), \quad \tau \in [1/3, 2/3), \\ x(0) &= x_0, \end{aligned} \tag{3.10}$$

where $x(\tau) = H(\tau, \cdot)$, $W_1(\tau) = w_1(\tau, \cdot)$ and $W_2(\tau) = w_2(\tau, \cdot)$.

The system (3.10) is trajectory controllable over the interval $[0, 1]$ if the nonlinear functions F and G satisfy the hypotheses of the theorem.

4 T-controllability with non-local conditions

Consider the system governed by the non-instantaneous impulsive evolution equation

$$\begin{aligned} x'(\tau) &= A(\tau)x(\tau) + F\left(\tau, x(\tau), \int_0^\tau h(s, x(s))\, ds\right) + W_1(\tau), \quad \tau \in [s_k, t_{k+1}), \\ x'(\tau) &= A(\tau)x(\tau) + G(\tau, x(\tau)) + W_2(\tau), \quad \tau \in [t_k, s_k), \\ x(0) &= p(x), \end{aligned} \tag{4.1}$$

over the interval $[0, T_0]$. Here, $x(\tau)$ denotes the state of the system at time $\tau \in [0, T_0]$ taking values in a Banach space $\mathcal{X}$. The operator $A(\tau)$ is a linear operator on $\mathcal{X}$. The mappings F, G, and h are nonlinear functions defined on appropriate Banach spaces, and $p : \mathcal{X} \to \mathcal{X}$ represents the non-local condition.

The mild solution of the equation (4.1) is given by

$$x(\tau) = \begin{cases} \mathcal{T}(\tau)p(x) + \displaystyle\int_0^\tau \mathcal{T}(\tau - \zeta)\Big[F\Big(\zeta, x(\zeta), \int_0^\zeta h(s, x(s))\, ds\Big) + W_1(\zeta)\Big] d\zeta, & \tau \in [0, t_1), \\ \mathcal{T}(\tau - t_k)x(t_k) + \displaystyle\int_{t_k}^\tau \mathcal{T}(\tau - \zeta)\big[G(\zeta, x(\zeta)) + W_2(\zeta)\big] d\zeta, & \tau \in [t_k, s_k), \\ \mathcal{T}(\tau - s_k)x(s_k) + \displaystyle\int_{s_k}^\tau \mathcal{T}(\tau - \zeta)\Big[F\Big(\zeta, x(\zeta), \int_0^\zeta h(s, x(s))\, ds\Big) + W_1(\zeta)\Big] d\zeta, & \tau \in [s_k, t_{k+1}), \end{cases} \tag{4.2}$$

where $\mathcal{T}(\tau)$ denotes the C_0-semigroup generated by the linear operator A.

The following theorem discusses the trajectory controllability of the system governed by the equation (4.1).

Theorem 4.1. *If,*

(A1) The linear operator A in the system (4.1) is the infinitesimal generator of a C_0 semigroup.

(A2) The nonlinear map $F : [0, T_0] \times \mathcal{X} \to \mathcal{X}$ is continuous such that there exist non-decreasing functions l_F and l_h from $\mathbb{R}_+ \to \mathbb{R}_+$ and a positive real number r_0 satisfying

$$\|F(\tau, x_1, y_1) - F(\tau, x_2, y_2)\| \le l_F(r)\|x_1 - x_2\| + l_h(r)\|y_1 - y_2\|,$$

for all $\tau \in [0, T_0]$, $x_1, x_2, y_1, y_2 \in B_r(\mathcal{X})$ and $r \le r_0$.

(A3) The nonlinear function h is continuous with respect to τ and there exists a non-decreasing function $m_h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|h(\tau, x_1) - h(\tau, x_2)\| \leq m_h(r)\|x_1 - x_2\|$$

for all $\tau \in [0, T_0]$, $x_1, x_2 \in B_r(\mathcal{X})$ and $r \leq r_0$.

(A4) The nonlinear map $G : [0, T_0] \times \mathcal{X} \to \mathcal{X}$ is continuous such that there exists a non-decreasing function $l_G : \mathbb{R}_+ \to \mathbb{R}_+$ and a positive real number r_0 satisfying

$$\|G(\tau, x_1) - G(\tau, x_2)\| \leq l_G(r)\|x_1 - x_2\|,$$

for all $\tau \in [0, T_0]$, $x_1, x_2 \in B_r(\mathcal{X})$ and $r \leq r_0$.

(A5) The function $p : \mathcal{X} \to \mathcal{X}$ is Lipschitz continuous with Lipschitz constant $0 \leq l_p \leq 1$.

Then the system (4.1) is trajectory controllable over the interval $[0, T_0]$.

Proof. Let $u(\tau)$ be any trajectory in $\mathcal{C}_\tau$ satisfying $x(t_k^+) = u(t_k^+)$ along which the system (4.1) is steered from the initial state $x(0) = p(x)$ at $\tau = 0$ to the desired final state x_1 at $\tau = T_0$.

Over the interval $[0, t_1)$, the system (4.1) becomes

$$\begin{aligned} x'(\tau) &= A(\tau)x(\tau) + F\left(\tau, x(\tau), \int_0^\tau h(s, x(s))ds\right) + W(\tau), \\ x(0) &= p(x). \end{aligned} \tag{4.3}$$

Consider

$$W(\tau) = u'(\tau) - A(\tau)u(\tau) - F\left(\tau, u(\tau), \int_0^\tau h(s, u(s))ds\right),$$

over $[0, t_1)$. Substituting into (4.3), we obtain

$$x'(\tau) = A(\tau)x(\tau) + F\left(\tau, x(\tau), \int_0^\tau h(s, x(s))ds\right) + u'(\tau) - A(\tau)u(\tau) - F\left(\tau, u(\tau), \int_0^\tau h(s, u(s))ds\right),$$

with initial condition $x(0) - u(0) = p(x) - p(u)$.

Let $z = x - u$. Then

$$\begin{aligned} z'(\tau) &= A(\tau)z(\tau) + F\left(\tau, x(\tau), \int_0^\tau h(s, x(s))ds\right) - F\left(\tau, u(\tau), \int_0^\tau h(s, u(s))ds\right), \\ z(0) &= p(x) - p(u). \end{aligned} \tag{4.4}$$

The mild solution is

$$z(\tau) = \mathcal{T}(\tau)[p(x) - p(u)] + \int_0^\tau \mathcal{T}(\tau - \zeta)\left[F\left(\zeta, x(\zeta), \int_0^\zeta h(s, x(s))ds\right) - F\left(\zeta, u(\zeta), \int_0^\zeta h(s, u(s))ds\right)\right]d\zeta. \tag{4.5}$$

Since $\|\mathcal{T}(\tau)\| \leq M$, we obtain

$$\|z(\tau)\| \leq Ml_p\|z(\tau)\| + M\int_0^\tau (l_F(r) + l_h(r)m_h(r)\tau)\|z(\zeta)\|d\zeta.$$

Hence

$$\|z(\tau)\| \leq \frac{(l_F(r) + l_h(r)m_h(r)\tau)}{1 - Ml_p}\int_0^\tau \|z(\zeta)\|d\zeta.$$

By Gronwall's inequality, $z(\tau) = 0$ on $[0, t_1)$. Thus $x(\tau) = u(\tau)$ and the system is trajectory controllable on $[0, t_1)$.

Over $[t_1, s_1)$,

$$\begin{aligned} x'(\tau) &= A(\tau)x(\tau) + G(\tau, x(\tau)) + W_2(\tau), \\ x(t_1) &= u(t_1). \end{aligned} \tag{4.6}$$

Choosing $W_2(\tau) = u'(\tau) - A(\tau)u(\tau) - G(\tau, u(\tau))$ and letting $z = x - u$, we get

$$z'(\tau) = A(\tau)z(\tau) + G(\tau, x(\tau)) - G(\tau, u(\tau)).$$

Then

$$\|z(\tau)\| \leq M \int_{t_1}^{\tau} l_G(r)\|z(\zeta)\|d\zeta.$$

By Gronwall's inequality, $z(\tau) = 0$ on $[t_1, s_1)$.
Continuing similarly over $[s_k, t_{k+1})$,

$$\begin{aligned} x'(\tau) &= A(\tau)x(\tau) + F\left(\tau, x(\tau), \int_0^{\tau} h(s, x(s))ds\right) + W_1(\tau), \\ x(s_k) &= u(s_k). \end{aligned} \tag{4.7}$$

With

$$W_1(\tau) = u'(\tau) - A(\tau)u(\tau) - F\left(\tau, u(\tau), \int_0^{\tau} h(s, u(s))ds\right),$$

we obtain

$$\begin{aligned} z'(\tau) &= A(\tau)z(\tau) + F\left(\tau, x(\tau), \int_0^{\tau} h(s, x(s))ds\right) - F\left(\tau, u(\tau), \int_0^{\tau} h(s, u(s))ds\right), \\ z(s_k) &= 0. \end{aligned} \tag{4.8}$$

Hence

$$\|z(\tau)\| \leq M \int_{s_k}^{\tau} (l_F(r) + l_h(r)m_h(r)\tau)\|z(\zeta)\|d\zeta,$$

and by Gronwall's inequality $z(\tau) = 0$.
Similarly over $[t_k, s_k)$,

$$\begin{aligned} x'(\tau) &= A(\tau)x(\tau) + G(\tau, x(\tau)) + W_2(\tau), \\ x(t_k) &= u(t_k). \end{aligned} \tag{4.9}$$

Proceeding as above yields $z(\tau) = 0$. Therefore, the system is T-controllable on each subinterval, and hence on $[0, T_0]$. □

Example 4.2. Let $\mathcal{X} = L^2([0, \pi], \mathbb{R})$ and consider the system

$$\begin{aligned} \frac{\partial H(\tau, \psi)}{\partial \tau} &= \partial_{\psi}^2 H(\tau, \psi) + F(\tau, H(\tau, \psi)) + w_1(\tau, \psi), && \tau \in \left[0, \tfrac{1}{3}\right) \cup \left[\tfrac{2}{3}, 1\right], \\ \frac{\partial H(\tau, \psi)}{\partial \tau} &= \partial_{\psi}^2 H(\tau, \psi) + G(\tau, H(\tau, \psi)) + w_2(\tau, \psi), && \tau \in \left[\tfrac{1}{3}, \tfrac{2}{3}\right), \\ H(\tau, 0) &= 0, \quad H(\tau, \pi) = 0, && \tau > 0, \\ H(0, \psi) &= H(\tau, \psi), && 0 < \psi < \pi. \end{aligned} \tag{4.10}$$

Here the nonlocal initial condition is given by

$$H(0, \psi) = \sum_{i=1}^{n} \alpha_i H(\tau_i, \psi).$$

Define $A = \partial_{\psi}^2$ with domain $D(A) = H^2(0, \pi) \cap H_0^1(0, \pi)$. Then A generates a C_0-semigroup $\mathcal{T}(\tau)$ on $\mathcal{X}$ defined by

$$\mathcal{T}(\tau)z = \sum_{m=1}^{\infty} e^{\mu_m \tau} \langle z, \phi_m \rangle \phi_m,$$

where $\phi_m = \sqrt{2}\sin(m\psi)$ and $\mu_m = -m^2$.

Consequently, system (4.10) can be rewritten in the abstract form

$$\begin{aligned} x'(\tau) &= Ax(\tau) + F(\tau, x(\tau)) + W_1(\tau), \quad \tau \in \left[0, \tfrac{1}{3}\right) \cup \left[\tfrac{2}{3}, 1\right], \\ x'(\tau) &= Ax(\tau) + G(\tau, x(\tau)) + W_2(\tau), \quad \tau \in \left[\tfrac{1}{3}, \tfrac{2}{3}\right), \\ x(0) &= p(x), \end{aligned} \tag{4.11}$$

where $x(\tau) = H(\tau, \cdot)$, $W_1(\tau) = w_1(\tau, \cdot)$, $W_2(\tau) = w_2(\tau, \cdot)$, and

$$p(x) = \sum_{i=1}^{n} \alpha_i x(\tau_i).$$

The system (4.11) is trajectory controllable on $[0, 1]$ provided that F, G, and p satisfy the hypotheses of the main theorem.

5 Conclusion

This article examined the trajectory controllability of systems governed by non-instantaneous impulsive evolution equations involving distinct perturbing forces F and G, under both classical and non-local conditions in a Banach space framework. By employing semigroup theory and Gronwall's inequality, sufficient conditions ensuring trajectory controllability were established. The obtained results extend existing controllability studies by incorporating non-instantaneous impulses together with non-local initial conditions, thereby providing a more flexible and realistic modelling framework for dynamical systems subject to time-distributed disturbances.

Illustrative examples were presented to verify the applicability and effectiveness of the theoretical results. The developed approach may be useful in the analysis and control of various physical and engineering processes where system dynamics are influenced by interval-based perturbations. Future work may focus on extending these results to fractional-order systems, stochastic models, and systems with state-dependent delays.

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