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Solvability for some weakly singular integral equations via Petryshyn's fixed-point theorem

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Abstract In this article, we study the solvability of weakly singular integral equations utilizing the techniques of measure of noncompactness and the Petryshyn's fixed point theorem in Banach algebra. Furthermore, we have examined our results with the help of example.

1 Introduction

Fractional calculus is widely used in different fields related to biological sciences, modern economics, physics, chemistry, biology, mathematical modeling, control theory, and medicine. The theory of fractional calculus generalizes the concepts of classical calculus from integer order to non-integer as well as complex order (cf.[28]). Many authors discuss the theory and development related to applications and have also provided different definitions for more accuracy and applicability of fractional derivatives and integrals based on singularities of the kernel (cf.[27, 30]. Lately, two classes of fractional derivatives have been designed in the literature to improve the quality of the fractional derivatives, including the singular and nonsingular kernel in the definition of the fractional order derivative (cf.[18]). On the other hand, the theory of integral equations is rapidly developing with the help of additional studies of fixed point theory, topology, and functional analysis (cf. [4, 5, 13, 14, 15, 19, 25, 26]. Weakly singular integrals are considered improper integrals. In contrast, singular integrals are interpreted in Cauchy as principal values, and hypersingular integrals are considered in Hadamard. Generally, singular integrals are classified based on the strength of the singularity in the integrand, with weakly singular integrals being one category. The treatment of weakly singular integrals falls into one of three categories: polar and modified polar approaches, Duffy or triangular coordinates, and extrapolation methods [1]. This paper is dedicated to studying the following equation

$$z(\varphi) = q\left(\varphi, f(\varphi, z(\beta(\varphi))), \int_0^{\varphi} g(\varphi,\nu,z(\zeta(\nu)))d\nu, \int_0^{\varphi} \ln|\nu-\varphi| h\big(\varphi,\nu,z(\gamma(\nu))\big)d\nu\right), \quad (1.1)$$

for all $\varphi \in I_b = [0,b]$. The functions $q \in C(I_b \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f \in C(I_b \times \mathbb{R}, \mathbb{R})$, $g, h \in C(I_b \times I_b \times \mathbb{R}, \mathbb{R})$ and $\beta, \zeta, \gamma : I_b \to I_b$, are continuous functions.
Equation (1.1) includes many particular cases; (see [3, 29]). The existence result of integral equations has been studied in many papers by using Darbo's fixed point theorem and Petryshyn's fixed point theorem in different function spaces (cf.[6, 7, 17, 22, 23, 24, 31, 32, 34]). Here, we deal with a class of fractional-order integral equations and give an effect on their solvability inside their domain of definition by Petryshyn's fixed point theorem. The idea of using Petryshyn's fixed point theorem in order to study the solvability of functional integral equations for the first

time was proposed by Kazemi and Ezzati [20]. The following statement describes the main reasons why we use equation [20] and the excellence of our work. The first aim is to simplify the conditions used by different authors in many papers. The second reason is that this paper summarises the relevant work in this field. The third reason is that the bounded condition describes that the "sub-linear condition" that has been managed in the literature (cf.[8, 9, 10])does not play a significant role in realising the solvability of functional integral equations. Finally, we do some special cases and examples that show the utilisation of fractional integral equations. "

1.1 Motivation

The result in this article is inspired by the wish to generalize the work introduced by Maleknejad et. al.[29]. They operated Darbo's fixed point theorem associated with measure of noncompactness for nonlinear integral equations and got the existence result about the solutions. In this paper, we have generalized their work in the form of fractional integral equations and obtained some results about the solvability in Banach algebra. We have released the restriction on the bounded condition explains the "sub-linear condition".

1.2 Outline

The paper is categorized into four sections including the introduction. In Section 1, a short introduction linked to the importance of literature on the fractional calculus and integral equations are presented. In Section 2, we show some preliminaries and defines about the concept of measure of noncompactness. In section 3, we prove solvability including condensing operators via Petryshyn's fixed point theorem and we test the utilization of this kind of fractional integral equations.

2 Preliminaries

In this paper, we have some notations as
• F : Banach space,
• $\bar{B}_r$: Closed ball at center 0 with radius r,
• $\partial\bar{B}_r$: Sphere in F around 0 with radius $r > 0$.

Definition 2.1. [2] Let E is a bounded subset of a Banach space F, and

$$\alpha(E) = \inf\left\{\sigma > 0 : E = \bigcup_{i=1}^{n} E_i \text{ with diam } E_i \leq \sigma,\ i = 1, 2, ..., n\right\}$$

is called the Kuratowski measure of noncompactness.

Definition 2.2. [2] The Hausdorff measure of noncompactness is defined as

$$\psi(E) = \inf\{\sigma > 0 : \textit{there exists} \text{ a finite } \sigma \text{ net for E in } F\}. \tag{2.1}$$

These measure of noncompactness are mutually alike as follows

$$\psi(E) \leq \alpha(E) \leq 2\psi(E)$$

for any bounded set $E \subset F$.

Theorem 2.3. *[36] Let $E, \bar{E} \in F$ and $\lambda \in \mathbb{R}$. Then*

(i) $\psi(E) = 0$ *if and only if E is pre-compact;*

(ii) $E \subseteq \bar{E} \implies \psi(E) \leq \psi(\bar{E})$*;*

(iii) $\psi(ConvE) = \psi(E)$*;*

(iv) $\psi(E \cup \bar{E}) = \max\{\psi(E), \psi(\bar{E})\}$*;*

(v) $\psi(\lambda E) = |\lambda|\psi(E)$*, where* $\lambda E = \{\lambda z : z \in E\}$*;*

(vi) $\psi(E+\bar{E}) \leq \psi(E) + \psi(\bar{E})$.

In what follows, we focus on the Banach algebra $F = C([0,b],\mathbb{R})$ consisting of all real-valued functions and continuous on the interval $I_b = [0,b]$. The space $C[0,b]$ is equipped with the usual norm

$$\|z\| = \sup\{|z(\varphi)| : \varphi \in [0,b]\}.$$

The modulus of continuity of $z \in C[0,b]$ is defined as

$$\omega_0(E) = \lim_{\sigma\to 0} \omega(E,\sigma)$$

such that

$$\omega(z,\sigma) = \sup\{|z(\varphi) - z(\bar{\varphi})| : \varphi, \bar{\varphi} \in [0,b], |\varphi - \bar{\varphi}| \leq \sigma\}$$

and

$$\omega(E,\sigma) = \sup\{\omega(z,\sigma) : z \in E\}.$$

Theorem 2.4. *[20] The Hausdorff measure of noncompactness (2.1) is equivalent to*

$$\psi(E) = \lim_{\sigma\to 0} \sup_{z\in E} \omega(z,\sigma) \tag{2.2}$$

for all bounded sets $E \subset C[0,b]$.

Definition 2.5. [35] Assume that $T : F \to F$ is a continuous mapping on F. T is called a k-set contraction if for all $E \subset F$ with E bounded, $T(E)$ bounded

$$\alpha(TE) \leq k\alpha(E), \text{ for } k \in (0,1).$$

Moreover, if

$$\alpha(TE) < \alpha(E), \ \forall\, \alpha(E) > 0,$$

then T is called condensing map.

Theorem 2.6. *[36] Let $T : \bar{B}_r \to F$ be a condensing mapping which satisfies the boundary condition,*

$$\textit{if } T(z) = kz, \quad \textit{for some } z \in \partial\bar{B}_r \textit{ then } k \leq 1.$$

Then the set of fixed points in $\bar{B}_r$ is non-empty. This is called Petryshyn's fixed point theorem.

3 Main Results

Now, we study the Eq.(1.1) under the following assumptions;

(1) $q \in C(I_b \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f \in C(I_b \times \mathbb{R}, \mathbb{R})$, $g, h \in C(I_b \times I_b \times \mathbb{R}, \mathbb{R})$, and $\beta, \zeta, \gamma : I_b \to I_b$, are continuous functions.

(2) There exist non-negative constants $k_i,\ i = 1, ..., 4$ with $k_1 k_4 < 1$ such that

$$|q(\varphi, u_1, u_2, u_3) - q(\varphi, \bar{u_1}, \bar{u_2}, \bar{u_3}| \leq k_1|u_1 - \bar{u_1}| + k_2|u_2 - \bar{u_2}| + k_3|u_3 - \bar{u_3}|;$$

$$|f(\varphi, z) - f(\varphi, \bar{z})| \leq k_4|z - \bar{z}|.$$

(3) There exists $r > 0$ such that q satisfy the inequality $\sup\{|q(\varphi, u_1, u_2, u_3)| : \varphi \in I_b, u_1 \in [-r,r], u_2 \in [-L_1 b, L_1 b],$ $u_3 \in [-L_2|b(\ln b - 1)|, L_2|b(\ln b - 1)|]\} \leq r$
where

$$L_1 = \sup\{|g(\varphi,\nu,z)| : \varphi, \nu \in I_b \ \textit{and} \ z \in [-r,r]\},$$

$$L_2 = \sup\{|h(\varphi,\nu,z)| : \varphi, \nu \in I_b \ \textit{and} \ z \in [-r,r]\}.$$

Theorem 3.1. *Under the assumptions* $(1) - (3)$ *the Eq. (1.1) has at least one solution in* $F = C(I_b)$.

Proof. Define the operator $T : B_r \to E$ in the following form

$$(Tz)(\varphi) = q\bigg(\varphi, f(\varphi, z(\beta(\varphi))), \int_0^{\varphi} g(\varphi,\nu,z(\zeta(\nu)))d\nu, \int_0^{\varphi} \ln|\nu-\varphi| h\big(\varphi,\nu,z(\gamma(\nu))\big)d\nu\bigg)$$

where $B_r = \{z \in C(I_b) : \|z\| \le r\}$. Now, we show that T is continuous on B_r. Choose $\sigma > 0$ and any $z, x \in B_r$ such that $\|z - x\| < \sigma$. Then

$$\begin{aligned}
& |(Tz)(\varphi) - (Tx)(\varphi)| \\
= & \bigg| q\bigg(\varphi, f(\varphi, z(\beta(\varphi))), \int_0^{\varphi} g(\varphi,\nu,z(\zeta(\nu)))d\nu, \int_0^{\varphi} \ln|\nu-\varphi| h\big(\varphi,\nu,z(\gamma(\nu))\big)d\nu\bigg) \\
& - q\bigg(\varphi, f(\varphi, x(\beta(\varphi))), \int_0^{\varphi} g(\varphi,\nu,x(\zeta(\nu)))d\nu, \int_0^{\varphi} \ln|\nu-\varphi| h\big(\varphi,\nu,x(\gamma(\nu))\big)d\nu\bigg)\bigg| \\
\le & \bigg| q\bigg(\varphi, f(\varphi, z(\beta(\varphi))), \int_0^{\varphi} g(\varphi,\nu,z(\zeta(\nu)))d\nu, \int_0^{\varphi} \ln|\nu-\varphi| h\big(\varphi,\nu,z(\gamma(\nu))\big)d\nu\bigg) \\
& - q\bigg(\varphi, f(\varphi, x(\beta(\varphi))), \int_0^{\varphi} g(\varphi,\nu,z(\zeta(\nu)))d\nu, \int_0^{\varphi} \ln|\nu-\varphi| h\big(\varphi,\nu,z(\gamma(\nu))\big)d\nu\bigg)\bigg| \\
& + \bigg| q\bigg(\varphi, f(\varphi, x(\beta(\varphi))), \int_0^{\varphi} g(\varphi,\nu,z(\zeta(\nu)))d\nu, \int_0^{\varphi} \ln|\nu-\varphi| h\big(\varphi,\nu,z(\gamma(\nu))\big)d\nu\bigg) \\
& - q\bigg(\varphi, f(\varphi, x(\beta(\varphi))), \int_0^{\varphi} g(\varphi,\nu,x(\zeta(\nu)))d\nu, \int_0^{\varphi} \ln|\nu-\varphi| h\big(\varphi,\nu,z(\gamma(\nu))\big)d\nu\bigg)\bigg| \\
& + \bigg| q\bigg(\varphi, f(\varphi, x(\beta(\varphi))), \int_0^{\varphi} g(\varphi,\nu,x(\zeta(\nu)))d\nu, \int_0^{\varphi} \ln|\nu-\varphi| h\big(\varphi,\nu,z(\gamma(\nu))\big)d\nu\bigg) \\
& - q\bigg(\varphi, f(\varphi, x(\beta(\varphi))), \int_0^{\varphi} g(\varphi,\nu,x(\zeta(\nu)))d\nu, \int_0^{\varphi} \ln|\nu-\varphi| h\big(\varphi,\nu,x(\gamma(\nu))\big)d\nu\bigg)\bigg| \\
\le & k_1|f(\varphi, z(\beta(\varphi))) - f(\varphi, x(\beta(\varphi)))| \\
& + k_2 \int_0^{\varphi} |g(\varphi,\nu,z(\zeta(\nu))) - g(\varphi,\nu,x(\zeta(\nu)))| d\nu \\
& + k_3 \int_0^{\varphi} \ln|\nu-\varphi| |h\big(\varphi,\nu,z(\gamma(\nu))\big) - h\big(\varphi,\nu,x(\gamma(\nu))\big)| d\nu \\
\le & k_1k_4|z(\beta(\varphi)) - x(\beta(\varphi))| + k_2 b\omega(g,\sigma) + k_3\omega(h,\sigma)\int_0^{\varphi} \ln|\nu-\varphi| d\nu \\
\le & k_1k_4\|z-x\| + k_2 b\omega(g,\sigma) + k_3\omega(h,\sigma)|\varphi\ln\varphi - \varphi| \\
\le & k_1k_4\|z-x\| + k_2 b\omega(g,\sigma) + k_3\omega(h,\sigma)|,
\end{aligned}$$

where

$$\omega(g,\sigma) = \sup\{|g(\varphi,\nu,z) - q(\varphi,\nu,x)| : \varphi,\nu \in I_b, z, x \in [-r,r], |z-x| \le \sigma\}.$$

and

$$\omega(h,\sigma) = \sup\{|h(\varphi,\nu,z) - h(\varphi,\nu,x)| : \varphi,\nu \in I_b, z, x \in [-r,r], |z-x| \le \sigma\}.$$

From the uniform continuity of $g(\varphi,\nu,z)$ and $h(\varphi,\nu,z)$ on the set $I_b \times I_b \times [-r,r]$, we infer that $\omega(g,\sigma) \to 0, \omega(h,\sigma) \to 0$ as $\sigma \to 0$. Thus, the above fact prove that the operator T is continuous on B_r.

Next, we prove that T satisfy the condensing map. Let E be bounded subset of F, $\varphi_1, \varphi_2 \in I_b$ with $\varphi_1 \leq \varphi_2$ and $|\varphi_2 - \varphi_1| \leq \sigma$. For arbitrary $\sigma > 0$ and $z \in E$ we have

$$\begin{aligned}
& |(Tz)(\varphi_2) - (Tz)(\varphi_1)| \\
= & \left| q\left(\varphi_2, f(\varphi_2, z(\beta(\varphi_2))), \int_0^{\varphi_2} g(\varphi_2, \nu, z(\zeta(\nu)))d\nu, \int_0^{\varphi_2} \ln|\nu - \varphi_2| h\big(\varphi_2, \nu, z(\gamma(\nu))\big)d\nu \right) \right. \\
& \left. -q\left(\varphi_1, f(\varphi_1, z(\beta(\varphi_1))), \int_0^{\varphi_1} g(\varphi_1, \nu, z(\zeta(\nu)))d\nu, \int_0^{\varphi_1} \ln|\nu - \varphi_1| h\big(\varphi_1, \nu, z(\gamma(\nu))\big)d\nu \right) \right| \\
\leq & \left| q\left(\varphi_2, f(\varphi_2, z(\beta(\varphi_2))), \int_0^{\varphi_2} g(\varphi_2, \nu, z(\zeta(\nu)))d\nu, \int_0^{\varphi_2} \ln|\nu - \varphi_2| h\big(\varphi_2, \nu, z(\gamma(\nu))\big)d\nu \right) \right. \\
& \left. -q\left(\varphi_2, f(\varphi_2, z(\beta(\varphi_2))), \int_0^{\varphi_2} g(\varphi_2, \nu, z(\zeta(\nu)))d\nu, \int_0^{\varphi_1} \ln|\nu - \varphi_1| h\big(\varphi_1, \nu, z(\gamma(\nu))\big)d\nu \right) \right| \\
& + \left| q\left(\varphi_2, f(\varphi_2, z(\beta(\varphi_2))), \int_0^{\varphi_2} g(\varphi_2, \nu, z(\zeta(\nu)))d\nu, \int_0^{\varphi_1} \ln|\nu - \varphi_1| h\big(\varphi_1, \nu, z(\gamma(\nu))\big)d\nu \right) \right. \\
& \left. -q\left(\varphi_2, f(\varphi_2, z(\beta(\varphi_2))), \int_0^{\varphi_1} g(\varphi_1, \nu, z(\zeta(\nu)))d\nu, \int_0^{\varphi_1} \ln|\nu - \varphi_1| h\big(\varphi_1, \nu, z(\gamma(\nu))\big)d\nu \right) \right| \\
& + \left| q\left(\varphi_2, f(\varphi_2, z(\beta(\varphi_2))), \int_0^{\varphi_1} g(\varphi_1, \nu, z(\zeta(\nu)))d\nu, \int_0^{\varphi_1} \ln|\nu - \varphi_1| h\big(\varphi_1, \nu, z(\gamma(\nu))\big)d\nu \right) \right. \\
& \left. -q\left(\varphi_2, f(\varphi_1, z(\beta(\varphi_1))), \int_0^{\varphi_1} g(\varphi_1, \nu, z(\zeta(\nu)))d\nu, \int_0^{\varphi_1} \ln|\nu - \varphi_1| h\big(\varphi_1, \nu, z(\gamma(\nu))\big)d\nu \right) \right| \\
& + \left| q\left(\varphi_2, f(\varphi_1, z(\beta(\varphi_1))), \int_0^{\varphi_1} g(\varphi_1, \nu, z(\zeta(\nu)))d\nu, \int_0^{\varphi_1} \ln|\nu - \varphi_1| h\big(\varphi_1, \nu, z(\gamma(\nu))\big)d\nu \right) \right. \\
& \left. -q\left(\varphi_1, f(\varphi_1, z(\beta(\varphi_1))), \int_0^{\varphi_1} g(\varphi_1, \nu, z(\zeta(\nu)))d\nu, \int_0^{\varphi_1} \ln|\nu - \varphi_1| h\big(\varphi_1, \nu, z(\gamma(\nu))\big)d\nu \right) \right| \\
\leq & k_3 \left| \int_0^{\varphi_2} \ln|\nu - \varphi_2| h\big(\varphi_2, \nu, z(\gamma(\nu))\big)d\nu - \int_0^{\varphi_1} \ln|\nu - \varphi_1| h\big(\varphi_1, \nu, z(\gamma(\nu))\big)d\nu \right| \\
& + k_2 \int_0^{\varphi_1} |g(\varphi_2, \nu, z(\zeta(\nu)) - g(\varphi_1, \nu, z(\zeta(\nu))| d\nu + k_2 \int_{\varphi_1}^{\varphi_2} |g(\varphi_2, \nu, z(\zeta(\nu))| d\nu \\
& + k_1 |f(\varphi_2, z(\beta(\varphi_2)) - f(\varphi_2, z(\beta(\varphi_1))| + k_1 |f(\varphi_2, z(\beta(\varphi_1)) - f(\varphi_1, z(\beta(\varphi_1))| \\
& + \omega_q(I_b, \sigma), \\
\leq & k_3 \left[\left| \int_0^{\varphi_2} \ln|\nu - \varphi_2| h\big(\varphi_2, \nu, z(\gamma(\nu))\big)d\nu - \int_0^{\varphi_2} \ln|\nu - \varphi_2| h\big(\varphi_1, \nu, z(\gamma(\nu))\big)d\nu \right| \right. \\
& \left. + \left| \int_0^{\varphi_2} \ln|\nu - \varphi_2| h\big(\varphi_1, \nu, z(\gamma(\nu))\big)d\nu - \int_0^{\varphi_1} \ln|\nu - \varphi_1| h\big(\varphi_1, \nu, z(\gamma(\nu))\big)d\nu \right| \right] \\
& + k_2 b \omega_g(I_b, \sigma) + k_2 L_1 |\varphi_2 - \varphi_1| + k_1 k_4 |z(\beta(\varphi_2) - z(\beta(\varphi_1)| + k_1 \omega_f(I_b, \sigma) \\
& + \omega_q(I_b, \sigma), \\
\leq & k_3 \Big[\int_0^{\varphi_2} \ln|\nu - \varphi_2| |h\big(\varphi_2, \nu, z(\gamma(\nu))\big) - h\big(\varphi_1, \nu, z(\gamma(\nu))\big)| d\nu \Big] \\
& + k_3 |h\big(\varphi_1, \nu, z(\gamma(\nu))\big)| \Big[\int_0^{\varphi_2} \ln|\nu - \varphi_2| d\nu - \int_0^{\varphi_1} \ln|\nu - \varphi_1| d\nu \Big]
\end{aligned}$$

$$+k_2 b\omega_g(I_b,\sigma)+k_2L_1\sigma+k_1k_4|z(\beta(\varphi_2)-z(\beta(\varphi_1)|+k_1\omega_f(I_b,\sigma)$$
$$+\omega_q(I_b,\sigma),$$

where

$$L_1=\sup\{|g(\varphi,\nu,z)|:\varphi,\nu\in I_b, z\in[-r,r]\},$$
$$L_2=\sup\{|h(\varphi,\nu,z)|:\varphi,\nu\in I_b, z\in[-r,r]\},$$
$$\omega_f(I_b,\sigma)=\sup\{|f(\varphi,z)-f(\bar\varphi,z)|:\varphi,\bar\varphi\in I_b,|\varphi-\bar\varphi|\le\sigma, z\in[-r,r]\},$$

$$\omega_q(I_b,\sigma)=\sup\{|q(\varphi,u_1,u_2,u_3)-q(\bar\varphi,u_1,u_2,u_3)|:\varphi,\bar\varphi\in I_b,|\varphi-\bar\varphi|\le\sigma,$$
$$u_1\in[-r,r],u_2\in[-L_1b,L_1b],u_3\in[-L_2|b(\ln b-1)|,L_2|b(\ln b-1)|]\},$$

$$\omega_g(I_b,\sigma)=\sup\{|g(\varphi,\nu,z)-g(\bar\varphi,\nu,z)|:\varphi,\bar\varphi,\nu\in I_b,|\varphi-\bar\varphi|\le\sigma, z\in[-r,r]\},$$

$$\omega_h(I_b,\sigma)=\sup\{|h(\varphi,\nu,z)-h(\bar\varphi,\nu,z)|:\varphi,\bar\varphi,\nu\in I_b,|\varphi-\bar\varphi|\le\sigma, z\in[-r,r]\}.$$

From above relations, we have

$$\begin{aligned}|(Tz)(\varphi_2)-(Tz)(\varphi_1)| &\le k_1k_4|z(\beta(\varphi_2)-z(\beta(\varphi_1)|+k_1\omega_f(I_b,\sigma)+k_2b\omega_g(I_b,\sigma)+k_2L_1\sigma\\ &+ k_3\omega_h(I_b,\sigma)|b(\ln b-1)|+k_3L_2|\varphi_2-\varphi_1|+\omega_q(I_b,\sigma)\end{aligned}$$

Thus, by the previous estimate, we have

$$\begin{aligned}\omega(Tz,\sigma) &\le k_1k_4\omega(z,\omega(\beta,\sigma))+k_1\omega_f(I_b,\sigma)+k_2b\omega_g(I_b,\sigma)+k_2L_1b+k_3\omega_h(I_b,\sigma)\\ &+k_3L_2\sigma+\omega_q(I_b,\sigma),\end{aligned}$$

After taking supremum for z on E, taking limit as $\sigma\to 0$, we get

$$\omega_0(TE)\le(k_1k_4)\omega_0(E).$$

Consequently

$$\psi(TE)\le(k_1k_4)\psi(E). \tag{3.1}$$

The inequality (3.1) implies that the operator T is a condensing map. Finally, investigation of condition 2.6 is remained. Now, let $z\in\partial\bar B_r$. If $Tz=kz$ then $\|Tz\|=k\|z\|=kr$ and by assumption (3), we can write

$$|Tz(\varphi)|=\left|q\left(\varphi,f(\varphi,z(\beta(\varphi))),\int_0^{\varphi}g(\varphi,\nu,z(\zeta(\nu)))d\nu,\int_0^{\varphi}\ln|\nu-\varphi|h(\varphi,\nu,z(\gamma(\nu)))d\nu\right)\right|\le r$$

for all $\varphi\in I_b$, hence $\|Tz\|\le r$ i.e $k\le 1$. The proof is complete. □

Corollary 3.2. *Assume that*

(1) $q\in C(I_b\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $g,h\in C(I_b\times I_b\times\mathbb{R},\mathbb{R})$, *and* $\beta,\zeta,\gamma:I_b\to I_b$, *are continuous functions.*

(2) *There exist non-negative constants* $k_i,\ i=1,...,3$ *with* $k_1<1$ *such that*

$$|q(\varphi,u_1,u_2,u_3)-q(\varphi,\bar{u_1},\bar{u_2},\bar{u_3}|\le k_1|u_1-\bar{u_1}|+k_2|u_2-\bar{u_2}|+k_3|u_3-\bar{u_3}|;$$

(3) *There exists* $r > 0$ *such that* q *satisfy the inequality*
$\sup\{|q(\varphi, u_1, u_2, u_3)| : \varphi \in I_b, u_1 \in [-r, r], u_2 \in [-L_1 b, L_1 b],$
$u_3 \in [-L_2|b(\ln b - 1)|, L_2|b(\ln b - 1)|]\} \leq r$
where

$$L_1 = \sup\{|g(\varphi, \nu, z)| : \varphi, \nu \in I_b \ \ and \ \ z \in [-r, r]\},$$

$$L_2 = \sup\{|h(\varphi, \nu, z)| : \varphi, \nu \in I_b \ \ and \ \ z \in [-r, r]\}.$$

Then

$$z(\varphi) = q\left(\varphi, z(\beta(\varphi)), \int_0^{\varphi} g(\varphi, \nu, z(\zeta(\nu)))d\nu, \int_0^{\varphi} \ln|\nu - \varphi| h\big(\varphi, \nu, z(\gamma(\nu))\big)d\nu\right), \qquad (3.2)$$

has at least one solution in $C(I_b)$.

Corollary 3.3. *Let*

(1) $q \in C(I_b \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $f \in C(I_b \times \mathbb{R}, \mathbb{R})$, $g, h \in C(I_b \times I_b \times \mathbb{R}, \mathbb{R})$, *and* $\beta, \zeta, \gamma : I_b \to I_b$, *are continuous functions.*

(2) *There exist non-negative constants* $k_i, \ i = 1, ..., 4$ *with* $k_1 + k_4 < 1$ *such that*

$$|q(\varphi, u_1, u_2, u_3) - q(\varphi, \bar{u_1}, \bar{u_2}, \bar{u_3}| \leq k_1|u_1 - \bar{u_1}| + k_2|u_2 - \bar{u_2}| + k_3|u_3 - \bar{u_3}|;$$

$$|f(\varphi, z) - f(\varphi, \bar{z})| \leq k_4|z - \bar{z}|.$$

(3) *There exists* $r > 0$ *such that* q *satisfy the inequality*
$\sup\{|q(\varphi, u_1, u_2, u_3)| : \varphi \in I_b, u_1 \in [-r, r], u_2 \in [-L_1 b, L_1 b],$
$u_3 \in [-L_2|b(\ln b - 1)|, L_2|b(\ln b - 1)|]\} \leq r$
where

$$L_1 = \sup\{|g(\varphi, \nu, z)| : \varphi, \nu \in I_b \ \ and \ \ z \in [-r, r]\},$$

$$L_2 = \sup\{|h(\varphi, \nu, z)| : \varphi, \nu \in I_b \ \ and \ \ z \in [-r, r]\}.$$

Then

$$z(\varphi) = f(\varphi, z(\beta(\varphi))) + q\left(\varphi, \int_0^{\varphi} g(\varphi, \nu, z(\zeta(\nu)))d\nu, \int_0^{\varphi} \ln|\nu - \varphi| h\big(\varphi, \nu, z(\gamma(\nu))\big)d\nu\right), \quad (3.3)$$

has at least one solution in $C(I_b)$.

Example 3.4. Let us consider the following integral equation in $C\,[0,1]$

$$\begin{aligned} z(\varphi) \quad &= \quad \frac{1}{5 + \varphi^{\frac{3}{2}}} e^{\varphi^2 - 2} + \frac{z(\varphi^2)\varphi^3}{3(1 + \varphi^4 + \varphi^5)} + \left(\int_0^{\varphi} \frac{z\,(\nu)}{2\sqrt{2}\left(1 + \varphi^{\frac{3}{2}}\nu^4\right)} d\nu \right) \\ &+ \frac{\cos(\varphi^2)}{5(1 + e^{\varphi^2 + 3} + 5\sin(\varphi^{\frac{5}{2}}))} \int_0^{\varphi} \ln|\nu - \varphi| \frac{1 + \cos\sqrt{\nu} + |z(\sqrt{\nu})|}{7 + \nu\varphi^2 + \ln(\varphi)} d\nu, \end{aligned} \qquad (3.4)$$

for all $\varphi \in [0, 1]$. Eq. (3.4) is particular form of Eq. (1.1) with

$$\zeta(\varphi) = \varphi, \gamma(\varphi) = \sqrt{\varphi}, \beta(\varphi) = \varphi^2, \forall\, \varphi \in [0, 1],$$

and

$$q(\varphi, u_1, u_2, u_3) = q_1(\varphi, u_1) + q_2(\varphi, u_2, u_3),$$

where

$$q_1(\varphi, u_1) = \frac{1}{5 + \varphi^{\frac{3}{2}}} e^{\varphi^2 - 2} + \frac{1}{3} u_1, \ \ u_1 = \frac{z(\varphi^2)\varphi^3}{(1 + \varphi^4 + \varphi^5)},$$

$$q_2(\varphi, u_2, u_3) = \frac{1}{2\sqrt{2}} u_2 + \frac{\cos(\varphi^2)}{5(1 + e^{\varphi^2+3} + 5\sin(\varphi^{\frac{5}{2}}))} u_3,$$

$$u_2 = \int_0^{\varphi} \frac{z(\nu)}{\left(1 + \varphi^{\frac{3}{2}} \nu^4\right)} d\nu,$$

$$u_3 = \int_0^{\varphi} \ln|\nu - \varphi| \frac{1 + \cos\sqrt{\nu} + |z(\sqrt{\nu})|}{7 + \nu\varphi^2 + \ln(\varphi)} d\nu, \quad h(\varphi, \nu, z) = \frac{1 + \cos\sqrt{\nu} + |z(\sqrt{\nu})|}{7 + \nu\varphi^2 + \ln(\varphi)}.$$

It is obvious that assumptions (1) and (2) of Theorem 3.1 are satisfied. We need to check that assumption (3) holds true. Suppose that $\|z\| \le \sigma, \sigma > 0$, then

$$\begin{aligned} |z(\varphi)| &= \Big| \frac{1}{5 + \varphi^{\frac{3}{2}}} e^{\varphi^2 - 2} + \frac{z(\varphi^2)\varphi^3}{3(1 + \varphi^4 + \varphi^5)} + \left(\int_0^{\varphi} \frac{z(\nu)}{2\sqrt{2}\left(1 + \varphi^{\frac{3}{2}}\nu^4\right)} d\nu \right) \\ &\quad + \frac{\cos(\varphi^2)}{5(1 + e^{\varphi^2+3} + 5\sin(\varphi^{\frac{5}{2}}))} \int_0^{\varphi} \ln|\nu - \varphi| \frac{1 + \cos\sqrt{\nu} + |z(\sqrt{\nu})|}{7 + \nu\varphi^2 + \ln(\varphi)} d\nu \Big| \le \sigma, \end{aligned}$$

for all $\varphi \in I_b$. Hence (3) holds if,

$$\frac{1}{5} + \frac{1}{3}\sigma + \frac{1}{5}(2 + \sigma) \le \sigma.$$

It can be verified that $\sigma = 1.28$ satisfies in the last inequality. Hence, all conditions of Theorem 3.1 are fulfill, then Eq. (3.4) has at least one solution in $C[0, 1]$.

4 Conclusion

In the present study, we analyzed the existence of a solutions for weakly singular integral equations by Petryshyn's fixed point theorem. We gave a example to confirm the efficiency of our results. The interested researchers may obtain the existence of the solutions of Eq.(1.1) in different Banach function spaces, e.g., Orlicz space, Sobolev space, Holder space, etc.

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