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APPLICATION OF DOUBLE GENERAL RANGAIG INTEGRAL TRANSFORM FOR SOLVING SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS.

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Abstract. In this study, the linear system of partial differential equations with initial and boundary conditions is solved with the application of the double general Rangaig integral transform. Additionally, a new iterative method is used with the double general Rangaig integral transform to yield solutions for non-linear systems of integer-order partial differential equations. Using the new iterative method and the double general Rangaig integral transform, the coupled Burgers equations may finally be solved exactly.

1 Introduction

Partial differential equations (PDEs) are a system of equations that have a broad range of applications in the fields of physical science, engineering, and mathematics. Research is now being done to establish an effective and precise method for handling nonlinear systems of PDEs. We briefly examine the several approaches that have been proposed for solving PDE systems. Eltayeb and Kilicman [1] coupled Sumudu transform and decomposition method to obtain approximate solutions of nonlinear system of partial differential equations.

In 2014, Rabie and Elzaki [2] used Adomian decomposition method and modified ADM for solving nonlinear PDEs subject to the general initial conditions. In 2016, Singh and Mehendra [3] applied reduced differential transform method to obtain an analytical solution of initial value system of time dependent linear and nonlinear PDEs. Chowdhury et al.[4] in 2010 and Hemeda [5] employed homotopy perturbation method to obtain approximate analytical solution to the linear and nonlinear systems of PDEs. In 2011, Biazar and Eslami [6]modified HPM for obtaining solutions of systems of nonlinear PDEs. Elzaki and Biazar [7] combined Elzaki transform and HPM to obtain the exact solution of nonlinear systems of PDEs.

In 2000, Wazwaz [8] applied Adomian decomposition method to obtain analytical solution of systems of PDEs and reaction-diffusion Brusselator model. In 2004, Ayaz [9] applied two and three-dimensional differential transform method for finding exact solutions of linear and nonlinear systems of PDEs. Abdou and Solimon [10] in 2005 used VIM to find the solution of a Burger's and coupled Burger's equations. Further, Wazwaz [11] in 2007 applied VIM to solve linear and nonlinear systems of PDEs. In 2022, Ranjit R. Dhunde and G. L. Waghmare [12] applied double Laplace transform method in solving system of linear , non-linear and fractional partial differential equations. In 2024 M.S Derle and Dr. D.P. Patil [13] applied double general Rangaig integral transform to obtain solution of integral-differential equations.

In 2017 N.A. Rangaig and etc. all[14] define a new integral transform Rangaig transform on another domain from minus infinity to zero and applied it for solving O.D.E., integral equations and Aabls integral equation.In 2022,E. A. Mansoura and E. A. Kuffi[15]generalize a Rangaig integral transform and applied it for solving higher order O.D.E. Derle M. S and etc. all[16]define double general rangaig integral transform , its properties and applied it to find exact solution of partial differential equations. In 2023 S. Hasan and etc. all [17] define the new integral transform Sum transform , its properties and applied to Wave equation , Klein-Gordan equation and Telegraph equation.

In this paper, The double general Rangaig integral transform will be used to a system of linear and non-linear PDEs , with a focus on the coupled Burgers equation.

2 Preliminaries

In this section, we introduced some basic concepts that are required [14],[15],[16].

Definition 2.1. Rangaig integral transform
The Rangaig integral transform can be written as:

$$\eta[h(t)] = \Lambda(\mu) = \frac{1}{\mu}\int_{t=-\infty}^{0} e^{-\mu t}h(t)dt, \frac{1}{\lambda_1} \leq \mu \leq \frac{1}{\lambda_2}$$

Definition 2.2. General Rangaig integral transform
The set of functions, an exponential order is

$$Hg = \Big\{h(t):\ \text{there exist } N, \lambda_1 \text{ and } \lambda_2 > 0, |h(t)| > Ne^{\lambda_j |t|}, t \in (-1)^{j-1} \times (-\infty, 0),\ \text{where } j = 1, 2\Big\}.$$

The General Rangaig integral transform can be written as:

$$\eta_g\{h(t)\} = \wedge_g(\mu) = \frac{1}{\mu^n}\int_{t=-\infty}^{0} e^{p(\mu)t}h(t)dt$$

Definition 2.3. Double general Rangaig integral transform
The set of functions, of an exponential order is defined as:

$$H_{2g} = \{h(t_1,t_2) : \exists N, M, \lambda_1, \lambda_2, \chi_1, \chi_{2_1} > 0, |h(t_1,0)| > N.e^{\lambda_i t_1}, t_1 \in (-1)^{i-1}\times$$
$$(-\infty,0), |h(0,t_2)| > M\cdot e^{\chi_j t_2}, t_2 \in (-1)^{j-1} \times (-\infty,0),\ \text{where } i,j = 1,2\}$$

where, $N, M\ \equiv$ finite constants $\lambda_1, \lambda_2, \chi_1, \chi_2$ are finite or infinite constants. Then the double general Rangaig integral transform for the set H_{2g} can be written as:

$$\eta_{2g}\{h(t_1,t_2)\} = \Lambda_{2g}(\mu_1,\mu_2) = \frac{1}{\mu_1 n_1 \cdot \mu_2^{n_2}}\int_{-\infty}^{0}\int_{-\infty}^{0} e^{p(\mu_1)t_1+q(\mu_2)t_2}h(t_1,t_2)\,dt_1dt_2$$

here $\Lambda_2 g$ denote the double general Rangaig integral transform of function $h(t_1,t_2) \in H_{2g}$, $p(\mu_1), q(\mu_2)$ are functions of parameters μ_1 and μ_2 The inverse of double general Rangaig transform $\eta_{2g}\{h(t_1,t_2)\}$ is defined as

$$\eta_{2g}^{-1}\{\wedge_D(\mu_1,\mu_2)\} = \eta_{t_1}^{-1}\eta_{t_2}^{-1}\{\wedge_D(\mu_1,\mu_2)\} = h(t_1,t_2)$$

3 Double general Rangaig integral transform for partial derivatives.

[16]

Theorem 3.1. *Consider $\Lambda_D(\mu_1,\mu_2)$ as the double general Rangaig integral transform of the function $h(t_1,t_2)$ and $\Lambda_g(0,\mu_2)$ is the general Rangaig integral transform of the function $h(0,t_2)$. then*

i

$$\eta_{2g}\left\{\frac{\partial h(t_1,t_2)}{\partial t_1}\right\} = \frac{1}{\mu_1^{n_1}}\Lambda_g(0,\mu_2) - p(\mu_1)\Lambda_D(\mu_1,\mu_2)$$

ii

$$\eta_{2g}\left\{\frac{\partial^2 h(t_1,t_2)}{\partial t_1^2}\right\} = \frac{1}{\mu_1^{n_1}}\frac{\partial \Lambda_g(0,\mu_2)}{\partial t_1} - p(\mu_1)\cdot\frac{1}{\mu 1^{n_1}}\Lambda_g(0,\mu_2) + [p(\mu_1)]^2\Lambda_D(\mu_1,\mu_2)$$

iii

$$\eta_{2g}\left\{\frac{\partial^n h(t_1,t_2)}{\partial t_1^n}\right\} = \frac{1}{\mu_1^{n_1}}\sum_{k=0}^{m-1}(-1)^k[p(\mu_1)]^k\frac{\partial^{(m-1-k)}\Lambda_g(0,\mu_2)}{\partial^{(m-1-k)}t_1}(-1)^m[p(\mu_1)]^m\Lambda_D(\mu_1,\mu_2)$$

Theorem 3.2. *Consider $\Lambda_D(\mu_1,\mu_2)$ as the double general Rangaig integral transform of the function $h(t_1,t_2)$ and $\Lambda_g(\mu_1,0)$ is the general Rangaig integral transform of the function $h(t_1,0)$. then*

i

$$\eta_{2g}\left\{\frac{\partial h(t_1,t_2)}{\partial t_2}\right\} = \frac{1}{\mu_2^{n_2}}\Lambda_g(\mu_1,0) - q(\mu_2)\Lambda_D(\mu_1,\mu_2)$$

ii

$$\eta_{2g}\left\{\frac{\partial^2 h(t_1,t_2)}{\partial t_2^2}\right\} = \frac{1}{\mu_2^{n_2}}\frac{\partial \Lambda_g(\mu_1,0)}{\partial t_2} - q(\mu_2)\cdot\frac{1}{\mu_2^{n_2}}\Lambda_g(\mu_1,0) + [q(\mu_2)]^2\Lambda_D(\mu_1,\mu_2)$$

iii

$$\eta_{2g}\left\{\frac{\partial^n h(t_1,t_2)}{\partial t_1^n}\right\} = \frac{1}{\mu_2^{n_2}}\sum_{k=0}^{m-1}(-1)^k[q(\mu_2)]\,k\frac{\partial^{(m-1-\kappa)}\Lambda_g(\mu_1,0)}{\partial^{(m-1-k)}t_2}+(-1)^m[q(\mu_2)]^m\Lambda_D(\mu_1,\mu_2)$$

4 Double General Rangaig Integral Transform of Some Fundamental Functions.

[16] The double general Rangaig transforms of some common functions are given in following Table 1.

Table 1.

$h(\mathbf{t_1},\mathbf{t_2})$	$\wedge_{\mathbf{2g}}(\mu_1,\mu_2)$
1	$\frac{1}{\mu_1^{n_1}\cdot\mu_2^{n_2}}\frac{1}{p(\mu_1)q(\mu_2)}$
t_1t_2	$\frac{1}{\mu_1^{n_1}\cdot\mu_2^{n_2}}\frac{1}{[p(\mu_1)]^2[q(\mu_2)]^2}$
${t_1}^m{t_2}^n m,n>0$	$\frac{1}{\mu_1^{n_1}\cdot\mu_2^{n}}\frac{(-1)^{m+n}m!n!}{[p(\mu_1)]^{m+1}[q(\mu_2)]^{n+1}}$
${t_1}^m{t_2}^n, m,n>-1$	$\frac{1}{\mu_1^{n_1}\cdot\mu_2^{n2}}\frac{(-1)^{m+n}\Gamma(m+1)\Gamma(n+1)}{[p(\mu_1)]^{m+1}[q(\mu_2)]^{n+1}}$
$e^{at_1+bt_2}$	$\frac{1}{\mu_1^{n_1}\cdot\mu_2^{n_2}}\frac{1}{(p(\mu_1)+a)}\frac{1}{(q(\mu_2)+b)}$
$\cos(at_1+bt_2)$	$\frac{1}{\mu_1^{n_1}\cdot\mu_2^{n_2}}\left(\frac{p(\mu_1)q(\mu_2)-ab}{\left(\left([p(\mu_1)]^2+a^2\right)\right)\left([q(\mu_2)]^2+b^2\right)}\right)$
$\sin(at_1+bt_2)$	$\frac{1}{\mu_1^{n_1}\cdot\mu_2^{n_2}}\left(\frac{-aq(\mu_2)-bp(\mu_1)}{\left(\left([p(\mu_1)]^2+a^2\right)\right)\left([q(\mu_2)]^2+b^2\right)}\right)$
$\cosh(at_1+bt_2)$	$\frac{1}{\mu_1^{n_1}\cdot\mu_2^{n_2}}\left(\frac{p(\mu_1)q(\mu_2)+ab}{\left(\left([p(\mu_1)]^2-a^2\right)\right)\left([q(\mu_2)]^2-b^2\right)}\right)$
$\sinh(at_1+bt_2)$	$\frac{1}{\mu_1^{n_1}\cdot\mu_2^{n_2}}\left(\frac{-aq(\mu_2)-bp(\mu_1)}{\left(\left([p(\mu_1)]^2-a^2\right)\right)\left([q(\mu_2)]^2-b^2\right)}\right)$
$e^{-t_2}\sin t_1$	$\frac{1}{\mu_1^{n_1}\cdot\mu_2^{n_2}}\frac{-1}{(p(\mu_1)^2+1)\cdot(q(\mu_2)-1)}$
$e^{t_1}\cosh t_2$	$\frac{1}{\mu_1^{n_1}\cdot\mu_2^{n_2}}\frac{q(\mu_2)}{(p(\mu_1)+1)\cdot(q(\mu_2)^2-1)}$
$e^{-t_1}\sinh t_2$	$\frac{1}{\mu_1^{n_1}\cdot\mu_2^{n_2}}\frac{-1}{(p(\mu_1)-1)\cdot(q(\mu_2)^2-1)}$

5 Application of D.G.R.I.T. for solving system of P.D.E.

5.1 Linear system of P.D.E

Consider Linear System
Example. 1

$$\left.\begin{aligned} u_{t_1} + v_{t_2} = 0 \\ v_{t_1} + u_{t_2} = 0 \end{aligned}\right\} \tag{5.1}$$

$$u(t_1, 0) = e^{t_1}, v(t_1, 0) = e^{-t_1} \tag{5.2}$$

$$u(0, t_2) = e^{t_2}, v(0, t_2) = e^{-t_2} \tag{5.3}$$

Applying D.G.R.I.T. to both sides of Eq. (5.1)and using(3.1),(3.2), yeilds

$$\frac{1}{\mu_2^{n_2}}\Lambda_{g_u}(\mu_1, 0) - q(\mu_2)\Lambda_{D_u}(\mu_1, \mu_2) + \frac{1}{\mu_1^{n_1}}\Lambda_{g_v}(0, \mu_2) - p(\mu_1)\Lambda_{D_v}(\mu_1, \mu_2) = 0 \tag{5.4}$$

$$\frac{1}{\mu_2^{n_2}}\Lambda_{g_v}(\mu_1, 0) - q(\mu_2)\Lambda_{D_v}(\mu_1, \mu_2) + \frac{1}{\mu_1^{n_1}}\Lambda_{g_u}(0, \mu_2) - p(\mu_1)\Lambda_{D_u}(\mu_1, \mu_2) = 0 \tag{5.5}$$

Taking Single G.R.I.T.of Eq.(5.2) and (5.3) i.e I.C and B.C,we get

$$\Lambda_{g_u}(\mu_1, 0) = \frac{1}{\mu_1^{n_1}(p(\mu_1) + 1)}, \Lambda_{g_v}(\mu_1, 0) = \frac{1}{\mu_1^{n_1}(p(\mu_1) - 1)} \tag{5.6}$$

$$\Lambda_{g_u}(0, \mu_2) = \frac{1}{\mu_2^{n_2}(q(\mu_2) + 1)}, \Lambda_{g_v}(0, \mu_2) = \frac{1}{\mu_2^{n_2}(q(\mu_2) - 1)} \tag{5.7}$$

After putting conditions (5.6) and(5.7) in Eq. (5.4) and (5.5), we get

$$\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{1}{(p(\mu_1) + 1)} + \frac{1}{(q(\mu_2) - 1)}\Big] - q(\mu_2)\Lambda_{D_u}(\mu_1, \mu_2) - p(\mu_1)\Lambda_{D_v}(\mu_1, \mu_2) = 0 \tag{5.8}$$

$$\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{1}{(p(\mu_1) - 1)} + \frac{1}{(q(\mu_2) + 1)}\Big] - q(\mu_2)\Lambda_{D_v}(\mu_1, \mu_2) - p(\mu_1)\Lambda_{D_u}(\mu_1, \mu_2) = 0 \tag{5.9}$$

Multiply Eq. (5.8) by $q(\mu_2)$ and Eq. (5.7) by $p(\mu_1)$ and substract it ,simplify,we get

$$\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{q(\mu_2)}{(p(\mu_1) + 1)} + \frac{q(\mu_2)}{(q(\mu_2) - 1)} - \frac{p(\mu_1)}{(p(\mu_1) - 1)} - \frac{p(\mu_1)}{(q(\mu_2) + 1)}\Big] = (q(\mu_2)^2 - p(\mu_1)^2)\Lambda_{D_u}(\mu_1, \mu_2)$$

After Simplification,we get

$$\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{(p(\mu_1) - 1)q(\mu_2) - (p(\mu_1) + 1)}{(q(\mu_2)^2 - 1)(p(\mu_1)^2 - 1)}\Big] = \Lambda_{D_u}(\mu_1, \mu_2)$$

$$\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{q(\mu_2)}{(q(\mu_2)^2 - 1)(p(\mu_1) + 1)} - \frac{1}{(q(\mu_2)^2 - 1)(p(\mu_1) - 1)}\Big] = \Lambda_{D_u}(\mu_1, \mu_2)$$

$$\Lambda_{D_u}(\mu_1, \mu_2) = \frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\frac{q(\mu_2)}{(q(\mu_2)^2 - 1)(p(\mu_1) + 1)} - \frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\frac{1}{(q(\mu_2)^2 - 1)(p(\mu_1) - 1)} \tag{5.10}$$

Inverse D. G. R. I. T. applied to both sides of equation (5.10) yields 1

$$u(t_1, t_2) = e^{t_1}\cosh t_2 + e^{-t_1}\sinh t_2$$

substituting Eq.(5.10) in Eq.(5.8),we get

$$\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{1}{(p(\mu_1) + 1)} + \frac{1}{(q(\mu_2) - 1)}\Big] - \frac{q(\mu_2)}{\mu_1^{n_1}\mu_2^{n_2}}\Big[\frac{q(\mu_2)}{(p(\mu_1) + 1)(q(\mu_2)^2 - 1)} - \frac{1}{(p(\mu_1) - 1)(q(\mu_2)^2 - 1)}\Big] = p(\mu_1)\Lambda_{D_v}(\mu_1, \mu_2)$$

After simplification, we get the following.

$$\Lambda_{D_v}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\frac{q(\mu_2)}{(p(\mu_1)-1)(q(\mu_2)^2-1)}+\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\frac{1}{(p(\mu_1)+1)(q(\mu_2)^2-1)}$$

Inverse D. G. R. I. T. applied on both sides,leads to1

$$v(t_1,t_2)=e^{-t_1}\cosh t_2-e^{t_1}\sinh t_2$$

Example. 2

$$\left.\begin{array}{l}u_{t_2}+u_{t_1}-2v=0\\ v_{t_2}+v_{t_1}+2u=0\end{array}\right\} \tag{5.11}$$

$$u(t_1,0)=sint_1, v(t_1,0)=cost_1 \tag{5.12}$$

$$u(0,t_2)=sint_2, v(0,t_2)=cost_2 \tag{5.13}$$

Applying D.G.R.I.T. to both sides of Eq.(5.11), yeilds

$$\frac{1}{\mu_2^{n_2}}\Lambda_{g_u}(\mu_1,0)-q(\mu_2)\Lambda_{D_u}(\mu_1,\mu_2)+\frac{1}{\mu_1^{n_1}}\Lambda_{g_u}(0,\mu_2)-p(\mu_1)\Lambda_{D_u}(\mu_1,\mu_2)-2\Lambda_{D_v}(\mu_1,\mu_2)=0 \tag{5.14}$$

$$\frac{1}{\mu_2^{n_2}}\Lambda_{g_v}(\mu_1,0)-q(\mu_2)\Lambda_{D_v}(\mu_1,\mu_2)+\frac{1}{\mu_1^{n_1}}\Lambda_{g_v}(0,\mu_2)-p(\mu_1)\Lambda_{D_v}(\mu_1,\mu_2)+2\Lambda_{D_u}(\mu_1,\mu_2)=0 \tag{5.15}$$

Taking Single G.R.I.T.of equation(5.12) and (5.13) i.e I.C and B.C,we get

$$\Lambda_{g_u}(\mu_1,0)=\frac{-1}{\mu_1^{n_1}(p(\mu_1)^2+1)},\Lambda_{g_v}(\mu_1,0)=\frac{p(\mu_1)}{\mu_1^{n_1}(p(\mu_1)^2+1)} \tag{5.16}$$

$$\Lambda_{g_u}(0,\mu_2)=\frac{-1}{\mu_2^{n_2}(q(\mu_2)^2+1)},\Lambda_{g_v}(0,\mu_2)=\frac{q(\mu_2}{\mu_2^{n_2}(q(\mu_2)^2+1)} \tag{5.17}$$

After putting condidtions (5.16) and(5.17) in equation (5.14) and (5.15), we get

$$\frac{-1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\left[\frac{1}{(p(\mu_1)^2+1)}+\frac{1}{(q(\mu_2)^2+1)}\right]-(q(\mu_2)+p(\mu_1))\Lambda_{D_u}(\mu_1,\mu_2)-2\Lambda_{D_v}(\mu_1,\mu_2)=0 \tag{5.18}$$

$$\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\left[\frac{p(\mu_1)}{(p(\mu_1)^2+1)}+\frac{q(\mu_2)}{(q(\mu_2)^2+1)}\right]-(q(\mu_2)+p(\mu_1))\Lambda_{D_v}(\mu_1,\mu_2)+2\Lambda_{D_u}(\mu_1,\mu_2)=0 \tag{5.19}$$

Multiply equation (5.18) by $(p(\mu_1)+q(\mu_2))$ and equation (5.19) by 2 , substract it & simplify, we get

$$\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\left[\frac{-(p(\mu_1)+q(\mu_2))}{(p(\mu_1)^2+1)}-\frac{(p(\mu_1)+q(\mu_2))}{(q(\mu_2)^2+1)}-\frac{-2p(\mu_1)}{(p(\mu_1)^2+1)}-\frac{2q(\mu_2)}{(q(\mu_2)^2+1)}\right]$$
$$=(4+(q(\mu_2)^2+p(\mu_1)^2))\Lambda_{D_u}(\mu_1,\mu_2)$$

After simplification, we get the following.

$$(4+(q(\mu_2)^2+p(\mu_1)^2))\Lambda_{D_u}(\mu_1,\mu_2)$$
$$=\frac{-1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{(p(\mu_1)+q(\mu_2))(q(\mu_2)^2+p(\mu_1)^2+2p(\mu_1)q(\mu_2)+4)}{(p(\mu_1)^2+1)(q(\mu_2)^2+1)}\Big]$$

$$\Lambda_{D_u}(\mu_1,\mu_2)=\frac{-1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{(p(\mu_1)+q(\mu_2))}{(p(\mu_1)^2+1)(q(\mu_2)^2+1)}\Big] \tag{5.20}$$

Applying inverse D. G. R. I.T. on both sides gives1

$$u(t_1,t_2)=sin(t_1+t_2)$$

substituting equation(5.20) in equation (5.18),we get

$$\frac{-1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{1}{(p(\mu_1)^2+1)}+\frac{1}{(q(\mu_2)^2+1)}\Big]-\frac{-1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{(q(\mu_2)+p(\mu_1))(p(\mu_1)+q(\mu_2))}{(p(\mu_1)^2+1)(q(\mu_2)^2+1)}\Big]-2\Lambda_{D_v}(\mu_1,\mu_2)=0$$

After simplification,we get the following.

$$\Lambda_{D_v}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{(p(\mu_1)q(\mu_2)-1)}{(p(\mu_1)^2+1)(q(\mu_2)^2+1)}\Big]$$

Apply in reverse D. G. R. I. T. on both sides,we get1

$$v(t_1,t_2)=cos(t_1+t_2)$$

5.2 Inhomogeneous Linear system of P.D.E

Consider the inhomogeneous linear system
Example. 3

$$\left.\begin{aligned} u_{t_2}-v_{t_1}-u+v&=-2\\ v_{t_2}+u_{t_1}-u+v&=-2 \end{aligned}\right\} \tag{5.21}$$

with I.C.

$$u(t_1,0)=1+e^{t_1},v(t_1,0)=-1+e^{t_1} \tag{5.22}$$

and B.C.

$$u(0,t_2)=1+e^{t_2},v(0,t_2)=-1+e^{-t_2} \tag{5.23}$$

Applying D.G.R.I.T. to both sides of Eq.(5.21), we get

$$\frac{1}{\mu_2^{n_2}}\Lambda_{g_u}(\mu_1,0)-q(\mu_2)\Lambda_{D_u}(\mu_1,\mu_2)-\frac{1}{\mu_1^{n_1}}\Lambda_{g_v}(0,\mu_2)+p(\mu_1)\Lambda_{D_v}(\mu_1,\mu_2)-\Lambda_{D_u}(\mu_1,\mu_2)+\Lambda_{D_v}(\mu_1,\mu_2)$$
$$=\frac{-1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{2}{p(\mu_1)q(\mu_2)} \tag{5.24}$$

$$\frac{1}{\mu_2^{n_2}}\Lambda_{g_v}(\mu_1,0)-q(\mu_2)\Lambda_{D_v}(\mu_1,\mu_2)+\frac{1}{\mu_1^{n_1}}\Lambda_{g_u}(0,\mu_2)-p(\mu_1)\Lambda_{D_u}(\mu_1,\mu_2)-\Lambda_{D_u}(\mu_1,\mu_2)+\Lambda_{D_v}(\mu_1,\mu_2)$$
$$=\frac{-1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{2}{p(\mu_1)q(\mu_2)} \tag{5.25}$$

Taking Single G.R.I.T.of Eq.n(5.22) and Eq. (5.23) i.e. I.C and B.C,we get

$$\Lambda_{g_u}(\mu_1,0)=\frac{1}{\mu_1^{n_1}(p(\mu_1)}+\frac{1}{\mu_1^{n_1}(p(\mu_1)+1)},\Lambda_{g_v}(\mu_1,0)=\frac{-1}{\mu_1^{n_1}(p(\mu_1)}+\frac{1}{\mu_1^{n_1}(p(\mu_1)+1)} \quad (5.26)$$

$$\Lambda_{g_u}(0,\mu_2)=\frac{1}{\mu_2^{n_2}q(\mu_2)}+\frac{1}{\mu_2^{n_2}(q(\mu_2)+1)},\Lambda_{g_v}(0,\mu_2)=\frac{-1}{\mu_2^{n_2}q(\mu_2)}+\frac{1}{\mu_2^{n_2}(q(\mu_2)-1)} \quad (5.27)$$

After putting condidtions (5.26) and (5.27) in Eq. (5.24) and Eq. (5.25), we get

$$\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{1}{p(\mu_1)}+\frac{1}{(p(\mu_1)+1}+\frac{1}{q(\mu_2)}-\frac{1}{(q(\mu_2)-1}\Big]-(q(\mu_2)+1)\Lambda_{D_u}(\mu_1,\mu_2)+(p(\mu_1)+1)\Lambda_{D_v}(\mu_1,\mu_2)$$
$$=\frac{-1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{2}{p(\mu_1)q(\mu_2)} \quad (5.28)$$

$$\frac{-1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{1}{p(\mu_1)}+\frac{1}{(p(\mu_1)+1}+\frac{1}{q(\mu_2)}+\frac{1}{(q(\mu_2)+1)}\Big]-(p(\mu_1)+1)\Lambda_{D_u}(\mu_1,\mu_2)-(q(\mu_2)-1)\Lambda_{D_v}(\mu_1,\mu_2)$$
$$=\frac{-1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{2}{p(\mu_1)q(\mu_2)} \quad (5.29)$$

Multiply Eq. (5.28) by $(q(\mu_2)-1)$ and Eq. (5.29) by $(p(\mu_1)+1)$ and subtract it, simplify, we get

$$\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{(q(\mu_2)-p(\mu_1)-2)}{p(\mu_1)}+\frac{(q(\mu_2)+p(\mu_1))}{q(\mu_2)}+\frac{2(q(\mu_2)+p(\mu_1))}{p(\mu_1)q(\mu_2)}+\frac{(q(\mu_2)^2-1)+(p(\mu_1)+1)^2}{(p(\mu_1)+1)(q(\mu_2)+1)}\Big]$$
$$=\big[(q(\mu_2)^2-1)+(p(\mu_1)+1)^2\big]\Lambda_{D_u}(\mu_1,\mu_2)$$

After simplification,

$$\Lambda_{D_u}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{1}{(p(\mu_1)q(\mu_2))}+\frac{1}{(p(\mu_1)+1)(q(\mu_2)+1)}\Big] \quad (5.30)$$

Apply in reverse D. G. R. I. T. on both sides,we get1

$$u(t_1,t_2)=1+e^{(t_1+t_2)}$$

substituting Eq.(5.30) in Eq.(5.29), we get

$$\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{-1}{p(\mu_1)}+\frac{1}{(p(\mu_1)+1}+\frac{1}{q(\mu_2)}+\frac{1}{(q(\mu_2)+1)}+\frac{2}{(p(\mu_1)q(\mu_2)}\Big]-$$
$$(p(\mu_1)+1)\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{1}{(p(\mu_1)q(\mu_2))}+\frac{1}{(p(\mu_1)+1)(q(\mu_2)+1)}\Big]=(q(\mu_2)-1)\Lambda_{D_v}(\mu_1,\mu_2)$$

After simplification,

$$\Lambda_{D_v}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}}\frac{1}{\mu_2^{n_2}}\Big[\frac{-1}{(p(\mu_1)q(\mu_2))}+\frac{1}{(p(\mu_1)+1)(q(\mu_2)-1)}\Big]$$

Apply inverse D. G. R. I.T. on both sides gives1

$$v(t_1,t_2)=-1+e^{(t_1-t_2)}$$

5.3 Inhomogeneous Non-linear system of P.D.E

Consider the inhomogeneous non-linear system.
Example.4

$$\left.\begin{aligned} u_{t_2}+vu_{t_1}+u=1 \\ v_{t_2}-uv_{t_1}-v=1 \end{aligned}\right\} \tag{5.31}$$

with I.C.

$$u(t_1,0)=e^{t_1}, v(t_1,0)=e^{-t_1} \tag{5.32}$$

Applying D.G.R.I.T. to both sides of Eq.(5.31),we get

$$\frac{1}{\mu_2^{n_2}}\Lambda_{g_u}(\mu_1,0)-q(\mu_2)\Lambda_{D_u}(\mu_1,\mu_2)+\Lambda_{D_u}(\mu_1,\mu_2)=\eta_{2g}[1-vu_{t_1}] \tag{5.33}$$

$$\frac{1}{\mu_2^{n_2}}\Lambda_{g_v}(\mu_1,0)-q(\mu_2)\Lambda_{D_v}(\mu_1,\mu_2)-\Lambda_{D_v}(\mu_1,\mu_2)=\eta_{2g}[1+uv_{t_1}] \tag{5.34}$$

Further taking Single G.R.I.T.of Eq.(5.32) i.e I.C ,we get

$$\Lambda_{g_u}(\mu_1,0)=\frac{1}{\mu_1^{n_1}(p(\mu_1)+1)},\Lambda_{g_v}(\mu_1,0)=\frac{1}{\mu_1^{n_1}(p(\mu_1)-1)} \tag{5.35}$$

subsituting condidtions (5.35) in equations (5.33) and (5.34), we get

$$\frac{1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{1}{(p(\mu_1)+1)}-q(\mu_2)\Lambda_{D_u}(\mu_1,\mu_2)+\Lambda_{D_u}(\mu_1,\mu_2)=\eta_{2g}[1-vu_{t_1}]$$

$$\frac{1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{1}{(p(\mu_1)-1)}-q(\mu_2)\Lambda_{D_v}(\mu_1,\mu_2)-\Lambda_{D_v}(\mu_1,\mu_2)=\eta_{2g}[1+uv_{t_1}]$$

After simplification, gives

$$\Lambda_{D_u}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{1}{(p(\mu_1)+1)(q(\mu_2)-1)}+\frac{1}{(q(\mu_2)-1)}\eta_{2g}[1-vu_{t_1}] \tag{5.36}$$

$$\Lambda_{D_v}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{1}{(p(\mu_1)-1)(q(\mu_2)+1)}-\frac{1}{(q(\mu_2)+1)}\eta_{2g}[1+uv_{t_1}] \tag{5.37}$$

Inverse D. G. R. I. T. applied on both sides of Eq.(5.36) and (5.37), we get1

$$u(t_1,t_2)=e^{(t_1-t_2)}+\eta_{2g}^{-1}\left[\frac{1}{(q(\mu_2)-1)}\eta_{2g}[1-vu_{t_1}]\right] \tag{5.38}$$

$$v(t_1,t_2)=e^{(-t_1+t_2)}-\eta_{2g}^{-1}\left[\frac{1}{(q(\mu_2)+1)}\eta_{2g}[1+uv_{t_1}]\right] \tag{5.39}$$

Now, we use new iterative method ,
Consider

$$u(t_1,t_2)=\sum_{i=0}^{\infty}u_i(t_1,t_2),\quad v(t_1,t_2)=\sum_{i=0}^{\infty}v_i(t_1,t_2) \tag{5.40}$$

Substituting Eq. (5.40) in Eq.(5.38) and (5.39), we get

$$\sum_{i=0}^{\infty}u_i(t_1,t_2)=e^{(t_1-t_2)}+\eta_{2g}^{-1}\left[\frac{1}{(q(\mu_2)-1)}\eta_{2g}\left[1-\Big(\sum_{i=0}^{\infty}v_i(t_1,t_2)\Big)\Big(\sum_{i=0}^{\infty}u_i(t_1,t_2)\Big)_{t_1}\right]\right] \tag{5.41}$$

$$\sum_{i=0}^{\infty}v_i(t_1,t_2)=e^{(-t_1+t_2)}-\eta_{2g}^{-1}\left[\frac{1}{(q(\mu_2)+1)}\eta_{2g}\left[1+\Big(\sum_{i=0}^{\infty}u_i(t_1,t_2)\Big)\Big(\sum_{i=0}^{\infty}v_i(t_1,t_2)\Big)_{t_1}\right]\right] \tag{5.42}$$

The nonlinear term N is decomposed as

$$N\Big(\sum_{i=0}^{\infty}u_i(t_1,t_2),\sum_{i=0}^{\infty}v_i(t_1,t_2)\Big)$$
$$=N\Big(u_0(t_1,t_2),v_0(t_1,t_2)\Big)+\sum_{i=1}^{\infty}\Big[N\Big(\sum_{k=0}^{i}u_k(t_1,t_2),\sum_{k=0}^{i}v_k(t_1,t_2)\Big)-N\Big(\sum_{k=0}^{i-1}u_k(t_1,t_2),\sum_{k=0}^{i-1}v_k(t_1,t_2)\Big)\Big] \tag{5.43}$$

Substitute eq.(5.43) in Eq.(5.41) and Eq. (5.42),gives

$$\sum_{i=0}^{\infty}u_i(t_1,t_2)=e^{(t_1-t_2)}+\eta_{2g}^{-1}\Big[\frac{1}{(q(\mu_2)-1)}\eta_{2g}\Big[1-v_0(t_1,t_2)\Big[u_0(t_1,t_2)\Big]_{t_1}\Big]\Big]$$
$$-\eta_{2g}^{-1}\Big[\frac{1}{(q(\mu_2)-1)}\eta_{2g}\Big[\sum_{i=1}^{\infty}\Big[\Big[\sum_{k=0}^{i}v_k(t_1,t_2)\Big]\Big[\sum_{k=0}^{i}u_k(t_1,t_2)\Big]_{t_1}-\Big[\sum_{k=0}^{i-1}v_k(t_1,t_2)\Big]\Big[\sum_{k=0}^{i-1}u_k(t_1,t_2)\Big]_{t_1}\Big]\Big]\Big] \tag{5.44}$$

Similarly,

$$\sum_{i=0}^{\infty}v_i(t_1,t_2)=e^{(-t_1+t_2)}-\eta_{2g}^{-1}\Big[\frac{1}{(q(\mu_2)+1)}\eta_{2g}\Big[1+u_0(t_1,t_2)\Big[v_0(t_1,t_2)\Big]_{t_1}\Big]\Big]$$
$$-\eta_{2g}^{-1}\Big[\frac{1}{(q(\mu_2)+1)}\eta_{2g}\Big[\sum_{i=1}^{\infty}\Big[\Big[\sum_{k=0}^{i}u_k(t_1,t_2)\Big]\Big[\sum_{k=0}^{i}v_k(t_1,t_2)\Big]_{t_1}-\Big[\sum_{k=0}^{i-1}u_k(t_1,t_2)\Big]\Big[\sum_{k=0}^{i-1}v_k(t_1,t_2)\Big]_{t_1}\Big]\Big]\Big] \tag{5.45}$$

Now using Eq.(5.44)and Eq. (5.45) by splitting the terms, we define Recursive relation as,
For $i=0$

$$u_0(t_1,t_2)=e^{(t_1-t_2)}\quad v_0(t_1,t_2)=e^{(-t_1+t_2)}$$

For $i=1$

$$\begin{aligned}u_1(t_1,t_2)&=\eta_{2g}^{-1}\Big[\frac{1}{(q(\mu_2)-1)}\eta_{2g}\Big[1-v_0(t_1,t_2)\Big[u_0(t_1,t_2)\Big]_{t_1}\Big]\Big]\\&=\eta_{2g}^{-1}\Big[\frac{1}{(q(\mu_2)-1)}\eta_{2g}\Big[1-e^{(-t_1+t_2)}\Big[e^{(t_1-t_2)}\Big]_{t_1}\Big]\Big]=0\\v_1(t_1,t_2)&=-\eta_{2g}^{-1}\Big[\frac{1}{(q(\mu_2)+1)}\eta_{2g}\Big[1+u_0(t_1,t_2)\Big[v_0(t_1,t_2)\Big]_{t_1}\Big]\Big]\\&=-\eta_{2g}^{-1}\Big[\frac{1}{(q(\mu_2)+1)}\eta_{2g}\Big[1+e^{(t_1-t_2)}\Big[e^{(-t_1+t_2)}\Big]_{t_1}\Big]\Big]=0\end{aligned}$$

For $i=2$

$$u_2(t_1,t_2)=-\eta_{2g}^{-1}\Big[\frac{1}{(q(\mu_2)-1)}\eta_{2g}\Big[\Big[v_0(t_1,t_2)+v_1(t_1,t_2)\Big]\Big[u_0(t_1,t_2)+u_1(t_1,t_2)\Big]_{t_1}-[v_0(t_1,t_2)]\Big[u_0(t_1,t_2)\Big]_{t_1}\Big]\Big]$$
$$=-\eta_{2g}^{-1}\Big[\frac{1}{(q(\mu_2)-1)}\eta_{2g}\Big[\Big[e^{(-t_1+t_2)}+0\Big]\Big[e^{(t_1-t_2)}+0\Big]_{t_1}-[e^{(-t_1+t_2)}][e^{(t_1-t_2)}]_{t_1}\Big]\Big]$$
$$=0$$

$$\begin{aligned}
v_2(t_1,t_2) &= -\eta_{2g}^{-1}\Big[\frac{1}{(q(\mu_2)+1)}\eta_{2g}\Big[\Big[u_0(t_1,t_2)+u_1(t_1,t_2)\Big]\Big[v_0(t_1,t_2)+v_1(t_1,t_2)\Big]_{t_1}-[u_0(t_1,t_2)]\Big[v_0(t_1,t_2)\Big]_{t_1}\Big]\Big]\\
&= -\eta_{2g}^{-1}\Big[\frac{1}{(q(\mu_2)+1)}\eta_{2g}\Big[\Big[e^{(t_1-t_2)}+0\Big]\Big[e^{(-t_1+t_2)}+0\Big]_{t_1}-[e^{(t_1-t_2)}][e^{(-t_1+t_2)}]_{t_1}\Big]\Big]\\
&= 0
\end{aligned}$$

And similarly, we get all other components $u_i(t_1,t_2), v_i(t_1,t_2)$ are zero.,For $i \geq 1$

$$\begin{aligned}
\therefore\quad u(t_1,t_2) &= u_0(t_1,t_2)+u_1(t_1,t_2)+u_2(t_1,t_2)+...\\
&= e^{(t_1-t_2)}+0+0...\\
&= e^{t_1-t_2}\\
\therefore\quad v(t_1,t_2) &= v_0(t_1,t_2)+v_1(t_1,t_2)+v_2(t_1,t_2)+...\\
&= e^{(-t_1+t_2)}+0+0...\\
&= e^{-t_1+t_2}
\end{aligned}$$

Consider the coupled Burgers Equation

Exaple. 5

$$\left.\begin{aligned}
\frac{\partial u}{\partial t_2} &= \frac{\partial^2 u}{\partial t_1^2}+2u\frac{\partial u}{\partial t_1}-\frac{\partial(uv)}{\partial t_1}\\
\frac{\partial v}{\partial t_2} &= \frac{\partial^2 v}{\partial t_1^2}+2v\frac{\partial v}{\partial t_1}-\frac{\partial(uv)}{\partial t_1}
\end{aligned}\right\} \tag{5.46}$$

with I.C.

$$u(t_1,0)=sint_1, v(t_1,0)=sint_1 \tag{5.47}$$

and B.C.

$$u(0,t_2)=v(0,t_2)=0, u_{t_1}(0,t_2)=v_{t_1}(0,t_2)=e^{-t_2} \tag{5.48}$$

Applying D.G.R.I.T. to both sides of Eq.(5.46),we get the following

$$\begin{aligned}
\frac{1}{\mu_2^{n_2}}\Lambda_{g_u}(\mu_1,0)-q(\mu_2)\Lambda_{D_u}(\mu_1,\mu_2) &= \frac{1}{\mu_1^{n_1}}\frac{\partial\Lambda_{g_u}(0,\mu_2)}{\partial t_1}-\frac{p(\mu_1)}{\mu_1^{n_1}}\Lambda_{g_u}(0,\mu_2)\\
&\quad+[p(\mu_1)]^2\Lambda_{D_u}(\mu_1,\mu_2)+\eta_{2g}\Big[2u\frac{\partial u}{\partial t_1}-\frac{\partial(uv)}{\partial t_1}\Big]
\end{aligned} \tag{5.49}$$

$$\begin{aligned}
\frac{1}{\mu_2^{n_2}}\Lambda_{g_v}(\mu_1,0)-q(\mu_2)\Lambda_{D_v}(\mu_1,\mu_2) &= \frac{1}{\mu_1^{n_1}}\frac{\partial\Lambda_{g_v}(0,\mu_2)}{\partial t_1}-\frac{p(\mu_1)}{\mu_1^{n_1}}\Lambda_{g_v}(0,\mu_2)\\
&\quad+[p(\mu_1)]^2\Lambda_{D_v}(\mu_1,\mu_2)+\eta_{2g}\Big[2v\frac{\partial v}{\partial t_1}-\frac{\partial(uv)}{\partial t_1}\Big]
\end{aligned} \tag{5.50}$$

Now, taking Single G.R.I.T.of Eq.(5.47) and (5.48) i.e I.C. and B.C. , we get

$$\Lambda_{g_u}(\mu_1,0)=\Lambda_{g_v}(\mu_1,0)=\frac{-1}{\mu_1^{n_1}}\frac{1}{[p(\mu_1)^2]+1},\ \Lambda_{g_u}(0,\mu_2)=\Lambda_{g_v}(0,\mu_2)=0 \tag{5.51}$$

$$\frac{\partial\Lambda_{g_u}(0,\mu_2)}{\partial t_1}=\frac{\partial\Lambda_{g_v}(0,\mu_2)}{\partial t_1}=\frac{1}{\mu_2^{n_2}}\frac{1}{(q(\mu_2)^2)-1} \tag{5.52}$$

Now, substituting conditions (5.51) and (5.52) in equations (5.49) and (5.50), simply, leads to

$$\big(-q(\mu_2)-p(\mu_1)^2\big)\Lambda_{D_u}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}\mu_2^{n_2}}\Big[\frac{1}{[p(\mu_1)^2]+1}+\frac{1}{q(\mu_2)-1}\Big]+\eta_{2g}\Big[2u\frac{\partial u}{\partial t_1}-\frac{\partial(uv)}{\partial t_1}\Big]$$

similarly,

$$\big(-q(\mu_2)-p(\mu_1)^2\big)\Lambda_{D_v}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}\mu_2^{n_2}}\Big[\frac{1}{[p(\mu_1)^2]+1}+\frac{1}{q(\mu_2)-1}\Big]+\eta_{2g}\Big[2v\frac{\partial v}{\partial t_1}-\frac{\partial(uv)}{\partial t_1}\Big]$$

After simplification, the above equations become

$$\Lambda_{D_u}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{-1}{\big([p(\mu_1)^2]+1\big)\big(q(\mu_2)-1\big)}+\frac{1}{\big(-q(\mu_2)-p(\mu_1)^2\big)}\eta_{2g}\Big[2u\frac{\partial u}{\partial t_1}-\frac{\partial(uv)}{\partial t_1}\Big] \tag{5.53}$$

$$\Lambda_{D_v}(\mu_1,\mu_2)=\frac{1}{\mu_1^{n_1}\mu_2^{n_2}}\frac{-1}{\big([p(\mu_1)^2]+1\big)\big(q(\mu_2)-1\big)}+\frac{1}{\big(-q(\mu_2)-p(\mu_1)^2\big)}\eta_{2g}\Big[2v\frac{\partial v}{\partial t_1}-\frac{\partial(uv)}{\partial t_1}\Big] \tag{5.54}$$

Applying Inverse D. G. R. I. T. on both sides of Eq.(5.53) and Eq.(5.54), gives1

$$u(t_1,t_2)=e^{-t_2}sint_1+\eta_{2g}^{-1}\Bigg[\frac{1}{\big(-q(\mu_2)-p(\mu_1)^2\big)}\eta_{2g}\Big[2u\frac{\partial u}{\partial t_1}-\frac{\partial(uv)}{\partial t_1}\Big]\Bigg] \tag{5.55}$$

$$v(t_1,t_2)=e^{-t_2}sint_1+\eta_{2g}^{-1}\Bigg[\frac{1}{\big(-q(\mu_2)-p(\mu_1)^2\big)}\eta_{2g}\Big[2v\frac{\partial v}{\partial t_1}-\frac{\partial(uv)}{\partial t_1}\Big]\Bigg] \tag{5.56}$$

Now, we Apply new iterative method ,
Consider

$$u(t_1,t_2)=\sum_{i=0}^{\infty}u_i(t_1,t_2),\quad v(t_1,t_2)=\sum_{i=0}^{\infty}v_i(t_1,t_2) \tag{5.57}$$

Use equation (5.57) in equations(5.55) and (5.56), we get

$$\sum_{i=0}^{\infty}u_i(t_1,t_2)=e^{-t_2}sint_1+\eta_{2g}^{-1}\Bigg[\frac{1}{\big(-q(\mu_2)-p(\mu_1)^2\big)}\eta_{2g}\Bigg[2\Big[\sum_{i=0}^{\infty}u_i(t_1,t_2)\Big]\Big[\sum_{i=0}^{\infty}u_i(t_1,t_2)\Big]_{t_1} - \Big[\Big(\sum_{i=0}^{\infty}u_i(t_1,t_2)\Big)\Big(\sum_{i=0}^{\infty}u_i(t_1,t_2)\Big)\Big]_{t_1}\Bigg]\Bigg] \tag{5.58}$$

$$\sum_{i=0}^{\infty}v_i(t_1,t_2)=e^{-t_2}sint_1+\eta_{2g}^{-1}\Bigg[\frac{1}{\big(-q(\mu_2)-p(\mu_1)^2\big)}\eta_{2g}\Bigg[2\Big[\sum_{i=0}^{\infty}v_i(t_1,t_2)\Big]\Big[\sum_{i=0}^{\infty}v_i(t_1,t_2)\Big]_{t_1} - \Big[\Big(\sum_{i=0}^{\infty}u_i(t_1,t_2)\Big)\Big(\sum_{i=0}^{\infty}u_i(t_1,t_2)\Big)\Big]_{t_1}\Bigg]\Bigg] \tag{5.59}$$

The nonlinear term N is decomposed as

$$N\Big(\sum_{i=0}^{\infty}u_i(t_1,t_2),\sum_{i=0}^{\infty}v_i(t_1,t_2)\Big)$$
$$=N\Big(u_0(t_1,t_2),v_0(t_1,t_2)\Big)+\sum_{i=1}^{\infty}\Big[N\Big(\sum_{k=0}^{i}u_k(t_1,t_2),\sum_{k=0}^{i}v_k(t_1,t_2)\Big)-N\Big(\sum_{k=0}^{i-1}u_k(t_1,t_2),\sum_{k=0}^{i-1}v_k(t_1,t_2)\Big)\Big] \tag{5.60}$$

Substituting equation(5.60) into equations(5.58) and (5.59), simplifying gives

$$\sum_{i=0}^{\infty} u_i(t_1,t_2) = e^{-t_2} sint_1 + \eta_{2g}^{-1}\left[\frac{1}{(-q(\mu_2)-p(\mu_1)^2)}\eta_{2g}\left[2u_0\frac{\partial u_0}{\partial t_1} - \frac{\partial(u_0v_0)}{\partial t_1}\right]\right]$$
$$+\eta_{2g}^{-1}\left[\frac{1}{(-q(\mu_2)-p(\mu_1)^2)}\eta_{2g}\left[2\sum_{i=1}^{\infty}\Big[\sum_{k=0}^{i}\left(u_k\frac{\partial u_k}{\partial t_1}\right)-\sum_{k=0}^{i-1}\left(u_k\frac{\partial u_k}{\partial t_1}\right)\Big]-\sum_{i=1}^{\infty}\Big[\sum_{k=0}^{i}\frac{\partial(u_kv_k)}{\partial t_1}-\sum_{k=0}^{i-1}\frac{\partial(u_kv_k)}{\partial t_1}\right.\right. \tag{5.61}$$

$$\sum_{i=0}^{\infty} v_i(t_1,t_2) = e^{-t_2} sint_1 + \eta_{2g}^{-1}\left[\frac{1}{(-q(\mu_2)-p(\mu_1)^2)}\eta_{2g}\left[2v_0\frac{\partial v_0}{\partial t_1} - \frac{\partial(u_0v_0)}{\partial t_1}\right]\right]$$
$$+\eta_{2g}^{-1}\left[\frac{1}{(-q(\mu_2)-p(\mu_1)^2)}\eta_{2g}\left[2\sum_{i=1}^{\infty}\Big[\sum_{k=0}^{i}\left(v_k\frac{\partial v_k}{\partial t_1}\right)-\sum_{k=0}^{i-1}\left(v_k\frac{\partial v_k}{\partial t_1}\right)\Big]-\sum_{i=1}^{\infty}\Big[\sum_{k=0}^{i}\frac{\partial(u_kv_k)}{\partial t_1}-\sum_{k=0}^{i-1}\frac{\partial(u_kv_k)}{\partial t_1}\Big]\right.\right. \tag{5.62}$$

Now, from equations(5.61)and (5.62) we define the recursive relation to find u_0, v_0, u_1, v_1 and... as

For $i = 0$

$$u_0(t_1,t_2) = e^{-t_2}\sin t_1 \quad v_0(t_1,t_2) = e^{-t_2}\sin t_1$$

For $i = 1$

$$\begin{aligned} u_1(t_1,t_2) &= \eta_{2g}^{-1}\left[\frac{1}{(-q(\mu_2)-p(\mu_1)^2)}\eta_{2g}\left[2u_0\frac{\partial u_0}{\partial t_1} - \frac{\partial(u_0v_0)}{\partial t_1}\right]\right] \\ &= \eta_{2g}^{-1}\left[\frac{1}{(-q(\mu_2)-p(\mu_1)^2)}\eta_{2g}\left[2e^{-t_2}\sin t_1 e^{-t_2}\cos t_1 - e^{-t_2}e^{-t_2}2\sin t_1\cos t_1\right]\right] \\ &= 0 \end{aligned}$$

$$\begin{aligned} v_1(t_1,t_2) &= \eta_{2g}^{-1}\left[\frac{1}{(-q(\mu_2)-p(\mu_1)^2)}\eta_{2g}\left[2v_0\frac{\partial v_0}{\partial t_1} - \frac{\partial(u_0v_0)}{\partial t_1}\right]\right] \\ &= \eta_{2g}^{-1}\left[\frac{1}{(-q(\mu_2)-p(\mu_1)^2)}\eta_{2g}\left[2e^{-t_2}\sin t_1 e^{-t_2}\cos t_1 - e^{-t_2}e^{-t_2}2\sin t_1\cos t_1\right]\right] \\ &= 0 \end{aligned}$$

For $i = 2$

$$\begin{aligned} u_2(t_1,t_2) = \eta_{2g}^{-1}&\left[\frac{1}{(-q(\mu_2)-p(\mu_1)^2)}\eta_{2g}\left[\Big[2(u_0+u_1)\frac{\partial(u_0+u_1)}{\partial t_1} - \frac{\partial[(u_0+u_1)(v_0+v_1)]}{\partial t_1}\Big]\right.\right. \\ &\left.\left.-\Big[2u_0\frac{\partial u_0}{\partial t_1} - \frac{\partial(u_0v_0)}{\partial t_1}\Big]\right]\right] \end{aligned}$$

$$\begin{aligned} u_2(t_1,t_2) = \eta_{2g}^{-1}&\left[\frac{1}{(-q(\mu_2)-p(\mu_1)^2)}\eta_{2g}\left[\Big[2e^{-t_2}\sin t_1\frac{\partial(e^{-t_2}\sin t_1)}{\partial t_1} - \frac{\partial[(e^{-t_2}\sin t_1)(e^{-t_2}\sin t_1)]}{\partial t_1}\Big]\right.\right. \\ &\left.\left.-\Big[2e^{-t_2}\sin t_1 e^{-t_2}\cos t_1 - e^{-t_2}e^{-t_2}2\sin t_1\cos t_1\Big]\right]\right] \\ = 0& \end{aligned}$$

$$v_2(t_1,t_2) = \eta_{2g}^{-1}\Bigg[\frac{1}{\big(-q(\mu_2)-p(\mu_1)^2\big)}\eta_{2g}\Bigg[\Big[2(v_0+v_1)\frac{\partial(v_0+v_1)}{\partial t_1} - \frac{\partial[(v_0+v_1)(u_0+u_1)]}{\partial t_1}\Big]$$

$$- \Big[2v_0\frac{\partial v_0}{\partial t_1} - \frac{\partial(u_0v_0)}{\partial t_1}\Big]\Bigg]\Bigg]$$

$$v_2(t_1,t_2) = \eta_{2g}^{-1}\Bigg[\frac{1}{\big(-q(\mu_2)-p(\mu_1)^2\big)}\eta_{2g}\Bigg[\Big[2e^{-t_2}\sin t_1\frac{\partial(e^{-t_2}\sin t_1)}{\partial t_1} - \frac{\partial[(e^{-t_2}\sin t_1)(e^{-t_2}\sin t_1)]}{\partial t_1}\Big]$$

$$- \Big[2e^{-t_2}\sin t_1 e^{-t_2}\cos t_1 - e^{-t_2}e^{-t_2}2\sin t_1\cos t_1\Big]\Bigg]\Bigg]$$

$$= 0$$

and so on
Now By adding all components to get solution as

$$\begin{aligned} u(t_1,t_2) &= u_0(t_1,t_2) + u_1(t_1,t_2) + u_2(t_1,t_2) + ... \\ &= e^{-t_2}\sin t_1 + 0 + 0 + ... \\ &= e^{-t_2}\sin t_1 \end{aligned}$$

and similarly

$$\begin{aligned} v(t_1,t_2) &= v_0(t_1,t_2) + v_1(t_1,t_2) + v_2(t_1,t_2) + ... \\ &= e^{-t_2}\sin t_1 + 0 + 0 + ... \\ &= e^{-t_2}\sin t_1 \end{aligned}$$

6 Conclusion remarks

Systems of PDEs of integer order have been successfully solved using the double general Rangaig integral transform. Additionally, D.G.R.I.T is used in conjunction with a new iterative technique to solve systems of nonlinear PDEs of an integer order.Using techniques like the Adomian decomposition method, variational iteration method, and differential transform method, the same solution has already been found for the illustrative examples taken into consideration in both linear and nonlinear scenarios.

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