Palestine Journal of Mathematics

Vol 15(1)(2026) , 369–377

RINGS IN WHICH EVERY S-ALMOST PRIME IDEAL IS S-WEAKLY PRIME

Chahrazade Bakkari and Rachid Hachache

Communicated by: Najib Mahdou

MSC 2020 Classifications: Primary 13A15; Secondary 13B99, 13B30, 13G05.

Keywords and phrases: S-almost prime, S-weakly prime, trivial extension, amalgamated algebra, S-AW-rings.

The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper

Corresponding Author: Rachid Hachache

Abstract. Let R be a commutative ring with identity and S be a multiplicative subset of R. In this paper, we study the S-AW-ring, that is a ring in which every S-almost prime ideal is S-weakly prime. Some results including the characterizations and the transfer of S-AW-ring property to homomorphic image and localization are given. Also, we study the possible transfer of the S-AW-ring property between R and $R \propto E$ and between R and $R \bowtie^f J$. Our results provide new classes of commutative rings satisfying the above property.

1 Introduction

Throughout this paper, all rings are commutative with unity, all modules are unital. It is well-known that a proper ideal P of a ring R is said to be prime if whenever $x, y \in R$ with $xy \in P$, then either $x \in P$ or $y \in P$. Equivalently, if whenever $IJ \subseteq P$ for some ideals I, J of R, then either $I \subseteq P$ or $J \subseteq P$. Since the notion of prime ideals plays an important role in commutative ring theory, several generalizations of the concept of prime ideals have been studied in the literature, for example: almost prime, strongly prime, weakly prime, S-prime and S-almost prime ideals. Additional related concepts are discussed in [6, 18, 24]. Recall from [2] that a proper ideal of a ring R is said to be weakly prime if for $x, y \in R$ with $0 \neq xy \in P$, then either $x \in P$ or $y \in P$. Clearly every prime ideal is a weakly prime ideal, but the converse is not true, obviously $\{0\}$ is always weakly prime, but not prime provided that R is a ring which is an integral domain. For non-trivial examples, refer to [2]. In 2005, Bhatwadekar and Sharma [10] said a proper ideal P of an integral domain R to be almost prime if for $a, b \in R$ with $ab \in P - P^2$, then either $a \in P$ or $b \in P$. This definition can clearly be defined for any commutative ring R. Thus, a weakly prime ideal is almost prime, and any proper idempotent ideal is almost prime. Moreover, an ideal P of R is almost prime if and only if P/P^2 is a weakly prime ideal of R/P^2. Consider a multiplicative set S of a ring R that satisfies $0 \notin S$, $1 \in S$, and $xy \in S$ for all $x, y \in S$. Recently, the concept of S-extensions of some ideal structures has been taken important place in commutative algebra and it has drawn attention by several authors. The notion of S-prime ideals (resp., weakly) is first introduced and studied in (resp., [1]) [18]. An ideal P of a ring R disjoint with a multiplicative set S is said to be (resp., weakly) S-prime ideal if there exists $s \in S$ such that for all $a, b \in R$ if $ab \in P$ (resp., $0 \neq ab \in P$) , then $sa \in P$ or $sb \in P$.
In this study, we will be mostly interested to S-almost prime ideals defined and studied in [6]. Let P be an ideal of R disjoint with a multiplicative set S. Then P is an S-almost prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in P - P^2$, then $sa \in P$ or $sb \in P$. Equivalently, if there exists $s \in S$ such that for all ideals I, J of R, if $IJ \subseteq P - P^2$, then $sI \subseteq P$ or $sJ \subseteq P$ (see [6]).
In this paper, we introduce a new type of ring of which every S-almost prime ideal is S-weakly prime (called S-AW-ring) as a generalization of ring in which every almost prime ideal is weakly

prime (called AW-ring). If $S \subseteq U(R)$, then the concepts of S-AW-ring and AW-ring coincide. We next investigate the possible transfer of the ring property that every S-almost prime ideal is S-weakly prime in direct product, homomorphic image, localization, trivial ring extensions, and the amalgamation rings along an ideal in order to built new interesting examples. We give characterizations for $(S \propto E)$-AW rings regarding the trivial ring extension of a ring R by an R-module E denoted by $R \propto E$ (see Theorem 2.19). Moreover, we investigate the relationship between S-AW-rings A and the ideals of the ring $A \bowtie^f J$ the amalgamation of A and B along J with respect to f (see Theorem 2.24 and Corollaries 2.28, 2.29).

2 Main Results

Let R be a ring. As it is frequently used in this sequel, we should recall from [10] that an ideal P of R is said to be an almost prime ideal of R if for all $a, b \in R$, $ab \in P - P^2$, then $a \in P$ or $b \in P$. Now the following is our key definition.

Definition 2.1. A ring R is called AW-ring if every almost prime ideal is weakly prime.

Example 2.2. The ring of integers $\mathbb{Z}$ is an AW-ring.

Proof. Let $P = n\mathbb{Z}$ be an almost prime ideal of $\mathbb{Z}$. Assume that P is not prime, then $n = ab$ with $a, b \geq 2$ thus $ab \in P - P^2$ but neither $a \in P$ nor $b \in P$, a contradiction. Hence n must be irreducible. Thus P is prime and R is an AW-ring. □

A ring R is a regular ring if every element in R is a regular element.

Example 2.3. Let R be a regular ring. Then R is an AW-ring.

Proof. Follows from [10, Theorem 2.15]. □

Example 2.4. Let (R, M) is quasilocal with $M^2 = 0$. Then R is an AW-ring.

Proof. From [7, Theorem 17] we have every proper ideal almost prime and since (R, M) is quasilocal with $M^2 = 0$, then every ideal is weakly prime by [2, Theorem 8]. So R is an AW-ring. □

Let R be a ring and S be a multiplicative set of R. As it is frequently used in this sequel, we should recall from [6] that an ideal P of R disjoint with S is said to be an S-almost prime ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in P - P^2$, then $sa \in P$ or $sb \in P$. In this case, P is said to be S-almost prime ideal associated with s. Now the following is our key definition.

Definition 2.5. A ring R with multiplicative set S is called S-AW-ring if every S-almost prime ideal is S-weakly prime.

Example 2.6. Consider the ring $R = \mathbb{Z}$ and $S = \{p^n / n \in \mathbb{N}\}$ for some prime p or $S = \mathbb{Z} \setminus \{0\}$. Then R is an S-AW-ring.

Example 2.7. Let (R, M) be a local ring with $M^2 = 0$ and S be a multiplicative set of R. Then R is an S-AW-ring.

Note that any AW-ring is an S-AW-ring for every multiplicative subset S of R. Moreover, if S consists of units of R, then the AW-ring and S-AW-ring coincide. However, the converse is not true in general as shown in the following example which illustrate non AW-ring which is an S-AW-ring.

Example 2.8. Let $R = K \times K \times K$ with K is a field, and let $S = \{(1, 1, 1), (1, 0, 0)\}$ be a multiplicative subset of R, then :

(i) R is not an AW-ring.

(ii) R is an S-AW-ring.

Proof. (i) Let $P = 0 \times 0 \times K$, then $P^2 = P$ and so P is almost prime but not weakly prime ideal. Indeed, we have $0 \neq (1,0,1)(0,1,1) \in P$ and neither $(1,0,1) \in P$ nor $(0,1,1) \in P$. So P is not weakly prime ideal of R. Therefore R is not AW-ring.

(ii) Now we clair that R is an S-AW-ring. Let P be an ideal of R disjoint with S, so $(1,0,0) \notin P$, thus $P = \{0\} \times P_1 \times P_2$ with $P_i = \{0\}$ or K for $i = 1,2$. Let $(a,b,c),(a',b',c') \in R$ such that $0 \neq (a,b,c)(a',b',c') \in P$, then $aa' = 0$ and so $a = 0$ or $a' = 0$, if $a = 0$, then $(1,0,0)(a,b,c) = (0,0,0) \in P$ and if $a' = 0$, then $(1,0,0)(a',b',c') = (0,0,0) \in P$. Thus P is an S-weakly prime ideal of R. Therefore R is an S-AW-ring.

□

We note that the second part of the [6, Theorem 2.9] does not hold in general. By the following theorem, we give a generalized and corrected version of [6, Theorem 2.9].

Theorem 2.9. *Let $f : R_1 \to R_2$ be a ring homomorphism, S be a multiplicative set of R_1. Then the following assertions hold.*

(i) If P is an S-almost prime ideal of R_1 containing Ker(f), then $f(P)$ is an $f(S)$-almost prime ideal of R_2.

(ii) Let f be an epimorphism. If J is an $f(S)$-almost prime ideal of R_2 and Ker$(f) \subseteq f^{-1}(J^2)$, then $f^{-1}(J)$ is an S-almost prime ideal of R_1.

Proof. First, we show that $f(S)$ is a multiplicative set of R_2. Clearly, for any $f(s_1), f(s_2) \in f(S)$, $f(s_1)f(s_2) = f(s_1s_2) \in f(S)$. Now, assume that $0 \in f(S)$. Then $f(x) = 0 = f(s)$ for some $x \in R$ and $s \in S$. This implies that $x - s \in Kerf$. Since $Kerf \subseteq P$ and $P \cap S = \emptyset$, we have $Kerf \cap S = \emptyset$. But, as $x - s \in Kerf$ and $x \in Kerf$, we conclude that $s \in Kerf \cap S$, a contradiction. Thus, $0 \notin f(S)$ and $f(S)$ is a multiplicative set of R_2.

(i) Assume that $f(s) \in f(P) \cap f(S)$ for some $s \in S$. Then $f(a) = f(s)$ for some $a \in P$ which yields $a - s \in K$erf $\subseteq P$, and so $s \in P \cap S$, a contradiction. Hence, $f(P) \cap f(S) = \emptyset$. Suppose that $f(a)f(b) \in f(P) - f(P)^2$. From Kerf $\subseteq P$, we have $ab \in P$ and clearly $ab \notin P^2$. Since P is an S-almost prime ideal of R_1, there exists $s \in S$ such that $sa \in P$ or $sb \in P$. Thus, there exists $f(s) \in f(S)$ satisfying $f(s)f(a) \in f(P)$ or $f(s)f(b) \in f(P)$, as required.

(ii) If $s \in f^{-1}(J) \cap S$, then $f(s) \in J\cap f(S)$, a contradiction. Hence, $f^{-1}(J) \cap S = \emptyset$. Let $a, b \in R_1$ such that $ab \in f^{-1}(J) - f^{-1}(J)^2$. It is clear that $f(a)f(b) \in J$. Now, we show that $f(a)f(b) \notin J^2$. If $f(ab) \in J^2$, then $f(ab) = f(c)$ for some $c \in f^{-1}(J^2)$. It implies that $ab - c \in K$erf $\subseteq f^{-1}(J^2)$ and thus $ab \in f^{-1}(J^2) = f^{-1}(J)^2$, a contradiction. Therefore, $f(a)f(b) \in J - J^2$ and there exists $f(s) \in f(S)$ such that $f(s)f(a) \in J$ or $f(s)f(b) \in J$. Thus, $sa \in f^{-1}(J)$ or $sb \in f^{-1}(J)$, we are done.

□

Let R be a ring, $S \subseteq R$ be multiplicative set and I an ideal of R such that $I \cap S = \emptyset$. Notice that $\overline{S} = \{\overline{s} \ / \ s \in S\}$ is a multiplicative subset of R/I.

Now, we are ready to give the following relationship between S-almost prime ideals of a ring and those of their quotient rings.

Corollary 2.10. *(The corrected version of [6, Theorem 2.9]) Let S be a multiplicative set of a ring R, P and I be ideals of R such that $I \subseteq P^2$. Then the following are equivalent:*

(i) P is an S- almost prime ideal of R.

(ii) P/I is an $\overline{S}$-almost prime ideal of R/I.

We are ready to give the following relationship between S-AW-ring and those of their quotient rings.

Proposition 2.11. *Let R be a ring, S be a multiplicative subset of R and I an ideal of R. If R is a ring of which every ideal P satisfying $I \subseteq P^2$ is an S-AW-ring, then R/I is an $\overline{S}$-AW-ring. The converse is true if I is an S-weakly prime ideal of R.*

Proof. Assume that R is an S-AW-ring, let J be an ideal of R/I disjoint with $\overline{S}$, then $J = P/I$ is $\overline{S}$-almost prime ideal of R/I where P is an ideal of R containing I and disjoint with S is S-almost prime ideal of R by Corollary 2.10. As R is an S-AW-ring, we get P is an S-weakly prime of R, and so $J = P/I$ is an $\overline{S}$-weakly prime ideal of R/I by [22, Proposition 2.6]. Therefore, R/I is an $\overline{S}$-AW-ring. Conversely, let P be an S-almost prime ideal of R, then P/I is an $\overline{S}$-almost prime of R/I by Corollary 2.10. As R/I an is $\overline{S}$-AW-ring, we get P/I is an $\overline{S}$-weakly prime of R, since I is an S-weakly prime ideal of R, hence P is an S-weakly prime ideal of R by [22, Proposition 2.6]. Therefore, R is an S-AW-ring. □

Proposition 2.12. *Let R be a ring and S be a multiplicative subset of R consist of regular elements. If $S^{-1}R$ is an AW-ring, then R is an S-AW-ring.*

Proof. Let P be an S-almost prime ideal of R such that $P \cap S = \emptyset$. Then $S^{-1}P$ is an almost prime ideal of $S^{-1}R$ and $S^{-1}P \cap R = (P : s) \cup (0_S \cap R)$ for some $s \in S$ by [6, Theorem 2.12]. Since $S^{-1}R$ is an AW-ring, we get $S^{-1}P$ is a weakly prime of $S^{-1}R$. We show that P is an S-weakly prime ideal of R. Let $a, b \in R$ such that $0 \neq ab \in P$, then $0 \neq \frac{a}{1}\frac{b}{1} \in S^{-1}P$ since S is consist of regular elements, and so $\frac{a}{1} \in S^{-1}P$ or $\frac{b}{1} \in S^{-1}P$. But If $\frac{a}{1} \in S^{-1}P$, then $\frac{a}{1} = \frac{x}{s}$ for some $x \in P$ and for some $s \in S$, thus $tas = tx \in P$ for some $t \in S$ and so $a = \frac{ast}{st} \in S^{-1}P \cap R$. If $a \in (P : s)$, we get $sa \in P$ for some $s \in S$, if $a \in (0_S \cap R)$, then $as' = 0 \in P$ for some $s' \in S$. Therefore, P is an S-weakly prime ideal of R. □

Theorem 2.13. *Let $R := R_1 \times R_2$ be a ring, where R_1 and R_2 are two rings and $S = S_1 \times S_2$ be a multiplicative set of R, with S_1 and S_2 are multiplicative sets of R_1 and R_2 respectively. Then R is an S-AW-ring if and only if every S_i-almost prime ideal of R_i is S_i-prime for $i = 1, 2$.*

It is noteworthy that the author in [6, Theorem 2.7] investigates the structure of almost prime ideals in the direct product $R_1 \times R_2$. However, their proof relies on the assumption that $(0, 0) \in P - P^2$, which does not hold in general. The proof of Theorem 2.13 requires the following lemmas.

Lemma 2.14. *Let $R := R_1 \times R_2$ be a ring, where R_1 and R_2 are two rings, $S = S_1 \times S_2$ be a multiplicative set of R, with S_1 and S_2 are multiplicative sets of R_1 and R_2 respectively and P_1 and P_2 be an nonzero ideals of R_1 and R_2 respectively. If $P := P_1 \times P_2$ is an S-almost prime of R, then P_1 is an S_1-almost prime of R_1 and $S_2 \cap P_2 \neq \emptyset$ or P_2 is an S_2-almost prime of R_2 and $S_1 \cap P_1 \neq \emptyset$.*

Proof. Let $(p, q) \in P - P^2$ with $p \in R_1$ and $q \in R_2$. Then, $(p, q) = (p, 1)(1, q) \in P - P^2$. Since, P is an S-almost prime of R, then there is $s = (s_1, s_2) \in S$ such that $s(p, 1) = (s_1 p, s_2) \in P$ or $s(1, q) = (s_1 1, s_2 q) \in P$. Thus, $S_1 \cap P_1 \neq \emptyset$ or $S_2 \cap P_2 \neq \emptyset$. Assume, $S_2 \cap P_2 \neq \emptyset$. As, $P \cap S = \emptyset$, we have $S_1 \cap P_1 = \emptyset$. Now, we show that P_1 is an S_1-prime ideal of R_1. Let $pp' \in P_1 - P_1^2$ for some $p, p' \in R_1$, then we have $(pp', 0) = (p, 0)\,(p', 0) \in P - P^2$. Hence, $s(p, 0) = (s_1 p, 0) \in P$ or $s\,(p', 0) = (s_1 p', 0) \in P$. So, we get $s_1 p \in P_1$ or $s_1 p' \in P_1$, as desired. □

Lemma 2.15. *Let R_1 and R_2 be commutative rings, S_1 and S_2 be multiplicative sets of R_1 and R_2, respectively, $R = R_1 \times R_2$ and $S = S_1 \times S_2$. Let P_1 and P_2 are proper ideals of R_1 and R_2, respectively, then the following statements hold.*

(i) P_1 is an S_1-almost prime ideal of R_1 if and only if $P_1 \times R_2$ is an S-almost prime ideal of R.

(ii) P_2 is an S_2-almost prime ideal of R_2 if and only if $R_1 \times P_2$ is an S-almost prime ideal of R.

Proof. (i) Assume that P_1 is an S_1-almost prime ideal of R_1. Let $(a, b), (a', b') \in R$ such that $(a, b)(a', b') = (aa', bb') \in P_1 \times R_2 - (P_1 \times R_2)^2$. Then $aa' \in P_1 - P_1^2$ which implies that there exists $s_1 \in S_1$ such that $s_1 a \in P_1$ or $s_1 a' \in P_1$. Put $s = (s_1, 1) \in S$. Hence $s(a, b) \in P_1 \times R_2$ or $s(a', b') \in P_1 \times R_2$, then $P_1 \times R_2$ is an S-almost prime ideal of R.
Conversely, suppose that $P_1 \times R_2$ is an S-almost prime ideal of R, and $a, b \in R_1$ such that $ab \in P_1 - P_1^2$. Then $(a, 0)(b, 0) \in P_1 \times R_2 - (P_1 \times R_2)^2$ and there exists $(s_1, s_2) \in S$ such that $(s_1, s_2)(a, 0) \in P_1 \times R_2$ or $(s_1, s_2)(b, 0) \in P_1 \times R_2$, hence $s_1 a \in P_1$ or $s_1 b \in P_1$. Thus, P_1 is an S_1-almost prime ideal of R_1.

(ii) Similar to (i).

□

***Proof of Theorem* 2.13.** Assume that R is an S-AW-ring. Let P_1 be an S_1-almost prime ideal of R_1 so $P_1 \times R_2$ is an S-almost prime ideal of R by Lemma 2.15. As R is an S-AW-ring, then $P_1 \times R_2$ is an S-weakly prime. Thus P_1 is an S_1-prime by [1, Proposition 2.24]. Likewise, we show that every S_2-almost prime ideal of R_2 is S_2-prime. Conversely. Let $P = P_1 \times P_2$ be an S-almost prime of R. Thus by Lemma 2.14, we have either P_1 is an S_1-almost prime of R_1 and $S_2 \cap P_2 \neq \emptyset$ or P_2 is an S_2-almost prime of R_2 and $S_1 \cap P_1 \neq \emptyset$, so P_1 is an S_1-prime ideal of R_1 and $S_2 \cap P_2 \neq \emptyset$ or P_2 is an S_2-prime ideal of R_2 and $S_1 \cap P_1 \neq \emptyset$, then P is an S-weakly prime ideal of R by [1, Proposition 2.24]. Thus R is an S-AW-ring. □

Proposition 2.16. *Let R be a ring, $S \subseteq R$ be a multiplicative set such that $P^2 = 0$ for every ideal P of R disjoint with S. Then R is an S-AW-ring.*

Proof. Let P be an S-almost prime ideal of R. Let I and J are ideals of R such that $0 \neq IJ \subset P$, so $0 \neq IJ \nsubseteq P^2 = 0$, hence there exists $s \in S$ such that $sI \subset P$ or $sJ \subset P$. Thus P is an S-weakly prime ideal of R. Therefore R is an S-AW-ring. □

Proposition 2.17. *Let $S_1 \subseteq S_2$ be multiplicative subsets of R such that for any $s \in S_2$, there is an element $t \in S_2$ satisfying $st \in S_1$. Then R is an S_2-AW-ring if and only if R is an S_1-AW-ring.*

Proof. Assume that R is an S_2-AW-ring. Let P be an S_1-almost prime ideal of R, then P is an S_2-almost prime ideal of R by [6, Remark 2.22]. Since R is an S_2-AW-ring hence P is an S_2-weakly prime ideal of R thus P is an S_1-weakly prime ideal of R by [1, Proposition 2.20]. On the other hand, assume that R is an S_1-AW-ring. Let P be an S_2-almost prime ideal of R and let $a, b \in R$ such that $ab \in P - P^2$. So, there is $s \in S_2$ such that $sa \in P$ or $sb \in P$. By the assumption, $s' = st \in S_1$ for some $t \in S_2$, and, then $s'a \in P$ or $s'b \in P$. Consequently, P is an S_1-almost prime ideal of R. Since R is an S_1-AW-ring, so P is an S_1-weakly prime ideal of R thus P is an S_2-weakly prime ideal of R by [1, Remark 2.19]. □

Let S be a multiplicative subset of R, $S^* = \{r \in R \mid \frac{r}{1}\, is\, unit\, in\, S^{-1}R\}$ denotes the saturation of S. Note that, S^* is a multiplicative subset containing S. A multiplicative subset S of R is called saturated if $S = S^*$. It is clear that S^* is always a saturated multiplicative subset of R.

Proposition 2.18. *Let R be a ring, S be a multiplicative subset of R. Then R is an S-AW-ring if and only if R is an S^*-AW-ring.*

Proof. It is clear that $S^* \cap P = \emptyset$. We will show that for any $r \in S^*$, there is $r' \in S^*$ such that $rr' \in S_1$. Let $r \in S^*$, then $\frac{r}{1}.\frac{a}{s} = 1$ for some $s \in S$ and $a \in R$. This implies that $tar = ts \in S_1$, for some $t \in S$. Now, take $r' = ta$. Then we have $r' \in S^*$ with $rr' \in S$, and so the desired condition is satisfied. Therefore, by putting $S_1 = S$ and $S_2 = S^*$, we conclude immediately the result from the Proposition 2.17. □

Let R be a ring and E an $R-$module. The trivial ring extension of R by E (also called the idealization of E over R) is a commutative ring

$$R \propto E := \{(a, e) | a \in R \ \ and \ \ e \in E\}$$

under the usual addition and the multiplication defined by $(a, e)(b, f) = (ab, af + be)$ for all $(a, e), (b, f) \in R \propto E$. It is clear that $(1, 0)$ is the identity of $R \propto E$, and if S is a multiplicative subset of R, then $S \propto E$, and $S \propto 0$ are multiplicative subsets of $R \propto E$. Trivial ring extensions play an important role in commutative ring theory due to their effectiveness of producing new classes of examples and counter-examples of rings subject to various ring theoretic properties. For more detail the author may refer to [3]- [8], [20, 21].

Theorem 2.19. *Let R be a ring, E an $R-$module, let $R \propto E$ be the trivial ring extension of R by E and S be a multiplicative set of R. Then the following statements hold :*

(1) If $R \propto E$ is an $(S \propto E)$-AW-ring, then R is an S-AW-ring.

(2) *Let R be an integral domain with quotient field K and E is a K-vector space. Then the following assertions are equivalent:*

i) *$R \propto E$ is an $(S \propto E)$-AW-ring.*

ii) *R is an S-AW-ring.*

Proof. (1) Assume that $R \propto E$ is an $(S \propto E)$-AW-ring. Let P be an S- almost prime ideal of R, then $P \propto E$ is an $(S \propto E)$-almost prime of $R \propto E$ by [6, Theorem 3.1]. Since $R \propto E$ is an $(S \propto E)$-AW-ring, then $P \propto E$ is an $(S \propto E)$-weakly prime of $R \propto E$, hence P is an S-weakly prime of R by [22, Theorem 3.1], and so we conclude that R is an S-AW-ring.

(2) $(i) \Rightarrow (ii)$. From (1).
$(ii) \Rightarrow (i)$. Recall that if R is an integral domain with quotient field K and E be a divisible R-module, the ideals of $R \propto E$ are of the form $P \propto E$ where P is an ideal of R, or $0 \propto F$ where F is a R-submodule of E. Assume that R is an S-AW ring.
Case 1 : Suppose that $J = P \propto E$ is an ideal of $R \propto E$ disjoint with $S \propto E$. Then cleary $P \cap S = \emptyset$. If J is an $(S \propto E)$-almost prime of $R \propto E$, then P is an S-almost prime of R by [6, Theorem 3.1]. Since R is an S-AW-ring, then P is an S-weakly prime of R, hence $(S \propto E)$-weakly prime of $R \propto E$ by [22, Corollary 3.3].
Case 2 : Assume that $J = 0 \propto F$ is an $(S \propto E)$-almost prime ideal of $R \propto E$, then $0 \propto F$ is an $(S \propto E)$-weakly prime ideal of $R \propto E$ since $J^2 = 0$. Thus $R \propto E$ is an $(S \propto E)$-AW-ring.

□

Example 2.20. Consider the ring $\mathbb{Z}$ and the multiplicative set $S = \mathbb{Z} \setminus \{0\}$ of $\mathbb{Z}$. Then $\mathbb{Z} \propto Q$ is an $(S \propto Q)$-AW- ring, by Theorem 2.19 (2).

Example 2.21. Let $T = \mathbb{Z} \setminus p\mathbb{Z}$ where p is a prime number. Consider $R = \mathbb{Z}_{(p)} = T^{-1}\mathbb{Z}$ which is a local domain with maximal ideal $M = p\mathbb{Z}_{(p)}$. Let E be an R/M-vector space and S be a multiplicative set of R. Then $R \propto E$ is an $(S \propto E)$-AW- ring.

Let (R, R') be a pair of rings, J be an ideal of R' and $f : R \longrightarrow R'$ be a homomorphism. In this section, we consider the following subring of $R \times R'$

$$R \bowtie^f J = \{(a, f(a) + j) : a \in R \text{ and } j \in J\}$$

is called the amalgamation of R and R' along J with respect to f. If f is the identity homomorphism on R, then we get the amalgamated duplication of R along an ideal J, $R \bowtie J = \{(a, a + j) : a \in R, j \in J\}$. As a natural generalization of the duplication construction in [12], the amalgamation ring was initiated by D'Anna, Finocchiaro and Fontana. For more details regarding amalgamation rings, we refer the reader to [11, 13, 14, 16, 17].

Theorem 2.22. *Let $f : R \longrightarrow R'$ be a ring homomorphism and J be a nonzero proper ideal of R'. Assume that $f^{-1}(J) \neq 0$, and $Nil(R') \cap J \neq \{0\}$. Then $R \bowtie^f J$ has every proper ideal almost prime if and only if $(R, \mathfrak{m})$ is a quasilocal ring with $\mathfrak{m}^2 = 0$ and $(f(R) + J, f(\mathfrak{m}) + J)$ is a quasilocal ring with $(f(\mathfrak{m}) + J)^2 = 0$.*

The proof of Theorem 2.22 requires the following lemma.

Lemma 2.23. *[7, Theorem 17]. A commutative ring R has every proper (principal) ideal almost prime if and only if R is von Neumann regular or (R, M) is quasilocal with $M^2 = 0$.*

Proof of Theorem 2.22. Assume that $R \bowtie^f J$ has every proper ideal almost prime. Then by Lemma 2.23, two cases are possible:
Case 1 : $(R \bowtie^f J, M)$ is quasilocal with $M^2 = 0$. By [14, Proposition 5.1 (3)], both $(R, \mathfrak{m})$ and $(f(R) + J, f(\mathfrak{m}) + J)$ are quasilocal rings. Notice that $f^{-1}(J) \neq R$ (as J is a proper ideal of R'). So, there exists a maximal ideal $\mathfrak{m}$ such that $f^{-1}(J) \subset \mathfrak{m}$. Combining [15, Proposition 2.6 (4)] and [15, Proposition 2.1 (2)], it follows that $\mathfrak{m} \bowtie^f J$ is the unique maximal ideal of $R \bowtie^f J$. Since $M^2 = (\mathfrak{m} \bowtie^f J)^2 = 0$, then one can easily check that $\mathfrak{m}^2 = 0$ and $(f(\mathfrak{m}) + J)^2 = 0$. Therefore, R and $f(R) + J$ have every proper ideal almost prime.

Case 2 : $R \bowtie^f J$ is von Neumann regular then $R \bowtie^f J$ is reduced and so $Nil(R') \cap J = \{0\}$, which is absurd, as $Nil(R') \cap J \neq \{0\}$, by the assumption.
Conversely, assume that R and $f(R)+J$ are quasilocal rings with $\mathfrak{m}^2 = 0$ and $(f(\mathfrak{m})+J)^2 = 0$. Since $f^{-1}(J) \neq 0$ and J is an nonzero proper ideal of R', then we may assume R and $f(R)+J$ are not fields. By [21, Corollary 5.5], $(R \bowtie^f J, M) = (R \bowtie^f J, \mathfrak{m} \bowtie^f J)$ is quasilocal. The fact that R and $f(R)+J$ have every proper ideal is weakly prime, so every proper ideal almost prime, then necessarily $\mathfrak{m}^2 = 0$ and $(f(\mathfrak{m})+J)^2 = 0$ and so $M^2 = (\mathfrak{m} \bowtie^f J)^2 \subset (\mathfrak{m} \times (f(\mathfrak{m})+J))^2 = 0$. Hence, $(\mathfrak{m} \bowtie^f J)^2 = 0$, as desired. □

Let S be a multiplicative set of a ring R. Notice that $S' = \{(s, f(s)) : s \in S\}$ is a multiplicative set of $R \bowtie^f J$. Also, if $0 \notin f(S)$ then $f(S)$ is a multiplicative set of R'.

Theorem 2.24. *Let R and R' be two rings, S be a multiplicative set of A, J an ideal of R' and $f : R \longrightarrow R'$ be a ring homomorphism. Then the following statements hold.*

(1) Let $f^{-1}(J) = \{0\}$. Then $R \bowtie^f J$ is an S'-AW-ring if and only if $f(R)+J$ is an $f(S)$-AW-ring.

(2) Let $f(a)J = 0$ for every nonunit $a \in R$. If $R \bowtie^f J$ is an S'-AW-ring, then R is an S-AW-ring.

(3) Assume that $f^{-1}(J) \neq 0$ and J be a nonzero proper ideal of R'. If $(R, \mathfrak{m})$ is a local ring with $\mathfrak{m}^2 = 0$ and $(f(R)+J, f(\mathfrak{m})+J))$ is a local ring with $(f(\mathfrak{m})+J)^2 = 0$, then $R \bowtie^f J$ is an S'-AW-ring.

Proof. (1) As $f^{-1}(J) = 0$, from the claim [14, Proposition 5.1(3)] we have the isomorphism $R \bowtie^f J \cong f(R)+J$. Let $\psi : R \bowtie^f J \to f(R)+J$ be the natural projection of $R \bowtie^f J \subseteq R \times (f(R)+J)$ into $f(R)+J$. Then ψ surjective ring homomorphism and its kernel is $\mathrm{Ker}(\psi) = f^{-1}(J) \times \{0\} = 0$. Thus ψ is an isomorphism ring homomorphism with $\psi(S') = f(S)$.

(2) Assume that $R \bowtie^f J$ is an S'-AW- ring. Let P be an S- almost prime ideal of R. We prove that $P \bowtie^f J$ is an S'-almost prime of $R \bowtie^f J$. Let $(a, f(a)+i), (b, f(b)+j) \in R \bowtie^f J$ such that $(a, f(a)+i)(b, f(b)+j) = (ab, f(ab)+f(a)j+f(b)i+ij) \in P \bowtie^f J - (P \bowtie^f J)^2$, then two cases are possible :
Case 1: If $ab \in P - P^2$, then there exists $s \in S$ such that $sa \in P$ or $sb \in P$. Hence $(s, f(s))(a, f(a)+i) \in P \bowtie^f J$ or $(s, f(s))(b, f(b)+j) \in P \bowtie^f J$. Hence, $P \bowtie^f J$ is an S'-almost prime ideal of $R \bowtie^f J$.
Case 2: If $ab \in P^2$, then $ab = \sum_{i=1}^{n} a_i b_i$ with $a_i, b_i \in P$ so $(ab, f(ab)+ij) = \sum_{i=2}^{n} (a_i b_i, f(a_i b_i)) + (a_1 b_1, f(a_1 b_1)+ij)$ hence $(ab, f(ab)+ij) = \sum_{i=2}^{n} (a_i, f(a_i)(b_i, f(b_i)) + (a_1, f(a_1)+i)(b_1, f(b_1)+j) \in (P \bowtie^f J)^2$ a contradiction. Then $P \bowtie^f J$ is an S'-almost prime ideal of $R \bowtie^f J$. Since $R \bowtie^f J$ is an S'-AW-ring, so $P \bowtie^f J$ is an S'-weakly prime ideal of $R \bowtie^f J$, then P is an S-weakly prime of R by [22, Theorem 3.6]. Thus R is an S-AW-ring.

(3) Let K be an S'-almost prime ideal of $R \bowtie^f J$, since $f^{-1}(J) \neq 0$ and $(A, \mathfrak{m})$ is a quasilocal ring with $\mathfrak{m}^2 = 0$ and $(f(A)+J, f(\mathfrak{m})+J)$ is a quasilocal ring with $(f(\mathfrak{m})+J)^2 = 0$, then K is an weakly prime ideal of $R \bowtie^f J$ by [23, Theorem 2.15]. Hence K is an S-weakly prime of $R \bowtie^f J$, thus $R \bowtie^f J$ is an S'-AW-ring.

□

In conclusion of the Theorem 2.24 (2), we have the following corollaries.

Corollary 2.25. *Let (R, M) be a local ring with $f(M)J = 0$. If $R \bowtie^f J$ is an S'-AW-ring, then R is an S-AW-ring.*

Corollary 2.26. *Let $f : R \longrightarrow R'$ be a ring homomorphism and J be a nonzero proper ideal of R', and $f^{-1}(J) \neq 0$. If $(R, \mathfrak{m})$ is a quasilocal ring with $\mathfrak{m}^2 = 0$ and $(f(R)+J, f(\mathfrak{m})+J)$ is a quasilocal ring with $(f(\mathfrak{m})+J)^2 = 0$. then $R \bowtie^f J$ is an AW-ring.*

Proof. It's sufficient to take $S = \{1\}$ in Theorem 2.24 (3). □

Let I be a proper ideal of a ring R. The amalgamated duplication of R along I is a special amalgamation given by

$$R \bowtie I := \{(a, a+i) \mid a \in R, i \in I\}$$

Note that if S is a multiplicative set of R, then $S' = \{(s,s) \mid s \in S\}$ is a multiplicative set of $R \bowtie I$. Set, V={P is S-almost prime ideal of R / $ab = 0$ with $sa \notin P$ and $sb \notin P$ for each $s \in S$, we have $aj + bi + ij = 0$ for each $i, j \in I$}.

Proposition 2.27. *Let R be a ring and S be a multiplicative set of R. If $R \bowtie I$ is an S'-AW-ring, then every ideal of V is S-weakly prime.*

Proof. Let P be an ideal of V, then $P \bowtie I$ is an S'-almost prime of $R \bowtie I$ by [6, Theorem 3.4]. Since $R \bowtie I$ is an S'-AW-ring hence $P \bowtie I$ is an S'-weakly prime of $R \bowtie I$ so P is an S-weakly peime of R by [22, Corollary 3.7]. □

Corollary 2.28. *Let R be a ring, S be a multiplicative set of R and I an ideal of R such that $aI = 0$ for every nonunit $a \in R$. If $R \bowtie I$ is an S'-AW-ring, then R is an S-AW-ring.*

Example 2.29. Let (R, M) be a local ring with $M^2 = 0$ (for instance $R = \mathbb{Z}/4\mathbb{Z}$) with $I = M$ and S be a multiplicative set of R, it follows from [9, Corollary 4] that $R \bowtie I$ is an S'-AW-ring thus R is an S-AW-ring.

References

[1] F. A. Almahdi, E. M. Bouba and M. Tamekkante, On weakly S-prime ideals of commutative rings, Analele Stiint. ale Univ. Ovidius Constanta Ser. Mat., 29(2), (2021), 173-186.

[2] D. D. Anderson and E. Smith, Weakly prime ideals, Houston J. Math. 29(4), (2003), 831-840.

[3] D. D. Anderson and M. Winders, Idealization of a module, J. Commun. Algebra, (2009), 3-56.

[4] M. M. Ali, Idealization and Theorems of D. D. Anderson, Commun. Algebra. 34, (2006), 4479-4501.

[5] M. M. Ali, Idealization and Theorems of D. D. Anderson II, Commun. Algebra. 35, (2007), 2767-2792.

[6] W. A. Alkasasbeh, M. Bataineh, Generalizations of S-Prime ideals. WSEAS Transactions on Mathematics, 20, (2021), 694-699.

[7] D. D. Anderson, M. Bataineh. Generalization of prime ideals, Comm algebra Vol. 36, 2008, pp. 686-696.

[8] C. Bakkari, S. Kabbaj and N. Mahdou, Trivial extension definided by Prüfer conditions, J. Pure App. Algebra. 214, (2010), 53-60.

[9] C. Bakkari and R. Hachache, Rings in Which Every Ideal is S-Weakly Prime Ideal, International Conference On Algebra And Its Applications, pp. 333-341, Springer Nature Singapore, (2023).

[10] S. M. Bhatwadekar, and P. K. Sharma, Unique factorization and birth of almost primes, Commun. Algebra, Vol.33, No.1, (2005), pp. 43-49.

[11] M. Chhiti, and S. Moindze, Nil-clean and weakly nil-clean properties in Bi-Amalgamated algebras along ideals, Palestine J. Math., Vol.11, No.2, (2022), pp. 551-560.

[12] M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, J. Algebra Appl. 6(3), (2007), 443-459.

[13] M. D'Anna and M. Fontana, The amalgamated duplication of a ring along a multiplicative-canonical ideal, Ark. Mat. 45(2), (2007), 241-252.

[14] M. D'Anna, C. A. Finacchiaro, and M. Fontana, Amalgamated algebras along an ideal, Commun. Algebra and Applications, Walter De Gruyter, (2009), 241-252.

[15] M. D'Anna, C.A. Finocchiaro and M. Fontana, Properties of chains of prime ideals in an amalgamated algebra along an ideal, J. Pure Appl. Algebra, 1633-1641, 2010.

[16] A. El Khalfi, H. Kim, and N. Mahdou, Amalgamation extension in commutative ring theory : a survey, Moroccan Journal of Algebra and Geometry with Applications 1(1) (2022), 139-182.

[17] M. El Ouarrachi, N. Mahdou, and A. Mimouni On Armendariz-like properties in Amalgamated algebras along ideals, Turkish J. Math, vol. 41, no. 6, (2017), 1673-1686.

[18] A. Hamed and A. Malek, S-prime ideals of a commutative ring, Contributions to Algebra and Geometry, 61(3) (2020), 533-542.

[19] J. A. Huckaba, Commutative Coherent Rings with Zero Divisors, Marcel Dekker, New York Basel, (1988).

[20] S. Kabbaj and N. Mahdou, Trivial Extensions Defined by Coherent-like Conditions, Commun. Algebra. 32, (2004), 3937-3953.

[21] S. Kabbaj, K. Louartiti and M. Tamekkante, Bi-amalgamated algebras along ideals, J. Commut. Algebra 9 (1), 65-87, 2017.

[22] N. Mahdou, M. A. S. Moutui and Y. Zahir, On S-Weakly prime ideals of commutative rings algebra, Hacettepe J. Math. Stat, Vol.49, No.3, (2020), 1159-1176.

[23] N. Mahdou, M. A. S. Moutui, and Y. Zahir, Weakly prime ideals issued from an amalgamated algebra,Hacet. J. Math. Stat., 49 (3), (2020), pp. 1159-1167.

[24] J. Y. Nehete, Generalizations of 2-absorbing δ-primary ideals in commutative rings, Palestine J. Math., Vol.13, No.4, (2024), pp. 396-402.

Author information

Chahrazade Bakkari, Department of Mathematics, Faculty of Science, University Moulay Ismail Meknes, Morocco.
E-mail: cbakkari@hotmail.com

Rachid Hachache, Department of Mathematics, Faculty of Science, University Moulay Ismail Meknes, Morocco.
E-mail: rachid.hachache@gmail.com

Received: 2025-03-23
Accepted: 2025-09-07