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QUATERNIONS WHOSE COMPONENTS ARE BLAISE AND BLAISE-LUCAS NUMBERS

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Abstract.In this paper, we introduce a new application of Blaise and Blaise-Lucas numbers. We define quaternions of the Blaise and Blaise-Lucas numbers. We calculate some important theorems; Cassini, Catalan, Vajda and d'Ocagne identities for quaternions of the Blaise and Blaise-Lucas numbers. We give the concepts of recurrence relation, characteristic equation, roots of the characteristic equation, Binet formula and generating function for these numbers. Finally, we give matrices representation of the Blaise and Blaise-Lucas Quaternions.

1 Introduction

Number sequences have been studied by many scientists until today. The most famous of these number sequences are the Fibonacci sequences and Fibonacci number sequences have found application in many branches of science, especially in algebra [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. The Blaise and Blaise-Lucas number sequences are also two of many different new number sequences have emerged. Blaise B_n and Blaise-Lucas C_n number sequences are defined by Soykan and examined many properties of these numbers [16]. Blaise and Blaise-Lucas number sequences satisfy fourth order linear recurrences. Blaise numbers are defined by

$$B_n = B_{n-1} + B_{n-2} + B_{n-3} - 2B_{n-4},\ n \geq 4 \tag{1.1}$$

with initial conditions $B_0 = 0,\ B_1 = 1,\ B_2 = 1$ and $B_3 = 2$. Similarly, the following equation is also satisfied.

$$B_n = B_{n-2} + 2B_{n-3} + 1\ with\ B_0 = 0,\ B_1 = 1,\ B_2 = 1 \tag{1.2}$$

Blaise-Lucas numbers are defined by

$$C_n = C_{n-1} + C_{n-2} + C_{n-3} - 2C_{n-4},\ n \geq 4 \tag{1.3}$$

with initial conditions $C_0 = 4,\ C_1 = 1,\ C_2 = 3$ and $C_3 = 7$. Note that the sequences $\{QB_n\}_{n\geq 0}$ and $\{QC_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} QB_{-n} &= \frac{1}{2}QB_{-(n-1)} + \frac{1}{2}QB_{-(n-2)} + \frac{1}{2}QB_{-(n-3)} - \frac{1}{2}QB_{-(n-4)}, \\ QC_{-n} &= \frac{1}{2}QC_{-(n-1)} + \frac{1}{2}QC_{-(n-2)} + \frac{1}{2}QC_{-(n-3)} - \frac{1}{2}QC_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

Blaise and Blaise-Lucas number sequences are closely related to many number sequences in the literature. Adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers can be given as examples of these series [18]. We recall that adjusted Jacobsthal-Padovan sequence $\{K_n\}_{n\geq 0}$, Jacobsthal-Perrin

(Jacobsthal-Perrin-Lucas) sequence $\{L_n\}_{n\geq 0}$, Jacobsthal-Padovan sequence $\{Q_n\}_{n\geq 0}$, and modified Jacobsthal-Padovan sequence $\{M_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} K_{n+3} &= K_{n+1}+2K_n, \quad K_0=0, K_1=1, K_2=0, &(1.4)\\ L_{n+3} &= L_{n+1}+2L_n, \quad L_0=3, L_1=0, L_2=2, &(1.5)\\ Q_{n+3} &= Q_{n+1}+2Q_n, \quad Q_0=1, Q_1=1, Q_2=1, &(1.6)\\ M_{n+3} &= M_{n+1}+2M_n, \quad M_0=3, M_1=1, M_2=3. &(1.7)\end{aligned}$$

For more information on Jacobsthal-Padovan sequence, see, for example, [**?**] and [**?**].

There are close relations between Blaise, Blaise-Lucas and Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan, modified Jacobsthal-Padovan numbers. For instance, they satisfy the following interrelations (see [16]):

$$\begin{aligned} 2C_n &= -K_{n+2}+6K_{n+1}+K_n+2,\\ 2B_n &= K_{n+2}+K_{n+1}+2K_n-1,\\ C_n &= L_n+1,\\ 52B_n &= 2L_n+9L_{n+1}+10L_{n+2}-26,\\ 2C_n &= -3Q_{n+2}+2Q_{n+1}+7Q_n+2,\\ 2B_n &= Q_{n+2}-1,\\ 46C_n &= -5M_{n+2}-6M_{n+1}+53M_n+46,\\ 46B_n &= 8M_{n+2}+5M_{n+1}-2M_n-23,\end{aligned}$$

and

$$\begin{aligned} 52K_n &= 9C_{n+2}-C_{n+1}-6C_n-2,\\ 2K_n &= -B_{n+2}+3B_n+1,\\ 2L_n &= C_{n+3}-C_{n+1},\\ 4L_n &= -3B_{n+2}+14B_{n+1}-9B_n+1,\\ 26Q_n &= 4C_{n+2}+C_{n+1}+6C_n-11,\\ 2Q_n &= -B_{n+2}+2B_{n+1}+B_n+1,\\ 26M_n &= 3C_{n+2}+4C_{n+1}+24C_n-31,\\ 2M_n &= -B_{n+2}+6B_{n+1}-3B_n+1.\end{aligned}$$

Quaternions are an important mathematical construct that is widely used in many engineering fields, especially in 3D computer graphics, robotics, quantum mechanics, and many other fields. In mathematics, quaternions are number systems that expand complex numbers into one real and three imaginary dimensions. They were first described by the Irish mathematician Sir William Rowan Hamilton in 1843 and applied to mathematics in 3D space [17]. Quaternions do not have the property of commutation (ab = ba). Although vectors and matrices have replaced quarters in many applications, they are still used in theoretical and applied mathematics. Quaternions have a wide range of uses, both theoretically and practically. Quaternions have been associated with number sequences in algebra and studies have been made by many scientists [18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. Quaternions are defined in the following form.

$$p = p_0 + p_1 i + p_2 j + p_3 k$$

where $p_0, p_1, p_2 and p_3$ are real numbers, and i, j, k are the main quaternions which are satisfy rules in Table 1,

.	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

Table 1. The main multiplications

In this study, we studied a new application of Blaise and Blaise-Lucas numbers with quaternions.We investigated both the quaternion properties and the sequence properties of these numbers and calculated some of their identities.Finally, we gave the matrix representations.

2 Quaternions of the Blaise numbers and Blaise-Lucas numbers

In this section, we define quaternions whose components are elements of the Blaise and Blaise-Lucas number sequences. We investigated some properties of these numbers.

Definition 2.1. Quaternions of the Blaise numbers and Blaise-Lucas numbers are defined by for $n \geq 0$,

$$(a)\ QB_n = B_n + iB_{n+1} + jB_{n+2} + kB_{n+3},$$

$$(b)\ QC_n = C_n + iC_{n+1} + jC_{n+2} + kC_{n+3},$$

where B_n is the nth Blaise numbers and C_n is the nth Blaise-Lucas numbers.

Some of the terms of the Quaternions of the Blaise numbers and Blaise-Lucas numbers are showed in Table 2 and Table 3.

n	QB_n
0	$i + j + 2k$
1	$1 + i + 2j + 4k$
2	$1 + 2i + 4j + 5k$
3	$2 + 4i + 5j + 9k$

Table 2. Some of the terms of the Quaternions of the Blaise numbers

n	QC_n
0	$4 + i + 3j + 7k$
1	$1 + i + 7j + 3k$
2	$3 + 7i + 3j + 11k$
3	$7 + 3i + 11j + 15k$

Table 3. Some of the terms of the Quaternions of the Blaise-Lucas numbers

Definition 2.2. The real part of the Quaternions of the Blaise numbers is denoted by $Re(QB_n)$ and defined by

$$Re\,(QB_n) = B_n.$$

The real part of the Quaternions of the Blaise numbers is denoted by $Im(QB_n)$ and defined by

$$Im\,(QB_n) = iB_{n+1} + jB_{n+2} + kB_{n+3}.$$

Similarly, for the Quaternions of the Blaise-Lucas numbers as follows.

$$Re\,(QC_n) = C_n$$

and

$$Im\,(QC_n) = iC_{n+1} + jC_{n+2} + kC_{n+3}.$$

Definition 2.3. The conjugate of the QB_n and QC_n as follows. The conjugate of the QB_n is denoted by $\overline{QB_n}$ and defined by

$$\overline{QB_n} = B_n - iB_{n+1} - jB_{n+2} - kB_{n+3}.$$

Similarly,

$$\overline{QC_n} = C_n - iC_{n+1} - jC_{n+2} - kC_{n+3}.$$

Theorem 2.4. *The following are provided.*

$$(a)\ \overline{QB_n} + QB_n = 2B_n = 2Re\left(QB_n\right),$$
$$(b)\ \overline{QC_n} + QC_n = 2C_n = Re\left(QC_n\right).$$

Proof. The proof can be easily demonstrated by using Definition 2.3. □

Theorem 2.5. *The norm of the QB_n and QC_n as follows.*

$$(a)\ N\left(QB_n\right) = \sqrt{{B_n}^2 + {B_{n+1}}^2 + {B_{n+2}}^2 + {B_{n+3}}^2},$$
$$(b)\ N\left(QC_n\right) = \sqrt{{C_n}^2 + {C_{n+1}}^2 + {C_{n+2}}^2 + {C_{n+3}}^2}.$$

Proof. We use Definition 2.3 for the proof.

Theorem 2.6. *(Recurrence relations) There is a relationship between the terms of the quaternions of Blaise and Blaise-Lucas numbers as follows:*

$$(a)\quad QB_n = QB_{n-1} + QB_{n-2} + QB_{n-3} - 2QB_{n-4}\ \ for\ n \geq 4,$$
$$(b)\quad QC_n = QC_{n-1} + QC_{n-2} + QC_{n-3} - 2QC_{n-4}\ \ for\ n \geq 4.$$

Proof. We use of the recurrence relation of the Blaise numbers for the proof. From (1.1), we have.

$$\begin{aligned}(a)\quad QB_n &= B_n + iB_{n+1} + jB_{n+2} + kB_{n+3} \\ &= \left(B_{n-1} + B_{n-2} + B_{n-3} - 2B_{n-4}\right) + i\left(B_n + B_{n-1} + B_{n-2} - 2B_{n-3}\right) \\ &\quad + j\left(B_{n+1} + B_n + B_{n-1} - 2B_{n-2}\right) + k\left(B_{n+2} + B_{n+1} + B_n - 2B_{n-1}\right) \\ &= \left(B_{n-1} + iB_n + jB_{n+1} + kB_{n+2}\right) + \left(B_{n-2} + iB_{n-1} + jB_n + kB_{n+1}\right) \\ &\quad + \left(B_{n-3} + iB_{n-2} + jB_{n-1} + kB_n\right) - 2\left(B_{n-4} + iB_{n-3} + jB_{n-2} + kB_{n-1}\right)\end{aligned}$$

$$QB_n = QB_{n-1} + QB_{n-2} + QB_{n-3} - 2QB_{n-4}.$$

Thus, the proof is completed. Similarly, the proof of b is shown using the recurrence relation of the Blaise-Lucas numbers. □

Theorem 2.7. *The following equations are provided for $n \geq 3$.*

$$(a)\quad QB_n = QB_{n-2} + 2QB_{n-3} + 1 + i + j + k,$$
$$(b)\quad QC_n = QC_{n-2} + 2QC_{n-3} - 2\left(1 + i + j + k\right).$$

Proof. $(a)\ QB_n = B_n + iB_{n+1} + jB_{n+2} + kB_{n+3}$, from (1.2), we have

$$\begin{aligned}QB_n &= \left(B_{n-2} + 2B_{n-3} + 1\right) + i\left(B_{n-1} + 2B_{n-2} + 1\right) + j(B_n + 2B_{n-1} + 1) \\ &\quad + k(B_{n+1} + 2B_n + 1) \\ &= \left(B_{n-2} + iB_{n-1} + jB_n + kB_{n+1}\right) + 2\left(B_{n-3} + iB_{n-2} + jB_{n-1} + kB_n\right) \\ &\quad + \left(1 + i + j + k\right) \\ &= QB_{n-2} + 2QB_{n-3} + 1 + i + j + k.\end{aligned}$$

The proof of (b) is shown similarly to (a). □

Theorem 2.8. *(Binet formula) We have*

$$(a)\quad QB_n=\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}\left(1+i+j+k\right)\ n\geq 3,$$

$$(b)\quad QC_n=\alpha^n\overline{\alpha}+\beta^n\overline{\beta}+\gamma^n\overline{\gamma}+\left(1+i+j+k\right),n\geq 0.$$

where

$$\widehat{\alpha}=2\alpha^2+4\alpha-6,\ \ \widehat{\beta}=2\beta^2+4\beta-6,\ \ \widehat{\gamma}=2\gamma^2+4\gamma-6$$
$$\overline{\alpha}=\left(1+i\alpha+j\alpha^2+k\alpha^3\right),\ \ \overline{\beta}=\left(1+i\beta+j\beta^2+k\beta^3\right)$$
$$\overline{\gamma}=(1+i\gamma+j\gamma^2+k\gamma^3).$$

and

$$\begin{aligned}\alpha &= \sqrt[3]{1+\frac{\sqrt{78}}{9}}+\sqrt[3]{1-\frac{\sqrt{78}}{9}}\simeq 1.521379706804568,\\ \beta &= \omega\sqrt[3]{1+\frac{\sqrt{78}}{9}}+\omega^2\sqrt[3]{1-\frac{\sqrt{78}}{9}},\\ \gamma &= \omega^2\sqrt[3]{1+\frac{\sqrt{78}}{9}}+\omega\sqrt[3]{1-\frac{\sqrt{78}}{9}},\\ \omega &= \frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3).\end{aligned}$$

Proof. (a) For the proof of the theorem, we use the Binet formula of the Blaise numbers.

$$\begin{aligned}QB_n &= B_n+iB_{n+1}+jB_{n+2}+kB_{n+3}\\ &=\left(\frac{\alpha^{n+3}}{\widehat{\alpha}}+\frac{\beta^{n+3}}{\widehat{\beta}}+\frac{\gamma^{n+3}}{\widehat{\gamma}}-\frac{1}{2}\right)+i\left(\frac{\alpha^{n+4}}{\widehat{\alpha}}+\frac{\beta^{n+4}}{\widehat{\beta}}+\frac{\gamma^{n+4}}{\widehat{\gamma}}-\frac{1}{2}\right)\\ &\quad+j\left(\frac{\alpha^{n+5}}{\widehat{\alpha}}+\frac{\beta^{n+5}}{\widehat{\beta}}+\frac{\gamma^{n+5}}{\widehat{\gamma}}-\frac{1}{2}\right)+k\left(\frac{\alpha^{n+6}}{\widehat{\alpha}}+\frac{\beta^{n+6}}{\widehat{\beta}}+\frac{\gamma^{n+6}}{\widehat{\gamma}}-\frac{1}{2}\right)\\ &=\frac{\alpha^{n+3}}{\widehat{\alpha}}\left(1+i\alpha+j\alpha^2+k\alpha^3\right)+\frac{\beta^{n+3}}{\widehat{\beta}}\left(1+i\beta+j\beta^2+k\beta^3\right)\\ &\quad+\frac{\gamma^{n+3}}{\widehat{\gamma}}\left(1+i\gamma+j\gamma^2+k\gamma^3\right)-\frac{1}{2}\left(1+i+j+k\right)\\ &=\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}\left(1+i+j+k\right).\end{aligned}$$

Thus, the proof is obtained. The proof of (b) is shown similarly to (a). □

Theorem 2.9. *(Generating functions) The following equations are provided.*

$$(a)\ \sum_{t=0}^{\infty}QB_tx^t=\frac{QB_0+(QB_1-QB_0)x+(QB_2-QB_1-QB_0)x^2+\left(QB_3-QB_2-QB_1-QB_0\right)x^3}{(1-x-x^2-x^3-2x^4)}$$

$$(b)\ \sum_{t=0}^{\infty}QC_tx^t=\frac{QC_0+(QC_1-QC_0)x+(QC_2-QC_1-QC_0)x^2+\left(QC_3-QC_2-QC_1-QC_0\right)x^3}{(1-x-x^2-x^3-2x^4)}.$$

Proof. (a) Let's

$$G=\sum_{t=0}^{\infty}QB_tx^t=QB_0+QB_1x+QB_2x^2+QB_3x^3+QB_4x^4+\cdots+QB_tx^t+\dots\ . \quad (2.1)$$

Now, if we multiply Equation (2.1) by $x,\ x^2, x^3$ and $2x^4$ respectively, we get the following.

$$xG = QB_0x + QB_1x^2 + QB_2x^3 + QB_3x^4 + QB_4x^5 + \cdots + QB_tx^{t+1} + \dots\ ,$$

$$x^2G = QB_0x^2 + QB_1x^3 + QB_2x^4 + QB_3x^5 + QB_4x^6 + \cdots + QB_tx^{t+2} + \dots\ ,$$

$$x^3G = QB_0x^3 + QB_1x^4 + QB_2x^5 + QB_3x^6 + QB_4x^7 + \cdots + QB_tx^{t+3} + \dots\ ,$$

$$2x^4G = QB_02x^4 + QB_12x^5 + QB_22x^6 + QB_32x^7 + QB_42x^8 + \cdots + QB_t2x^{t+4} + \dots\ .$$

Thus, if we do the necessary mathematical operations, we have.

$$G = \frac{QB_0 + (QB_1 - QB_0)\,x + (QB_2 - QB_1 - QB_0)\,x^2 + (QB_3 - QB_2 - QB_1 - QB_0)\,x^3}{(1 - x - x^2 - x^3 + 2x^4)}$$

Thus, the proof is obtained. The proof of (b) is shown similarly to (a). □

Theorem 2.10. *The exponential generating functions of the QB_n and QC_n as follows.*

(a) $\sum_{t=0}^{\infty} QB_t \frac{x^t}{t!} = \frac{\overline{\alpha}\alpha^3 e^{\alpha x}}{\widehat{\alpha}} + \frac{\overline{\beta}\beta^3 e^{\beta x}}{\widehat{\beta}} + \frac{\overline{\gamma}\gamma^3 e^{\gamma x}}{\widehat{\gamma}} - \frac{1}{2}\left(1+i+j+k\right)e^x$

(b) $\sum_{t=0}^{\infty} QC_t x^t = \overline{\alpha}e^{\alpha x} + \overline{\beta}e^{\beta x} + \overline{\gamma}\gamma^3 e^{\gamma x} + \left(1+i+j+k\right)e^x.$

Proof. (a)

$$\begin{aligned}\sum_{t=0}^{\infty} QB_t \frac{x^t}{t!} &= \sum_{t=0}^{\infty}\Big(\frac{\alpha^{t+3}\overline{\alpha}}{\widehat{\alpha}} + \frac{\beta^{t+3}\overline{\beta}}{\widehat{\beta}} + \frac{\gamma^{t+3}\overline{\gamma}}{\widehat{\gamma}} - \frac{1}{2}(1+i+j+k)\Big)\frac{x^t}{t!}\\ &= \frac{\overline{\alpha}\alpha^3}{\widehat{\alpha}}\sum_{t=0}^{\infty}\frac{(\alpha x)^t}{t!} + \frac{\overline{\beta}\beta^3}{\widehat{\beta}}\sum_{t=0}^{\infty}\frac{(\beta x)^t}{t!} + \frac{\overline{\gamma}\gamma^3}{\widehat{\gamma}}\sum_{t=0}^{\infty}\frac{(\gamma x)^t}{t!} - \frac{1}{2}\left(1+i+j+k\right)\sum_{t=0}^{\infty}\frac{x^t}{t!}\\ &= \frac{\overline{\alpha}\alpha^3 e^{\alpha x}}{\widehat{\alpha}} + \frac{\overline{\beta}\beta^3 e^{\beta x}}{\widehat{\beta}} + \frac{\overline{\gamma}\gamma^3 e^{\gamma x}}{\widehat{\gamma}} - \frac{1}{2}\left(1+i+j+k\right)e^x.\end{aligned}$$

Thus, the proof is completed. The proof of (b) is done similarly to (a). □

3 Some of The Identities of These Numbers

In this section, we calculate some identities of quaternions of the Blaise numbers and Blaise-Lucas numbers.

Theorem 3.1. *(Cassini identities) For $n \geq 1$,*

(a) $QB_{n-1}QB_{n+1} - {QB_n}^2$

$$\begin{aligned}&= 2^{n-2}\left[\frac{\gamma^{-n-2}\left(\alpha-\beta\right)\left(\beta\overline{\alpha}\overline{\beta} - \alpha\overline{\beta}\overline{\alpha}\right)}{\widehat{\alpha}\widehat{\beta}} + \frac{\beta^{-n-2}\left(\alpha-\gamma\right)\left(\alpha\overline{\gamma}\overline{\alpha} - \gamma\overline{\alpha}\overline{\gamma}\right)}{\widehat{\alpha}\widehat{\gamma}}\right.\\ &\quad\left. + \frac{\alpha^{-n-2}\left(\gamma-\beta\right)\left(\gamma\overline{\beta}\overline{\gamma} - \beta\overline{\gamma}\overline{\beta}\right)}{\widehat{\gamma}\widehat{\beta}}\right] + \left(1+i+j+k\right)\left(QB_n - \frac{1}{2}\left(QB_{n-1} + QB_{n+1}\right)\right)\end{aligned}$$

(b) $QC_{n-1}QC_{n+1} - {QC_n}^2$

$$\begin{aligned}&= 2^{n-1}\ \left[\gamma^{1-n}\left(\alpha-\beta\right)\left(\alpha\overline{\beta}\overline{\alpha} - \beta\overline{\alpha}\overline{\beta}\right) + \beta^{1-n}\left(\gamma-\alpha\right)\left(\gamma\overline{\alpha}\overline{\gamma} - \alpha\overline{\gamma}\overline{\alpha}\right)\right.\\ &\quad\left. +\alpha^{1-n}\left(\gamma-\beta\right)\left(\gamma\overline{\beta}\overline{\gamma} - \beta\overline{\gamma}\overline{\beta}\right)\right] + \left(1+i+j+k\right)\left[QC_{n-1} - 2QC_n + QC_{n+1}\right].\end{aligned}$$

Proof. We use the Binet formula of the quaternions of the Blaise numbers for the proof.

$$
\begin{aligned}
&(a)\ QB_{n-1}QB_{n+1}-QB_n{}^2=\\
&\left(\frac{\alpha^{n+2}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+2}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{n+2}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}(1+i+j+k)\right)\\
&\left(\frac{\alpha^{n+4}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+4}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{n+4}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}(1+i+j+k)\right)\\
&-\left(\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}(1+i+j+k)\right)^2\\
&=\left(\frac{\alpha^{n+2}\beta^{n+4}\overline{\alpha}\overline{\beta}}{\widehat{\alpha}\widehat{\beta}}+\frac{\alpha^{n+2}\gamma^{n+4}\overline{\alpha}\overline{\gamma}}{\widehat{\alpha}\widehat{\gamma}}-\frac{1}{2}\frac{\alpha^{n+2}\overline{\alpha}}{\widehat{\alpha}}(1+i+j+k)\right)\\
&+\left(\frac{\beta^{n+2}\alpha^{n+4}\overline{\beta}\overline{\alpha}}{\widehat{\beta}\widehat{\alpha}}+\frac{\beta^{n+2}\gamma^{n+4}\overline{\beta}\overline{\gamma}}{\widehat{\beta}\widehat{\gamma}}-\frac{1}{2}\frac{\beta^{n+2}\overline{\beta}}{\widehat{\beta}}(1+i+j+k)\right)\\
&+\left(\frac{\gamma^{n+2}\alpha^{n+4}\overline{\gamma}\overline{\alpha}}{\widehat{\gamma}\widehat{\alpha}}+\frac{\gamma^{n+2}\beta^{n+4}\overline{\gamma}\overline{\beta}}{\widehat{\gamma}\widehat{\beta}}-\frac{1}{2}\frac{\gamma^{n+2}\overline{\gamma}}{\widehat{\gamma}}(1+i+j+k)\right)\\
&-\frac{1}{2}(1+i+j+k)\frac{\alpha^{n+4}\overline{\alpha}}{\widehat{\alpha}}-\frac{1}{2}(1+i+j+k)\frac{\beta^{n+4}\overline{\beta}}{\widehat{\beta}}\\
&-\frac{1}{2}(1+i+j+k)\frac{\gamma^{n+4}\overline{\gamma}}{\widehat{\gamma}}-\frac{\alpha^{n+3}\beta^{n+3}\overline{\alpha}\overline{\beta}}{\widehat{\alpha}\widehat{\beta}}\\
&-\frac{\alpha^{n+3}\gamma^{n+3}\overline{\alpha}\overline{\gamma}}{\widehat{\alpha}\widehat{\gamma}}+\frac{1}{2}\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}(1+i+j+k)\\
&-\frac{\beta^{n+3}\alpha^{n+3}\overline{\beta}\overline{\alpha}}{\widehat{\beta}\widehat{\alpha}}-\frac{\beta^{n+3}\gamma^{n+3}\overline{\beta}\overline{\gamma}}{\widehat{\beta}\widehat{\gamma}}+\frac{1}{2}\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}(1+i+j+k)\\
&-\frac{\gamma^{n+3}\alpha^{n+3}\overline{\gamma}\overline{\alpha}}{\widehat{\gamma}\widehat{\alpha}}-\frac{\gamma^{n+3}\beta^{n+3}\overline{\gamma}\overline{\beta}}{\widehat{\gamma}\widehat{\beta}}+\frac{1}{2}\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}(1+i+j+k)\\
&+\frac{1}{2}(1+i+j+k)\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{1}{2}(1+i+j+k)\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}+\frac{1}{2}(1+i+j+k)\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}\\
&=\frac{\alpha^{n+2}\beta^{n+3}\overline{\alpha}\overline{\beta}}{\widehat{\alpha}\widehat{\beta}}(\beta-\alpha)+\frac{\alpha^{n+2}\gamma^{n+3}\overline{\alpha}\overline{\gamma}}{\widehat{\alpha}\widehat{\gamma}}(\gamma-\alpha)+\frac{\beta^{n+2}\alpha^{n+3}\overline{\beta}\overline{\alpha}}{\widehat{\beta}\widehat{\alpha}}(\alpha-\beta)\\
&+\frac{\beta^{n+2}\gamma^{n+2}\overline{\beta}\overline{\gamma}}{\widehat{\beta}\widehat{\gamma}}(\gamma-\beta)+\frac{\gamma^{n+2}\alpha^{n+3}\overline{\gamma}\overline{\alpha}}{\widehat{\gamma}\widehat{\alpha}}(\alpha-\gamma)+\frac{\gamma^{n+2}\beta^{n+3}\overline{\gamma}\overline{\beta}}{\widehat{\gamma}\widehat{\beta}}(\beta-\gamma)\\
&+(1+i+j+k)\left(\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}\frac{\alpha^{n+4}\overline{\alpha}}{\widehat{\alpha}}-\frac{1}{2}\frac{\beta^{n+4}\overline{\beta}}{\widehat{\beta}}-\frac{1}{2}\frac{\gamma^{n+4}\overline{\gamma}}{\widehat{\gamma}}\right.\\
&\left.-\frac{1}{2}\frac{\alpha^{n+2}\overline{\alpha}}{\widehat{\alpha}}-\frac{1}{2}\frac{\beta^{n+2}\overline{\beta}}{\widehat{\beta}}-\frac{1}{2}\frac{\gamma^{n+2}\overline{\gamma}}{\widehat{\gamma}}\right)\\
&=\frac{\alpha^{n+2}\beta^{n+2}(\alpha-\beta)}{\widehat{\alpha}\widehat{\beta}}\left(\beta\overline{\alpha}\overline{\beta}-\alpha\overline{\beta}\overline{\alpha}\right)+\frac{\alpha^{n+2}\gamma^{n+2}(\alpha-\gamma)}{\widehat{\alpha}\widehat{\gamma}}\left(\alpha\overline{\gamma}\overline{\alpha}-\gamma\overline{\alpha}\overline{\gamma}\right)\\
&+\frac{\gamma^{n+2}\beta^{n+2}\overline{\gamma}\overline{\beta}(\gamma-\beta)}{\widehat{\gamma}\widehat{\beta}}\left(\gamma\overline{\beta}\overline{\gamma}-\beta\overline{\gamma}\overline{\beta}\right)+(1+i+j+k)\\
&\left(QB_n-\frac{1}{2}\left(QB_{n-1}+QB_{n+1}\right)\right).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&QB_{n-1}QB_{n+1}-{QB_n}^2\\
&\quad=2^{n-2}\left[\frac{\gamma^{-n-2}(\alpha-\beta)\left(\beta\overline{\alpha}\overline{\beta}-\alpha\overline{\beta}\overline{\alpha}\right)}{\widehat{\alpha}\widehat{\beta}}+\frac{\beta^{-n-2}(\alpha-\gamma)(\alpha\overline{\gamma}\overline{\alpha}-\gamma\overline{\alpha}\overline{\gamma})}{\widehat{\alpha}\widehat{\gamma}}\right.\\
&\quad\left.+\ \frac{\alpha^{-n-2}(\gamma-\beta)\left(\gamma\overline{\beta}\overline{\gamma}-\beta\overline{\gamma}\overline{\beta}\right)}{\widehat{\gamma}\widehat{\beta}}\right]\\
&\quad+(1+i+j+k)\left(QB_n-\frac{1}{2}\left(QB_{n-1}+QB_{n+1}\right)\right).
\end{aligned}
$$

Thus, the proof is completed. The proof of (b) is done similarly to (a). □

Theorem 3.2. *(Catalan identities) For* $n\geq t$*,*

(a) $QB_{n-t}QB_{n+t}-{QB_n}^2$

$$
\begin{aligned}
&=2^{n+3-t}\left[\frac{\gamma^{-n-3+t}(\alpha^t-\beta^t)\left(\alpha^t\overline{\beta}\overline{\alpha}-\beta^t\overline{\alpha}\overline{\beta}\right)}{\widehat{\alpha}\widehat{\beta}}+\frac{\beta^{-n-3+t}(\alpha^t-\gamma^t)(\alpha^t\overline{\gamma}\overline{\alpha}-\gamma^t\overline{\alpha}\overline{\gamma})}{\widehat{\alpha}\widehat{\gamma}}\right.\\
&\left.+\ \frac{\alpha^{-n-3+t}(\gamma^t-\beta^t)\left(\gamma^t\overline{\beta}\overline{\gamma}-\beta^t\overline{\gamma}\overline{\beta}\right)}{\widehat{\gamma}\widehat{\beta}}\right]\\
&+(1+i+j+k)\left(QB_n-\frac{1}{2}\left(QB_{n-t+3}+QB_{n+t+3}\right)\right),
\end{aligned}
$$

(b) $QC_{n-t}QC_{n+t}-{QC_n}^2$

$$
\begin{aligned}
&=2^{n-t}\ \left[\gamma^{t-n}\left(\alpha^t-\beta^t\right)\left(\alpha^t\overline{\beta}\overline{\alpha}-\beta^t\overline{\alpha}\overline{\beta}\right)+\beta^{t-n}\left(\alpha^t-\gamma^t\right)\left(\alpha^t\overline{\gamma}\overline{\alpha}-\gamma^t\overline{\alpha}\overline{\gamma}\right)\right.\\
&\left.+\alpha^{t-n}\left(\gamma^t-\beta^t\right)\left(\gamma^t\overline{\beta}\overline{\gamma}-\beta^t\overline{\gamma}\overline{\beta}\right)\right]+(1+i+j+k)\left[QC_{n-t}-2QC_n+QC_{n+t}\right].
\end{aligned}
$$

Proof. We use the Binet formula of the quaternions of the Blaise numbers for the proof.

(a) $QB_{n-t}QB_{n+t}-{QB_n}^2$

$$
\begin{aligned}
&=\left(\frac{\alpha^{n-t+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n-t+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{n-t+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}(1+i+j+k)\right)\\
&\left(\frac{\alpha^{n+t+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+t+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{n+t+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}(1+i+j+k)\right)\\
&-\left(\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}(1+i+j+k)\right)^2\\
&=\left(\frac{\alpha^{n-t+3}\beta^{n+t+3}\overline{\alpha}\overline{\beta}}{\widehat{\alpha}\widehat{\beta}}+\frac{\alpha^{n-t+3}\gamma^{n+t+3}\overline{\alpha}\overline{\gamma}}{\widehat{\alpha}\widehat{\gamma}}-\frac{1}{2}\frac{\alpha^{n-t+3}\overline{\alpha}}{\widehat{\alpha}}(1+i+j+k)\right)\\
&+\left(\frac{\beta^{n-t+3}\alpha^{n+t+3}\overline{\beta}\overline{\alpha}}{\widehat{\beta}\widehat{\alpha}}+\frac{\beta^{n-t+3}\gamma^{n+t+3}\overline{\beta}\overline{\gamma}}{\widehat{\beta}\widehat{\gamma}}-\frac{1}{2}\frac{\beta^{n-t+3}\overline{\beta}}{\widehat{\beta}}(1+i+j+k)\right)\\
&+\left(\frac{\gamma^{n-t+3}\alpha^{n+t+3}\overline{\gamma}\overline{\alpha}}{\widehat{\gamma}\widehat{\alpha}}+\frac{\gamma^{n-t+3}\beta^{n+t+3}\overline{\gamma}\overline{\beta}}{\widehat{\gamma}\widehat{\beta}}-\frac{1}{2}\frac{\gamma^{n-t+3}\overline{\gamma}}{\widehat{\gamma}}(1+i+j+k)\right)\\
&-\frac{1}{2}(1+i+j+k)\frac{\alpha^{n+t+3}\overline{\alpha}}{\widehat{\alpha}}-\frac{1}{2}(1+i+j+k)\frac{\beta^{n+t+3}\overline{\beta}}{\widehat{\beta}}\\
&-\frac{1}{2}(1+i+j+k)\frac{\gamma^{n+t+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{\alpha^{n+3}\beta^{n+3}\overline{\alpha}\overline{\beta}}{\widehat{\alpha}\widehat{\beta}}-\frac{\alpha^{n+3}\gamma^{n+3}\overline{\alpha}\overline{\gamma}}{\widehat{\alpha}\widehat{\gamma}}\\
&+\frac{1}{2}\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}(1+i+j+k)-\frac{\beta^{n+3}\alpha^{n+3}\overline{\beta}\overline{\alpha}}{\widehat{\beta}\widehat{\alpha}}-\frac{\beta^{n+3}\gamma^{n+3}\overline{\beta}\overline{\gamma}}{\widehat{\beta}\widehat{\gamma}}\\
&+\frac{1}{2}\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}(1+i+j+k)-\frac{\gamma^{n+3}\alpha^{n+3}\overline{\gamma}\overline{\alpha}}{\widehat{\gamma}\widehat{\alpha}}-\frac{\gamma^{n+3}\beta^{n+3}\overline{\gamma}\overline{\beta}}{\widehat{\gamma}\widehat{\beta}}\\
&+\frac{1}{2}\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}(1+i+j+k)+\frac{1}{2}(1+i+j+k)\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{1}{2}(1+i+j+k)\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}\\
&+\frac{1}{2}(1+i+j+k)\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}\\
&=\frac{\alpha^{n+3}\beta^{n+3}\overline{\alpha}\overline{\beta}}{\widehat{\alpha}\widehat{\beta}}\left(\alpha^{-t}\beta^{t}-1\right)+\frac{\alpha^{n+3}\gamma^{n+3}\overline{\alpha}\overline{\gamma}}{\widehat{\alpha}\widehat{\gamma}}\left(\alpha^{-t}\gamma^{t}-1\right)\\
&+\frac{\beta^{n+3}\alpha^{n+3}\overline{\beta}\overline{\alpha}}{\widehat{\beta}\widehat{\alpha}}\left(\beta^{-t}\alpha^{t}-1\right)+\frac{\beta^{n+3}\gamma^{n+3}\overline{\beta}\overline{\gamma}}{\widehat{\beta}\widehat{\gamma}}\left(\beta^{-t}\gamma^{t}-1\right)\\
&+\frac{\gamma^{n+3}\alpha^{n+3}\overline{\gamma}\overline{\alpha}}{\widehat{\gamma}\widehat{\alpha}}\left(\gamma^{-t}\alpha^{t}-1\right)+\frac{\gamma^{n+2}\beta^{n+3}\overline{\gamma}\overline{\beta}}{\widehat{\gamma}\widehat{\beta}}\left(\gamma^{-t}\beta^{t}-1\right)\\
&+(1+i+j+k)\Big(\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}\frac{\alpha^{n-t+3}\overline{\alpha}}{\widehat{\alpha}}-\frac{1}{2}\frac{\beta^{n-t+3}\overline{\beta}}{\widehat{\beta}}\\
&-\frac{1}{2}\frac{\gamma^{n-t+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}\frac{\alpha^{n+t+3}\overline{\alpha}}{\widehat{\alpha}}-\frac{1}{2}\frac{\beta^{n+t+3}\overline{\beta}}{\widehat{\beta}}-\frac{1}{2}\frac{\gamma^{n+t+3}\overline{\gamma}}{\widehat{\gamma}}\Big)\\
&=\frac{(\alpha\beta)^{n+3-t}\left(\alpha^{t}-\beta^{t}\right)}{\widehat{\alpha}\widehat{\beta}}\left(\alpha^{t}\overline{\alpha}\overline{\beta}-\beta^{t}\overline{\beta}\overline{\alpha}\right)+\frac{(\alpha\gamma)^{n+3-t}\left(\alpha^{t}-\gamma^{t}\right)}{\widehat{\alpha}\widehat{\gamma}}\left(\alpha^{t}\overline{\gamma}\overline{\alpha}-\gamma^{t}\overline{\alpha}\overline{\gamma}\right)\\
&+\frac{(\gamma\beta)^{n+3-t}\left(\gamma^{t}-\beta^{t}\right)}{\widehat{\gamma}\widehat{\beta}}\left(\gamma^{t}\overline{\beta}\overline{\gamma}-\beta^{t}\overline{\gamma}\overline{\beta}\right)\\
&+(1+i+j+k)\left(QB_n-\frac{1}{2}\left(QB_{n-t+3}+QB_{n+t+3}\right)\right).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&QB_{n-t}QB_{n+t} - {QB_n}^2 \\
&\quad = 2^{n+3-t}\left[\frac{\gamma^{-n-3+t}\left(\alpha^t-\beta^t\right)\left(\beta^t\overline{\alpha}\overline{\beta}-\alpha^t\overline{\beta}\overline{\alpha}\right)}{\widehat{\alpha}\widehat{\beta}} + \frac{\beta^{-n-3+t}\left(\alpha^t-\gamma^t\right)\left(\alpha^t\overline{\gamma}\overline{\alpha}-\gamma^t\overline{\alpha}\overline{\gamma}\right)}{\widehat{\alpha}\widehat{\gamma}}\right. \\
&\quad \left. + \frac{\alpha^{-n-3+t}\left(\gamma^t-\beta^t\right)\left(\gamma^t\overline{\beta}\overline{\gamma}-\beta^t\overline{\gamma}\overline{\beta}\right)}{\widehat{\gamma}\widehat{\beta}}\right] \\
&\quad + (1+i+j+k)\left(QB_n - \frac{1}{2}\left(QB_{n-t+3}+QB_{n+t+3}\right)\right).
\end{aligned}
$$

Note that, if $t = 1$ is taken, then Cassini identity is obtained. Thus, the proof is completed. The proof of (b) is done similarly to (a). □

Theorem 3.3. *(d' Ocagne identities) For $n \leqslant m$, we have*

(a) $QB_mQB_{n+1} - QB_nQB_{m+1}$

$$
\begin{aligned}
&= 2^{n+3}\left[\frac{\gamma^{-n-3}(\alpha^{m-n}-\beta^{m-n})(\beta\overline{\alpha}\overline{\beta}-\alpha\overline{\beta}\overline{\alpha})}{\widehat{\alpha}\widehat{\beta}}\right. \\
&+ \frac{\beta^{-n-3}(\alpha^{m-n}-\gamma^{m-n})(\gamma\overline{\alpha}\overline{\gamma}-\alpha\overline{\gamma}\overline{\alpha})}{\widehat{\alpha}\widehat{\gamma}} + \left.\frac{\alpha^{-n-3}(\gamma^{m-n}-\beta^{m-n})(\beta\overline{\gamma}\overline{\beta}-\gamma\overline{\beta}\overline{\gamma})}{\widehat{\gamma}\widehat{\beta}}\right] \\
&+ \frac{1}{2}(1+i+j+k)(QB_n + QB_{m+1} - QB_m - QB_{n+1}),
\end{aligned}
$$

(b) $QC_mQC_{n+1} - QC_nQC_{m+1}$

$$
\begin{aligned}
&= 2^n\left[\gamma^{-n}\left(\alpha^{m-n}-\beta^{m-n}\right)\left(\beta\overline{\alpha}\overline{\beta}-\alpha\overline{\beta}\overline{\alpha}\right)+\beta^{-n}\left(\alpha^{m-n}-\gamma^{m-n}\right)\left(\gamma\overline{\alpha}\overline{\gamma}-\alpha\overline{\gamma}\overline{\alpha}\right)\right. \\
&\left.+\alpha^{-n}\left(\gamma^{m-n}-\beta^{m-n}\right)\left(\beta\overline{\gamma}\overline{\beta}-\gamma\overline{\beta}\overline{\gamma}\right)\right] \\
&+ (\ 1+i+j+k)\ (QB_m + QB_{n+1} - QB_n - QB_{m+1}).
\end{aligned}
$$

Proof.

(a) $QB_mQB_{n+1} - QB_nQB_{m+1}$

$$= \left(\frac{\alpha^{m+3}\overline{\alpha}}{\widehat{\alpha}} + \frac{\beta^{m+3}\overline{\beta}}{\widehat{\beta}} + \frac{\gamma^{m+3}\overline{\gamma}}{\widehat{\gamma}} - \frac{1}{2}(1+i+j+k)\right)$$
$$\left(\frac{\alpha^{n+4}\overline{\alpha}}{\widehat{\alpha}} + \frac{\beta^{n+4}\overline{\beta}}{\widehat{\beta}} + \frac{\gamma^{n+4}\overline{\gamma}}{\widehat{\gamma}} - \frac{1}{2}(1+i+j+k)\right)$$
$$- \left(\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}} + \frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}} + \frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}} - \frac{1}{2}(1+i+j+k)\right)$$
$$\left(\frac{\alpha^{m+4}\overline{\alpha}}{\widehat{\alpha}} + \frac{\beta^{m+4}\overline{\beta}}{\widehat{\beta}} + \frac{\gamma^{m+4}\overline{\gamma}}{\widehat{\gamma}} - \frac{1}{2}(1+i+j+k)\right)$$
$$= \left(\frac{\alpha^{m+3}\beta^{n+1}\overline{\alpha}\overline{\beta}}{\widehat{\alpha}\widehat{\beta}} + \frac{\alpha^{m+3}\gamma^{n+4}\overline{\alpha}\overline{\gamma}}{\widehat{\alpha}\widehat{\gamma}} - \frac{1}{2}\frac{\alpha^{m+3}\overline{\alpha}}{\widehat{\alpha}}(1+i+j+k)\right)$$
$$+ \left(\frac{\beta^{m+3}\alpha^{n+4}\overline{\beta}\overline{\alpha}}{\widehat{\beta}\widehat{\alpha}} + \frac{\beta^{m+3}\gamma^{n+4}\overline{\beta}\overline{\gamma}}{\widehat{\beta}\widehat{\gamma}} - \frac{1}{2}\frac{\beta^{m+3}\overline{\beta}}{\widehat{\beta}}(1+i+j+k)\right)$$

$$+ \left(\frac{\gamma^{m+3}\alpha^{n+4}\overline{\gamma}\overline{\alpha}}{\widehat{\gamma}\widehat{\alpha}} + \frac{\gamma^{m+3}\beta^{n+4}\overline{\gamma}\overline{\beta}}{\widehat{\gamma}\widehat{\beta}} - \frac{1}{2}\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}(1+i+j+k)\right)$$
$$- \frac{1}{2}(1+i+j+k)\frac{\alpha^{n+4}\overline{\alpha}}{\widehat{\alpha}} - \frac{1}{2}(1+i+j+k)\frac{\beta^{n+4}\overline{\beta}}{\widehat{\beta}} - \frac{1}{2}(1+i+j+k)\frac{\gamma^{n+4}\overline{\gamma}}{\widehat{\gamma}}$$
$$- \frac{\alpha^{n+3}\beta^{m+4}\overline{\alpha}\overline{\beta}}{\widehat{\alpha}\widehat{\beta}} - \frac{\alpha^{n+3}\gamma^{m+4}\overline{\alpha}\overline{\gamma}}{\widehat{\alpha}\widehat{\gamma}} + \frac{1}{2}\frac{\alpha^{n+4}\overline{\alpha}}{\widehat{\alpha}}(1+i+j+k)$$
$$- \frac{\beta^{n+3}\alpha^{m+4}\overline{\beta}\overline{\alpha}}{\widehat{\beta}\widehat{\alpha}} - \frac{\beta^{n+3}\gamma^{m+4}\overline{\beta}\overline{\gamma}}{\widehat{\beta}\widehat{\gamma}} + \frac{1}{2}\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}(1+i+j+k)$$
$$- \frac{\gamma^{n+3}\alpha^{m+4}\overline{\gamma}\overline{\alpha}}{\widehat{\gamma}\widehat{\alpha}} - \frac{\gamma^{n+3}\beta^{m+4}\overline{\gamma}\overline{\beta}}{\widehat{\gamma}\widehat{\beta}} + \frac{1}{2}\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}(1+i+j+k)$$
$$+ \frac{1}{2}(1+i+j+k)\frac{\alpha^{m+4}\overline{\alpha}}{\widehat{\alpha}} + \frac{1}{2}(1+i+j+k)\frac{\beta^{m+4}\overline{\beta}}{\widehat{\beta}} + \frac{1}{2}(1+i+j+k)\frac{\gamma^{m+4}\overline{\gamma}}{\widehat{\gamma}}$$
$$= \frac{\alpha^{n+3}\beta^{n+4}\overline{\alpha}\overline{\beta}}{\widehat{\alpha}\widehat{\beta}}\left(\alpha^{m-n} - \beta^{m-n}\right) + \frac{\alpha^{n+3}\gamma^{n+4}\overline{\alpha}\overline{\gamma}}{\widehat{\alpha}\widehat{\gamma}}\left(\alpha^{m-n} - \gamma^{m-n}\right)$$
$$+ \frac{\alpha^{n+3}\beta^{n+4}\overline{\beta}\overline{\alpha}}{\widehat{\beta}\widehat{\alpha}}\left(\beta^{m-n} - \alpha^{m-n}\right) + \frac{\beta^{n+3}\gamma^{n+4}\overline{\beta}\overline{\gamma}}{\widehat{\beta}\widehat{\gamma}}\left(\beta^{m-n} - \gamma^{m-n}\right)$$
$$+ \frac{\gamma^{n+3}\alpha^{n+4}\overline{\gamma}\overline{\alpha}}{\widehat{\gamma}\widehat{\alpha}}\left(\gamma^{m-n} - \alpha^{m-n}\right) + \frac{\gamma^{n+3}\beta^{n+4}\overline{\gamma}\overline{\beta}}{\widehat{\gamma}\widehat{\beta}}\left(\gamma^{m-n} - \beta^{m-n}\right)$$
$$+ \frac{1}{2}(1+i+j+k)\left(\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}} + \frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}} + \frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}} - \frac{\alpha^{m+3}\overline{\alpha}}{\widehat{\alpha}}\right.$$
$$\left. - \frac{\beta^{m+3}\overline{\beta}}{\widehat{\beta}} - \frac{\gamma^{m+3}\overline{\gamma}}{\widehat{\gamma}} - \frac{\alpha^{n+4}\overline{\alpha}}{\widehat{\alpha}} - \frac{\beta^{n+4}\overline{\beta}}{\widehat{\beta}} - \frac{\gamma^{n+4}\overline{\gamma}}{\widehat{\gamma}}\right)$$

Thus, if the necessary actions are taken, the desired is achieved.

$$QB_mQB_{n+1} - QB_nQB_{m+1} = 2^{n+3}\left[\frac{\gamma^{-n-3}(\alpha^{m-n} - \beta^{m-n})(\beta\overline{\alpha}\overline{\beta} - \alpha\overline{\beta}\overline{\alpha})}{\widehat{\alpha}\widehat{\beta}}\right.$$

$$\left. + \frac{\beta^{-n-3}(\alpha^{m-n} - \gamma^{m-n})(\gamma\overline{\alpha}\overline{\gamma} - \alpha\overline{\gamma}\overline{\alpha})}{\widehat{\alpha}\widehat{\gamma}} + \frac{\alpha^{-n-3}(\gamma^{m-n} - \beta^{m-n})(\beta\overline{\gamma}\overline{\beta} - \gamma\overline{\beta}\overline{\gamma})}{\widehat{\gamma}\widehat{\beta}}\right]$$

$$+ \frac{1}{2}(1 + i + j + k)(QB_n + QB_{m+1} - QB_m - QB_{n+1})$$

The proof of (b) is done similarly to (a). □

Theorem 3.4. *(Vajda identities) For $n, k \leq m$, we have*

(a) $QB_{n+k}QB_{m-k} - QB_nQB_m$

$$= 2^{n+3}\left[\frac{\gamma^{-n-3}(\alpha^k - \beta^k)(\beta^{m-k-n}\overline{\alpha}\overline{\beta} - \alpha^{m-k-n}\overline{\beta}\overline{\alpha})}{\widehat{\alpha}\widehat{\beta}}\right.$$

$$+ \frac{\beta^{-n-3}(\alpha^k - \gamma^k)(\gamma^{m-k-n}\overline{\alpha}\overline{\gamma} - \alpha^{m-k-n}\overline{\gamma}\overline{\alpha})}{\widehat{\alpha}\widehat{\gamma}}$$

$$\left. + \frac{\alpha^{-n-3}(\gamma^k - \beta^k)(\beta^{m-k-n}\overline{\gamma}\overline{\beta} - \gamma^{m-k-n}\overline{\beta}\overline{\gamma})}{\widehat{\gamma}\widehat{\beta}}\right]$$

$$+ \frac{1}{2}(1 + i + j + k)(QB_n + QB_m - QB_{m-k} - QB_{n+k}).$$

(b) $QC_{n+k}QC_{m-k} - QC_nQC_m$

$$= 2^n[\gamma^{-n}(\alpha^k - \beta^k)(\beta^{m-k-n}\overline{\alpha}\overline{\beta} - \alpha^{m-k-n}\overline{\beta}\overline{\alpha})$$

$$+ \beta^{-n}(\alpha^k - \gamma^k)(\gamma^{m-k-n}\overline{\alpha}\overline{\gamma} - \alpha^{m-k-n}\overline{\gamma}\overline{\alpha})$$

$$+ \alpha^{-n}\left(\gamma^k - \beta^k\right)\left(\beta^{m-k-n}\overline{\gamma}\overline{\beta} - \gamma^{m-k-n}\overline{\beta}\overline{\gamma}\right)]$$

$$+ (\ 1 + i + j + k)\left(QB_{m-k} + QB_{n+k} - QB_n - QB_m\right).$$

Proof.

(a) $QB_{n+k}QB_{m-k}-QB_nQB_m$

$$
\begin{aligned}
&=\left(\frac{\alpha^{n+k+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+k+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{n+k+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}(1+i+j+k)\right)\\
&\left(\frac{\alpha^{m-k+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{m-k+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{m-k+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}(1+i+j+k)\right)\\
&-\left(\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}(1+i+j+k)\right)\\
&\left(\frac{\alpha^{m+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{m+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{m+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{1}{2}(1+i+j+k)\right)\\
&=\left(\frac{\alpha^{n+k+3}\beta^{m-k+3}\overline{\alpha}\overline{\beta}}{\widehat{\alpha}\widehat{\beta}}+\frac{\alpha^{n+k+3}\gamma^{m-k+3}\overline{\alpha}\overline{\gamma}}{\widehat{\alpha}\widehat{\gamma}}-\frac{1}{2}\frac{\alpha^{n+k+3}\overline{\alpha}}{\widehat{\alpha}}(1+i+j+k)\right)\\
&+\left(\frac{\beta^{n+k+3}\alpha^{m-k+3}\overline{\beta}\overline{\alpha}}{\widehat{\beta}\widehat{\alpha}}+\frac{\beta^{n+k+3}\gamma^{m-k+3}\overline{\beta}\overline{\gamma}}{\widehat{\beta}\widehat{\gamma}}-\frac{1}{2}\frac{\beta^{n+k+3}\overline{\beta}}{\widehat{\beta}}(1+i+j+k)\right)\\
&+\left(\frac{\gamma^{n+k+3}\alpha^{m-k+3}\overline{\gamma}\overline{\alpha}}{\widehat{\gamma}\widehat{\alpha}}+\frac{\gamma^{n+k+3}\beta^{m-k+3}\overline{\gamma}\overline{\beta}}{\widehat{\gamma}\widehat{\beta}}-\frac{1}{2}\frac{\gamma^{n+k+3}\overline{\gamma}}{\widehat{\gamma}}(1+i+j+k)\right)\\
&-\frac{1}{2}(1+i+j+k)\frac{\alpha^{m-k+3}\overline{\alpha}}{\widehat{\alpha}}-\frac{1}{2}(1+i+j+k)\frac{\beta^{m-k+3}\overline{\beta}}{\widehat{\beta}}\\
&-\frac{1}{2}(1+i+j+k)\frac{\gamma^{m-k+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{\alpha^{n+3}\beta^{m+3}\overline{\alpha}\overline{\beta}}{\widehat{\alpha}\widehat{\beta}}-\frac{\alpha^{n+3}\gamma^{m+3}\overline{\alpha}\overline{\gamma}}{\widehat{\alpha}\widehat{\gamma}}\\
&+\frac{1}{2}\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}(1+i+j+k)-\frac{\beta^{n+3}\alpha^{m+3}\overline{\beta}\overline{\alpha}}{\widehat{\beta}\widehat{\alpha}}-\frac{\beta^{n+3}\gamma^{m+3}\overline{\beta}\overline{\gamma}}{\widehat{\beta}\widehat{\gamma}}\\
&+\frac{1}{2}\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}(1+i+j+k)-\frac{\gamma^{n+3}\alpha^{m+3}\overline{\gamma}\overline{\alpha}}{\widehat{\gamma}\widehat{\alpha}}-\frac{\gamma^{n+3}\beta^{m+3}\overline{\gamma}\overline{\beta}}{\widehat{\gamma}\widehat{\beta}}\\
&+\frac{1}{2}\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}}(1+i+j+k)+\frac{1}{2}(1+i+j+k)\frac{\alpha^{m+3}\overline{\alpha}}{\widehat{\alpha}}\\
&+\frac{1}{2}(1+i+j+k)\frac{\beta^{m+3}\overline{\beta}}{\widehat{\beta}}+\frac{1}{2}(1+i+j+k)\frac{\gamma^{m+3}\overline{\gamma}}{\widehat{\gamma}}\\
&=\frac{\alpha^{n+3}\beta^{m-k+3}\overline{\alpha}\overline{\beta}}{\widehat{\alpha}\widehat{\beta}}\left(\alpha^{k}-\beta^{k}\right)+\frac{\alpha^{n+3}\gamma^{m-k+3}\overline{\alpha}\overline{\gamma}}{\widehat{\alpha}\widehat{\gamma}}\left(\alpha^{k}-\gamma^{k}\right)\\
&+\frac{\beta^{n+3}\alpha^{m-k+3}\overline{\beta}\overline{\alpha}}{\widehat{\beta}\widehat{\alpha}}\left(\beta^{k}-\alpha^{k}\right)+\frac{\beta^{n+3}\gamma^{m-k+3}\overline{\beta}\overline{\gamma}}{\widehat{\beta}\widehat{\gamma}}\left(\beta^{k}-\gamma^{k}\right)\\
&+\frac{\gamma^{n+3}\alpha^{m-k+3}\overline{\gamma}\overline{\alpha}}{\widehat{\gamma}\widehat{\alpha}}\left(\gamma^{k}-\alpha^{k}\right)+\frac{\gamma^{n+3}\beta^{m-k+3}\overline{\gamma}\overline{\beta}}{\widehat{\gamma}\widehat{\beta}}\left(\gamma^{k}-\beta^{k}\right)\\
&+\frac{1}{2}(1+i+j+k)\left(\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}+\frac{\gamma^{m+3}\overline{\gamma}}{\widehat{\gamma}}+\frac{\alpha^{n+3}\overline{\alpha}}{\widehat{\alpha}}+\frac{\beta^{n+3}\overline{\beta}}{\widehat{\beta}}\right.\\
&+\frac{\gamma^{n+3}\overline{\gamma}}{\widehat{\gamma}})-\frac{\alpha^{m-k+3}\overline{\alpha}}{\widehat{\alpha}}-\frac{\beta^{m-k+3}\overline{\beta}}{\widehat{\beta}}-\frac{\gamma^{m-k+3}\overline{\gamma}}{\widehat{\gamma}}-\frac{\alpha^{n+k+3}\overline{\alpha}}{\widehat{\alpha}}\\
&\left.-\frac{\beta^{n+k+3}\overline{\beta}}{\widehat{\beta}}-\frac{\gamma^{n+k+3}\overline{\gamma}}{\widehat{\gamma}}\right)
\end{aligned}
$$

Thus, if the necessary actions are taken, the desired is achieved.

$$\begin{aligned} &QB_mQB_{n+1} - QB_nQB_{m+1} \\ &\quad = 2^{n+3}[\frac{\gamma^{-n-3}(\alpha^k - \beta^k)(\beta^{m-k-n}\overline{\alpha\beta} - \alpha^{m-k-n}\overline{\beta\alpha})}{\widehat{\alpha}\widehat{\beta}} \\ &\quad + \frac{\beta^{-n-3}(\alpha^k - \gamma^k)(\gamma^{m-k-n}\overline{\alpha\gamma} - \alpha^{m-k-n}\overline{\gamma\alpha})}{\widehat{\alpha}\widehat{\gamma}} \\ &\quad + \frac{\alpha^{-n-3}(\gamma^k - \beta^k)(\beta^{m-k-n}\overline{\gamma\beta} - \gamma^{m-k-n}\overline{\beta\gamma})}{\widehat{\gamma}\widehat{\beta}} \\ &\quad + \frac{1}{2}(1+i+j+k)(QB_n + QB_m - QB_{m-k} - QB_{n+k}). \end{aligned}$$

The proof of (b) is done similarly to (a). □

4 Matrices related with Blaise and Blaise-Lucas Quaternions

If we define the square matrix A of order 4 as

$$A = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$K_n = \begin{pmatrix} B_{n+1} & B_n + B_{n-1} - 2B_{n-2} & B_n - 2B_{n-1} & -2B_n \\ B_n & B_{n-1} + B_{n-2} - 2B_{n-3} & B_{n-1} - 2B_{n-2} & -2B_{n-1} \\ B_{n-1} & B_{n-2} + B_{n-3} - 2B_{n-4} & B_{n-2} - 2B_{n-3} & -2B_{n-2} \\ B_{n-2} & B_{n-3} + B_{n-4} - 2B_{n-5} & B_{n-3} - 2B_{n-4} & -2B_{n-3} \end{pmatrix}$$

then we get the following Theorem.

Theorem 4.1. *[1, Theorem 8.1. (a)] For all integers n, we have*

$$K_n = A^n,$$

i.e.,

$$\begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} B_{n+1} & B_n + B_{n-1} - 2B_{n-2} & B_n - 2B_{n-1} & -2B_n \\ B_n & B_{n-1} + B_{n-2} - 2B_{n-3} & B_{n-1} - 2B_{n-2} & -2B_{n-1} \\ B_{n-1} & B_{n-2} + B_{n-3} - 2B_{n-4} & B_{n-2} - 2B_{n-3} & -2B_{n-2} \\ B_{n-2} & B_{n-3} + B_{n-4} - 2B_{n-5} & B_{n-3} - 2B_{n-4} & -2B_{n-3} \end{pmatrix}.$$

Now, we define the matrices M_B and M_C as

$$M_B = \begin{pmatrix} QB_1 & QB_0 + QB_{0-1} - 2QB_{-2} & QB_0 - 2QB_{-1} & -2QB_0 \\ QB_0 & QB_{-1} + QB_{-2} - 2QB_{-3} & QB_{-1} - 2QB_{-2} & -2QB_{-1} \\ QB_{-1} & QB_{-2} + QB_{-3} - 2QB_{-4} & QB_{-2} - 2QB_{-3} & -2QB_{-2} \\ QB_{-2} & QB_{-3} + QB_{-4} - 2QB_{-5} & QB_{-3} - 2QB_{-4} & -2QB_{-3} \end{pmatrix},$$

and

$$M_C = \begin{pmatrix} QC_1 & QC_0 + QC_{0-1} - 2QC_{-2} & QC_0 - 2QC_{-1} & -2QC_0 \\ QC_0 & QC_{-1} + QC_{-2} - 2QC_{-3} & QC_{-1} - 2QC_{-2} & -2QC_{-1} \\ QC_{-1} & QC_{-2} + QC_{-3} - 2QC_{-4} & QC_{-2} - 2QC_{-3} & -2QC_{-2} \\ QC_{-2} & QC_{-3} + QC_{-4} - 2QC_{-5} & QC_{-3} - 2QC_{-4} & -2QC_{-3} \end{pmatrix},$$

respectively. These matrices M_B and M_C are called Blaise quaternion matrix and Blaise-Lucas quaternion matrix, respectively.

Theorem 4.2. *For all integers n, the following identities are valid:*

$$M_B\begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} QB_{n+1} & QB_n+QB_{n-1}-2QB_{n-2} & QB_n-2QB_{n-1} & -2QB_n \\ QB_n & QB_{n-1}+QB_{n-2}-2QB_{n-3} & QB_{n-1}-2QB_{n-2} & -2QB_{n-1} \\ QB_{n-1} & QB_{n-2}+QB_{n-3}-2QB_{n-4} & QB_{n-2}-2QB_{n-3} & -2QB_{n-2} \\ QB_{n-2} & QB_{n-3}+QB_{n-4}-2QB_{n-5} & QB_{n-3}-2QB_{n-4} & -2QB_{n-3} \end{pmatrix} \tag{4.1}$$

and

$$M_C\begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} QC_{n+1} & QC_n+QC_{n-1}-2QC_{n-2} & QC_n-2QC_{n-1} & -2QC_n \\ QC_n & QC_{n-1}+QC_{n-2}-2QC_{n-3} & QC_{n-1}-2QC_{n-2} & -2QC_{n-1} \\ QC_{n-1} & QC_{n-2}+QC_{n-3}-2QC_{n-4} & QC_{n-2}-2QC_{n-3} & -2QC_{n-2} \\ QC_{n-2} & QC_{n-3}+QC_{n-4}-2QC_{n-5} & QC_{n-3}-2QC_{n-4} & -2QC_{n-3} \end{pmatrix} \tag{4.2}$$

Proof. We prove by mathematical induction on $n \geq 0$. If $n = 0$, then the result is clear. Now, we assume it is true for $n = k$, that is

$$M_BK^k = \begin{pmatrix} QB_{k+1} & QB_k+QB_{k-1}-2QB_{k-2} & QB_k-2QB_{k-1} & -2QB_k \\ QB_k & QB_{k-1}+QB_{k-2}-2QB_{k-3} & QB_{k-1}-2QB_{k-2} & -2QB_{k-1} \\ QB_{k-1} & QB_{k-2}+QB_{k-3}-2QB_{k-4} & QB_{k-2}-2QB_{k-3} & -2QB_{k-2} \\ QB_{k-2} & QB_{k-3}+QB_{k-4}-2QB_{k-5} & QB_{k-3}-2QB_{k-4} & -2QB_{k-3} \end{pmatrix}.$$

If we use Theorem 2.6 a. i.e., $QB_k = QB_{k-1}+QB_{k-2}+QB_{k-3}-2QB_{k-4}$, then, by induction hypothesis, we obtain

$$\begin{aligned} & M_BK^{k+1} \\ = & (M_BK^k)K \\ = & \begin{pmatrix} QB_{k+1} & QB_k+QB_{k-1}-2QB_{k-2} & QB_k-2QB_{k-1} & -2QB_k \\ QB_k & QB_{k-1}+QB_{k-2}-2QB_{k-3} & QB_{k-1}-2QB_{k-2} & -2QB_{k-1} \\ QB_{k-1} & QB_{k-2}+QB_{k-3}-2QB_{k-4} & QB_{k-2}-2QB_{k-3} & -2QB_{k-2} \\ QB_{k-2} & QB_{k-3}+QB_{k-4}-2QB_{k-5} & QB_{k-3}-2QB_{k-4} & -2QB_{k-3} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ = & \begin{pmatrix} QB_{k+1+1} & QB_{k+1}+QB_{k+1-1}-2QB_{k+1-2} & QB_{k+1}-2QB_{k+1-1} & -2QB_{k+1} \\ QB_{k+1} & QB_{k+1-1}+QB_{k+1-2}-2QB_{k+1-3} & QB_{k+1-1}-2QB_{k+1-2} & -2QB_{k+1-1} \\ QB_{k+1-1} & QB_{k+1-2}+QB_{k+1-3}-2QB_{k+1-4} & QB_{k+1-2}-2QB_{k+1-3} & -2QB_{k+1-2} \\ QB_{k+1-2} & QB_{k+1-3}+QB_{k+1-4}-2QB_{k+1-5} & QB_{k+1-3}-2QB_{k+1-4} & -2QB_{k+1-3} \end{pmatrix}. \end{aligned}$$

Thus, (4.1) holds for all non-negative integers n. The case $n < 0$ can be proved similary by induction. So (4.1) is true for all integers n.

The case (4.2) can be proved similarly. □

We need the following Corollary which give identities for B_{n+m} and C_{n+m}.

Corollary 4.3. *[1, Corollary 8.4.] For all integers m, n, we have*

$$\begin{aligned} B_{n+m} &= B_{m+1}B_n+(B_m+B_{m-1}-2B_{m-2})B_{n-1}+(B_m-2B_{m-1})B_{n-2}-2B_mB_{n-3}, \\ C_{n+m} &= B_{m+1}C_n+(B_m+B_{m-1}-2B_{m-2})C_{n-1}+(B_m-2B_{m-1})C_{n-2}-2B_mC_{n-3}. \end{aligned}$$

Next, we present a Corollary of Theorem 4.2.

Corollary 4.4. *For all integers n, the following identities hold:*

$$\begin{aligned} QB_{n+m} &= QB_{m+1}B_n+(QB_m+QB_{m-1}-2QB_{m-2})B_{n-1} \\ &\quad +(QB_m-2QB_{m-1})B_{n-2}-2QB_mB_{n-3}, \\ QC_{n+m} &= QB_{m+1}C_n+(QB_m+QB_{m-1}-2QB_{m-2})C_{n-1} \\ &\quad +(QB_m-2QB_{m-1})C_{n-2}-2QB_mC_{n-3}. \end{aligned}$$

Proof. The proof can be seen by the coefficient of the matrices M_B and M_C and Corollary 4.3. □

5 Conclusion

In this study, we defined quaternions of Blaise and Blaise-Lucas numbers. We investigated both the quaternion and sequence properties of these numbers. Next, we calculated the important identities for these numbers. In the last section, we gave the matrix properties of these numbers.

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