

POINTWISE SEMINORM APPROXIMATION VIA BIHARMONIC POISSON INTEGRALS

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Abstract. In this paper, we investigate the rate of convergence of 2π -periodic functions within a seminormed space. Our analysis employs biharmonic Poisson integrals as a tool for measuring approximation in this setting. The degree of approximation is characterized in terms of two modulus-type functions.

1 Introduction

The approximation of functions $f \in \text{Lip}(\alpha, L^p)$ by k -th partial sums $s_k f(x)$ of their Fourier series (at the point x) in the L^p -norm ($p \in [1, \infty)$) was first studied by Quade [17], who established conditions under which the approximation order is $O(n^{-\alpha})$. His results have since inspired numerous generalizations. Gajewski and Łenski [5] extended the work of Prössdorf [16] and Stypiński [22] to functions in $W^r H^\alpha$, using generalized de la Vallée-Poussin means. Further contributions include studies by Leindler [13], Singh and Sonker [20], Değer and Bayindir [4], and others, who examined various summability methods and relaxed conditions on monotonicity. Additional results on this topic can be found in [8]-[9]. Results on approximation in Besov spaces and by deferred Cesàro means can be found in [2] and [21]. The rate of convergence of Fourier series via deferred Cesàro means of $s_k f(x)$ in the generalized Hölder metric $H_{L^p}^{(\omega)}$ was established by Nayak et al. [3]. Recently, several results on approximation in seminormed spaces using specific means of Fourier series have been published in [7], [14], and [15].

We adopt essential notation, fundamental concepts, and key results from [7], which will be used throughout the subsequent discussion.

Let L^p ($1 \leq p < \infty$) (C) be the class of all 2π -periodic real-valued functions integrable in the Lebesgue sense with p -th power (continuous) over $Q = [-\pi, \pi]$ and let $X^p = L^p$ when $1 \leq p < \infty$ or $X^p = C$ when $p = \infty$.

Let us define the norms of $f \in X^p$ as

$$\|f\|_{X^p} = \|f(\cdot)\|_{X^p} := \begin{cases} \left\{ \frac{1}{2\pi} \int_Q |f(x)|^p dx \right\}^{\frac{1}{p}} & \text{when } 1 \leq p < \infty, \\ \sup_{x \in Q} |f(x)| & \text{when } p = \infty. \end{cases}$$

For a seminorm P , we define the following seminormed space:

$$(X^p, P) = \{f \in X^p : P(f) < \infty\},$$

with the property that $f(\cdot + h) \in (X^p, P)$ for any $h \in \mathbb{R}$.

In this paper we will consider the seminorms P satisfying, for all $f, g \in (X^p, P)$, the following conditions:

1. for any $h \in \mathbb{R}$

$$P(f(\cdot + h)) = P(f(\cdot)), \tag{1.1}$$

2. if $|f(x)| \leq |g(x)|$ for every $x \in [-\pi, \pi]$, then

$$P(f) \leq P(g). \tag{1.2}$$

Let ω be a function of modulus of continuity type on the interval $[0, 2\pi]$, i.e., a non-decreasing continuous function having the following properties:

$$\omega(0) = 0, \omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$$

for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$.

For $f \in (X^p, P)$, the seminormed space $H_P^{(\omega)}$ is defined (see [7]) as

$$H_P^{(\omega)} = \{f \in (X^p, P) : M(f; \omega)_P < \infty\}$$

endowed with

$$P(f)^{(\omega)} := P(f) + M(f; \omega)_P,$$

where

$$M(f; \omega)_P := \sup_{t \neq 0} \frac{P(f(\cdot + t) - f(\cdot))}{\omega(|t|)}.$$

Here and throughout the paper, the modulus of continuity $\omega f(\delta)_P$ of f , is defined as

$$\omega f(\delta)_P = \sup_{|t| \leq \delta} P(f(\cdot + t) - f(\cdot)).$$

We observe that a function w of modulus of continuity type satisfies the condition

$$\frac{w(\delta)}{\delta} \leq 2 \frac{w(\eta)}{\eta}$$

for $0 < \eta \leq \delta$. However, we will focus on functions v and w of modulus of continuity type where the ratio w/v is almost non-decreasing, meaning that there exists a positive constant C such that

$$\frac{w(\delta)}{v(\delta)} \leq C \frac{w(\eta)}{v(\eta)}$$

for $0 < \delta \leq \eta$.

Motivated by the aforementioned results on approximation in seminormed spaces, we aim to conduct an advanced study in this direction by utilizing the so-called biharmonic Poisson integrals.

These integrals extend the classical Poisson integrals and are vital for solving approximation problems in mathematics and physics. They are especially important for solving the biharmonic equation

$$\Delta^2 u = \Delta(\Delta u) = 0, \tag{1.3}$$

where

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{r \partial r} + \frac{\partial^2 u}{r^2 \partial x^2}$$

is the Laplace operator in polar coordinates.

The solution of the equation (1.3) that satisfies the boundary conditions

$$u(r, x)|_{r=1} = f(x), \quad \frac{\partial u(r, x)}{\partial r} \Big|_{r=1} = 0, \quad x \in [-\pi, \pi],$$

is denoted by $B_r(f; x) := u(r, x)$, where $f(x)$ is an integrable 2π -function, and can be expressed in the form (called the biharmonic Poisson integrals of f)

$$B_r(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + t) P_2(r, t) dt, \tag{1.4}$$

where

$$P_2(r, t) = \frac{1}{2} + \sum_{k=1}^{\infty} \left(1 + \frac{k}{2}(1 - \rho^2)\right) \rho^k \cos kt, \quad 0 \leq r < 1,$$

is the biharmonic Poisson kernel (see [24] p. 400).

To further complete this article, we mention that the biharmonic Poisson integrals have been used in approximation theory by various authors. Notable contributions in this direction are due to Kharkevych and Shutovskiy [6], Shutovskiy and Pryt [18], Shutovskiy [19], and the present author [10]–[12].

The objective of our paper is to estimate the deviation of the biharmonic Poisson integrals $B_r(f; x)$ from a 2π -periodic function f within the space $H_P^{(w)}$, and measured by $P(\cdot)^{(v)}$.

To do this, we use mathematical approach to find the results. Along the way, we also depend on a few supporting results, which are explained in the next section.

To conclude this section, for two positive quantities v_1 and v_2 , we write $v_1 = \mathcal{O}(v_2)$ to mean that there exists a constant $K > 0$ such that $v_1 \leq K v_2$.

2 Some auxiliary Lemmas

In this section, we will use the well-known Abel’s transformation, also called summation by parts, a discrete analogue of integration by parts. Introduced by the Norwegian mathematician Niels Henrik Abel in the early 19th century, its form, as recalled from [1], page 1, is given as follows: If (a_k) and (b_k) are two real sequences, then

$$\sum_{k=m}^n a_k b_k = A_n b_n - A_{m-1} b_m + \sum_{k=m}^{n-1} A_k (b_k - b_{k+1}),$$

where

$$A_k = \sum_{j=m}^k a_j, \quad 0 \leq m \leq n,$$

with the convention that $A_{-1} = 0$ when $m = 0$.

We state below several lemmas that play a key role in proving our results.

Lemma 2.1 ([14]). (Lemma 3, p. 96) Let $f \in (L^p, P)$ with $1 \leq p < \infty$ and $K \in L^1$. If the seminorm P satisfies the conditions (1.1) and (1.2), then

$$P \left(\int_{-\pi}^{\pi} f(\cdot + t) K(t) dt \right) \leq \int_{-\pi}^{\pi} P(f) |K(t)| dt.$$

Remark 2.2. Lemma 2.1 is also true for $f \in (C, P)$ and thus for $f \in (X^p, P)$ because $(C, P) \subset (L^p, P)$ for $1 \leq p < \infty$.

Lemma 2.3 ([7]). Let $f \in (X^p, P)$, where the seminorm P satisfy the condition (1.1). If v and w be functions of modulus of continuity type such that $\frac{w}{v}$ is an almost non-decreasing function, then for $f \in Lip(w, P) := \{f \in (X^p, P) : \omega f(\delta)_P = \mathcal{O}(w(\delta))\}$:

(i) For $t \geq 0$,

$$P(\varphi_{\cdot+t}) = \mathcal{O}(1) \omega f(t)_P = O(w(t)),$$

(ii) For $t, y > 0$,

$$P(\varphi_{\cdot+y}(t) - \varphi_{\cdot}(t)) = \mathcal{O} \left(v(y) \frac{w(t)}{v(t)} \right).$$

Lemma 2.4. Let $r \in (0, 1)$ be fixed number and $0 < t \leq \pi$. Then, the following identities are valid:

(i)

$$P_2(r, t) = \frac{1-r}{2} \sum_{k=0}^{\infty} \left\{ 1 + \frac{k}{2}(1-r^2) - \frac{r(r+1)}{2} \right\} r^k \frac{\sin(k + \frac{1}{2})t}{\sin(\frac{t}{2})},$$

(ii)

$$P_2(r, t) = \frac{(1-r)^2}{2} \sum_{k=0}^{\infty} \left[1 + \frac{k}{2} (1-r^2) - r(1+r) \right] r^k \left(\frac{\sin \frac{(k+1)t}{2}}{\sin \frac{t}{2}} \right)^2.$$

Proof. By convention, we set $\cos 0 := \frac{1}{2}$ solely to express the kernel $P_2(r, t)$ as a single infinite sum. This is a technical choice made for notational convenience and does not alter the classical meaning of the cosine function. Given a fixed $r \in (0, 1)$, it follows that $\lim_{m \rightarrow \infty} r^m = 0$ and $\lim_{m \rightarrow \infty} mr^m = 0$.

(i) Applying Abel’s transformation and using the identity

$$\sum_{j=0}^k \cos jt = \frac{\sin \left(k + \frac{1}{2}\right) t}{2 \sin \left(\frac{t}{2}\right)}$$

we obtain

$$\begin{aligned} P_2(r, t) &= \lim_{m \rightarrow \infty} \sum_{k=0}^m \left[1 + \frac{k}{2} (1-r^2) \right] r^k \cos kt \\ &= \lim_{m \rightarrow \infty} \sum_{k=0}^{m-1} \left\{ \left[1 + \frac{k}{2} (1-r^2) \right] r^k - \left[1 + \frac{k+1}{2} (1-r^2) \right] r^{k+1} \right\} \sum_{j=0}^k \cos jt \\ &\quad + \lim_{m \rightarrow \infty} \left[1 + \frac{m}{2} (1-r^2) \right] r^m \sum_{j=0}^m \cos jt \\ &= \sum_{k=0}^{\infty} \left\{ \left[1 + \frac{k}{2} (1-r^2) \right] - \left[1 + \frac{k+1}{2} (1-r^2) \right] r \right\} r^k \frac{\sin \left(k + \frac{1}{2}\right) t}{2 \sin \left(\frac{t}{2}\right)} \\ &= (1-r) \sum_{k=0}^{\infty} \left\{ 1 + \frac{k}{2} (1-r^2) - \frac{r(r+1)}{2} \right\} r^k \frac{\sin \left(k + \frac{1}{2}\right) t}{2 \sin \left(\frac{t}{2}\right)}. \end{aligned}$$

(ii) From part (i), we apply Abel’s transformation and utilize the identity

$$\sum_{i=0}^k \frac{\sin \left(i + \frac{1}{2}\right) t}{\sin \left(\frac{t}{2}\right)} = \left(\frac{\sin \frac{(k+1)t}{2}}{\sin \left(\frac{t}{2}\right)} \right)^2$$

to derive

$$\begin{aligned} P_2(r, t) &= (1-r) \sum_{k=0}^{\infty} \left\{ 1 + \frac{k}{2} (1-r^2) - \frac{r(r+1)}{2} \right\} r^k \frac{\sin \left(k + \frac{1}{2}\right) t}{2 \sin \left(\frac{t}{2}\right)} \\ &= \frac{1-r}{2} \lim_{m \rightarrow \infty} \sum_{k=0}^{m-1} \left\{ \left[1 + \frac{k}{2} (1-r^2) - \frac{r(r+1)}{2} \right] r^k \right. \\ &\quad \left. - \left[1 + \frac{k+1}{2} (1-r^2) - \frac{r(r+1)}{2} \right] r^{k+1} \right\} \sum_{i=0}^k \frac{\sin \left(i + \frac{1}{2}\right) t}{\sin \left(\frac{t}{2}\right)} \\ &\quad + \lim_{m \rightarrow \infty} \left[1 + \frac{m}{2} (1-r^2) - \frac{r(r+1)}{2} \right] r^m \sum_{i=0}^m \frac{\sin \left(i + \frac{1}{2}\right) t}{\sin \left(\frac{t}{2}\right)} \\ &= \frac{(1-r)^2}{2} \sum_{k=0}^{\infty} \left[1 + \frac{k}{2} (1-r^2) - r(1+r) \right] r^k \left(\frac{\sin \frac{(k+1)t}{2}}{\sin \left(\frac{t}{2}\right)} \right)^2. \end{aligned}$$

This concludes the proof. □

Now we are in able to pass to the section main results.

3 Main Results

In this section, we present and prove the following result.

Theorem 3.1. *Let w and v be functions of the modulus of continuity type, where w/v is an almost non-decreasing function. Assume that $f \in H_P^{(w)}$, with the seminorm P satisfying conditions (1.1) and (1.2). Then,*

$$P(B_r(f) - f)^{(v)} = \mathcal{O}\left((1-r)\frac{w(1-r)}{v(1-r)} + (1-r)^2 \int_{1-r}^\pi \frac{w(t)dt}{t^2v(t)}\right), \tag{3.1}$$

with $r \in (0, 1)$.

Proof. We use the equality in (1.4) and the fact that

$$\frac{1}{\pi} \int_{-\pi}^\pi P_2(r, t) dt = \frac{1}{\pi} \int_{-\pi}^\pi \left\{ \frac{1}{2} + \sum_{k=1}^\infty \left[1 + \frac{k}{2}(1-r^2) \right] r^k \cos kt \right\} dt = 1$$

to derive

$$B_r(f)(x) := B_r(f; x) - f(x) = \frac{1}{\pi} \int_{-\pi}^\pi [f(x+t) - f(x)]P_2(r, t) dt.$$

Utilizing the symmetry property $P_2(r, -t) = P_2(r, t)$, we rewrite the above as

$$B_r(f)(x) = \frac{1}{\pi} \int_0^\pi \varphi_x(t)P_2(r, t)dt, \tag{3.2}$$

where

$$\varphi_x(t) := f(x+t) - 2f(x) + f(x-t).$$

By applying Lemma 2.1 to equation (3.2), we obtain

$$P(B_r(f)(\cdot)) \leq \frac{1}{\pi} \int_0^\pi P(\varphi(\cdot)) P_2(r, t)dt, \tag{3.3}$$

because of the non-negativity of the kernel $P_2(r, t)$, which can be rewritten as

$$P_2(r, t) = \frac{(1-r^2)^2(1-r \cos t)}{2\pi(1-2r \cos t+r^2)^2},$$

(see [23], p. 1113).

The integral in (3.3) can be decomposed as follows

$$\begin{aligned} P(B_r(f)(\cdot)) &= \mathcal{O}(1) \int_0^\pi P(\varphi(\cdot)) P_2(r, t)dt \\ &= \mathcal{O}(1) \left(\underbrace{\int_0^{1-r} P(\varphi(\cdot)) P_2(r, t)dt}_{=:B_{11}(r)} + \underbrace{\int_{1-r}^\pi P(\varphi(\cdot)) P_2(r, t)dt}_{=:B_{12}(r)} \right). \end{aligned} \tag{3.4}$$

Using standard inequalities for trigonometric functions, $\pi \sin \theta \geq 2\theta$ for $\theta \in [0, \pi/2]$, and $|\sin \theta| \leq |\theta|$ for any θ , in the Lemma 2.4 (i), we easily arrive at

$$P_2(r, t) \leq \frac{\pi}{2}(1-r) \sum_{k=0}^\infty \left(k + \frac{1}{2}\right) \left\{ 1 + \frac{k}{2}(1-r^2) - \frac{r(r+1)}{2} \right\} r^k.$$

The equalities

$$\sum_{k=0}^\infty r^k = \frac{1}{1-r} \quad \text{for } |r| < 1,$$

$$\sum_{k=0}^{\infty} kr^k = \frac{r}{(1-r)^2} \quad \text{for } |r| < 1,$$

$$\sum_{k=0}^{\infty} k^2 r^k = \frac{r(1+r)}{(1-r)^3}, \quad \text{for } |r| < 1,$$

imply

$$\begin{aligned} & \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) \left\{1 + \frac{k}{2}(1-r^2) - \frac{r(r+1)}{2}\right\} r^k \\ &= \sum_{k=0}^{\infty} kr^k + \frac{1}{2}(1-r^2) \sum_{k=0}^{\infty} k^2 r^k - \frac{r(r+1)}{2} \sum_{k=0}^{\infty} kr^k \\ &+ \frac{1}{2} \left\{ \sum_{k=0}^{\infty} r^k + \frac{1}{2}(1-r^2) \sum_{k=0}^{\infty} kr^k - \frac{r(r+1)}{2} \sum_{k=0}^{\infty} r^k \right\} \\ &= \frac{r}{(1-r)^2} + \frac{1}{2}(1-r^2) \frac{r(1+r)}{(1-r)^3} - \frac{r(r+1)}{2} \frac{r}{(1-r)^2} \\ &+ \frac{1}{2} \left\{ \frac{1}{1-r} + \frac{1}{2}(1-r^2) \frac{r}{(1-r)^2} - \frac{r(r+1)}{2} \frac{1}{1-r} \right\} \end{aligned}$$

and thus

$$\begin{aligned} P_2(r, t) &\leq (1-r)^2 \sum_{k=0}^{\infty} \left(k + \frac{1}{2}\right) \left\{1 + \frac{k}{2}(1-r^2) - \frac{r(r+1)}{2}\right\} r^k \\ &= r + \frac{1}{2}r(1+r)^2 - \frac{r^2(r+1)}{2} \\ &+ \frac{1}{2} \left\{ (1-r) + \frac{r}{2}(1-r^2) - \frac{r(r+1)}{2}(1-r) \right\} \\ &= \frac{1}{2}(1+r)^2. \end{aligned} \tag{3.5}$$

Now, using Lemma 2.3 (i) and the estimate in (3.5), we obtain

$$\begin{aligned} B_{11}(r) &= \mathcal{O}(1) \int_0^{1-r} w(t) P_2(r, t) dt \\ &= \mathcal{O}(1) w(1-r) \int_0^{1-r} (1/2)(1+r)^2 dt \\ &= \mathcal{O}(1)(1-r)w(1-r) \\ &= \mathcal{O}(v(\pi))(1-r) \frac{w(1-r)}{v(1-r)}, \end{aligned} \tag{3.6}$$

since $w(t)$ is an increasing function.

On the other hand, using the inequality $\pi \sin \theta \geq 2\theta$ for $\theta \in [0, \pi/2]$, and $|\sin \theta| \leq 1$ for any θ , in the Lemma 2.4 (ii), we deduce that

$$\begin{aligned} P_2(r, t) &\leq \frac{\pi^2}{2t^2} (1-r)^2 \sum_{k=0}^{\infty} \left[1 + \frac{k}{2}(1-r^2) - r(1+r)\right] r^k \\ &= \frac{\pi^2}{2t^2} (1-r)^2 \left[\frac{1}{1-r} + \frac{1}{2}(1-r^2) \frac{r}{(1-r)^2} - r(1+r) \frac{1}{1-r} \right] \\ &= \frac{\pi^2}{2t^2} \left[1-r + \frac{1}{2}(1-r^2)r - r(1-r^2) \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{\pi^2}{4t^2} (2 - 3r + r^3) \\
 &= \frac{\pi^2}{4t^2} (2 + r) (1 - r)^2.
 \end{aligned} \tag{3.7}$$

Using Lemma 2.3 (i) and (3.7) we derive the following estimate

$$\begin{aligned}
 B_{12}(r) &= \mathcal{O}(1) \int_{1-r}^{\pi} w(t)P_2(r, t)dt \\
 &= \mathcal{O}(v(\pi)) (1 - r)^2 \int_{1-r}^{\pi} \frac{w(t)dt}{t^2v(t)}.
 \end{aligned} \tag{3.8}$$

Combining (3.4), (3.6), and (3.8), we get

$$P(B_r(f)(\cdot)) = \mathcal{O} \left((1 - r) \frac{w(1 - r)}{v(1 - r)} + (1 - r)^2 \int_{1-r}^{\pi} \frac{w(t)dt}{t^2v(t)} \right). \tag{3.9}$$

It remains to estimate the quantity

$$\frac{P(B_r(f)(\cdot + y) + B_r(f)(\cdot))}{w(y)}$$

from above.

Indeed, using the representation in (3.2) we have

$$B_r(f)(x + y) - B_r(f)(x) = \frac{1}{\pi} \int_0^{\pi} [\varphi_{x+y}(t) - \varphi_x(t)]P_2(r, t)dt,$$

in which we apply Lemma 2.1 to obtain

$$P(B_r(f)(\cdot + y) - B_r(f)(\cdot)) \leq \frac{1}{\pi} \int_0^{\pi} P(\varphi_{\cdot+y}(t) - \varphi_{\cdot}(t)) P_2(r, t)dt.$$

After applying Lemma 2.3 (ii) to the right-hand side of equation (3.10), we can subsequently decompose the integral as follows

$$\begin{aligned}
 &P(B_r(f)(\cdot + y) - B_r(f)(\cdot)) \\
 &= \mathcal{O}(v(y)) \int_0^{\pi} \frac{w(t)}{v(t)} P_2(r, t)dt \\
 &= \mathcal{O}(v(y)) \left(\underbrace{\int_0^{1-r} \frac{w(t)}{v(t)} P_2(r, t)dt}_{=:B_1(r)} + \underbrace{\int_{1-r}^{\pi} \frac{w(t)}{v(t)} P_2(r, t)dt}_{=:B_2(r)} \right).
 \end{aligned} \tag{3.10}$$

Utilizing the estimate for $P_2(r, t)$ from (3.5) and the assumption that the function w/v is an almost non-increasing, we obtain

$$B_1(r) = \mathcal{O} \left(\frac{w(1 - r)}{v(1 - r)} \right) \int_0^{1-r} \frac{1}{2} (1 + r)^2 dt = \mathcal{O} \left((1 - r) \frac{w(1 - r)}{v(1 - r)} \right). \tag{3.11}$$

Using the inequality established in (3.7), we derive the following bound

$$\begin{aligned}
 B_2(r) &= \mathcal{O} \left(\frac{\pi^2}{4} (2 + r) (1 - r)^2 \right) \int_{1-r}^{\pi} \frac{w(t)}{t^2v(t)} dt \\
 &= \mathcal{O} \left((1 - r)^2 \int_{1-r}^{\pi} \frac{w(t)}{t^2v(t)} dt \right).
 \end{aligned} \tag{3.12}$$

Combining (3.10), (3.11), and (3.12), we get

$$\frac{P(B_r(f)(\cdot + y) - B_r(f)(\cdot))}{v(y)} = \mathcal{O} \left((1-r) \frac{w(1-r)}{v(1-r)} + (1-r)^2 \int_{1-r}^{\pi} \frac{w(t)dt}{t^2v(t)} \right). \tag{3.13}$$

Finally, integrating the results from equations (3.9) and (3.13) in definition of the seminorm $P(\cdot)^{(v)}$, we obtain

$$\begin{aligned} P(B_r(f) - f)^{(v)} &= P(B_r(f)(\cdot) - f(\cdot)) + \frac{P(B_r(f)(\cdot + y) - B_r(f)(\cdot))}{v(y)} \\ &= \mathcal{O} \left((1-r) \frac{w(1-r)}{v(1-r)} + (1-r)^2 \int_{1-r}^{\pi} \frac{w(t)dt}{t^2v(t)} \right) \end{aligned}$$

which is (3.1).

This completes the proof. □

Observe that the first term in (3.1) is dominated by the second term. Indeed, since $\frac{w(t)}{v(t)}$ is a non-decreasing function and $r \in (0, 1)$, it follows that

$$\begin{aligned} (1-r)^2 \int_{1-r}^{\pi} \frac{w(t)}{t^2v(t)} dt &\geq (1-r)^2 \frac{w(1-r)}{v(1-r)} \int_{1-r}^{\pi} t^{-2} dt \\ &= (1-r)^2 \frac{w(1-r)}{v(1-r)} \left(\frac{1}{1-r} - \frac{1}{\pi} \right) \\ &= (1-r) \frac{w(1-r)}{v(1-r)} \left(1 - \frac{1-r}{\pi} \right) \\ &\geq C(1-r) \frac{w(1-r)}{v(1-r)}, \end{aligned}$$

where $C := 1 - \pi^{-1}$.

This implies the following.

Corollary 3.2. *Assume that all the conditions in Theorem 3.1 are satisfied. Then,*

$$P(B_r(f) - f)^{(v)} = \mathcal{O} \left((1-r)^2 \int_{1-r}^{\pi} \frac{w(t)dt}{t^2v(t)} \right).$$

If, in addition, $\frac{w(t)}{tv(t)}$ is a non-increasing function, then

$$\int_{1-r}^{\pi} \frac{w(t)dt}{t^2v(t)} = \frac{w(1-r)}{(1-r)v(1-r)} \ln \left(\frac{\pi}{1-r} \right).$$

So, we derive the following.

Corollary 3.3. *Assume that all the conditions in Theorem 3.1 are satisfied. If, in addition, $\frac{w(t)}{tv(t)}$ is a non-increasing function, then*

$$P(B_r(f) - f)^{(v)} = \mathcal{O} \left(\frac{w(1-r)}{v(1-r)} (1-r) \ln \left(\frac{\pi}{1-r} \right) \right).$$

We examine the particular selections $w(t) = t^\alpha$ and $v(t) = t^\beta$ in Theorem 3.1, where the parameters α and β satisfy $0 \leq \beta < \alpha \leq 1$. By applying these conditions within the Corollary 3.3, we observe that the function $\frac{w(t)}{v(t)} = t^{\alpha-\beta}$ is non-decreasing, while $\frac{w(t)}{tv(t)} = \frac{1}{t^{1-\alpha+\beta}}$ is non-increasing. Consequently, we establish the following corollary.

Corollary 3.4. *Assume that all the conditions in Theorem 3.1 are satisfied. If $0 \leq \beta < \alpha \leq 1$, then*

$$P(B_r(f) - f)^{(\beta)} = \mathcal{O} \left((1-r)^{1+\alpha-\beta} \ln \left(\frac{\pi}{1-r} \right) \right).$$

For the special case $\beta = 0$ in Corollary 3.4, we derive the following result.

Corollary 3.5. *Assume that all the conditions in Theorem 3.1 are satisfied. If $0 < \alpha \leq 1$, then*

$$P(B_r(f) - f) = \mathcal{O}\left((1-r)^{1+\alpha} \ln\left(\frac{\pi}{1-r}\right)\right).$$

4 Conclusion

In this study, we analyzed the convergence of 2π -periodic functions in the seminormed space $H_P^{(w)}$, using biharmonic Poisson integrals $B_r(f)(x)$ as a tool for approximation. By describing the degree of approximation with two modulus-type functions, we provided a clearer understanding of the convergence behavior in this space. Our results contribute to the theory of function approximation and demonstrate how biharmonic Poisson integrals can be useful in this area. However, there are some limitations. Our results are valid only in the specific space $H_P^{(w)}$ and may not apply directly to other function spaces. They also depend on the particular modulus-type functions used; therefore, using different measures may lead to different results. Finally, for future work, we plan to extend these results to other function spaces, such as weighted L^p spaces, spaces with variable exponents, or Morrey spaces.

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