

GENERALIZED ROUGH STATISTICAL CONVERGENCE OF ORDER α FOR DOUBLE SEQUENCES IN NEUTROSOPHIC FUZZY NORMED SPACES VIA COMPACT LINEAR OPERATORS

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Abstract This paper introduces the concept of rough ideal statistical convergence of order α ($0 < \alpha < 1$) for double sequences in neutrosophic fuzzy normed spaces using a compact linear operator. We define and examine rough \mathcal{I}_2 -statistical boundedness, rough \mathcal{I}_2 -statistical cluster points, and rough \mathcal{I}_2 -statistical limit points in this context. Several properties and relationships among these notions are established, providing new insights into convergence behavior in neutrosophic fuzzy normed spaces.

1 Introduction

The concept of statistical convergence was initially proposed by Fast [12], and later, it was further explored from the perspective of sequence spaces by several researchers (see, for instance, [13, 31, 34, 37, 38, 39, 41, 42, 44]). Statistical convergence has emerged as a significant area of investigation due to its broad applicability across various fields of mathematics, including number theory, mathematical analysis, and probability theory, among others. The exploration of statistical convergence in double sequences was initiated by Mursaleen and Edely [33], and Tripathy [43], each working independently. The concept of ideal convergence serves as a more general extension of both classical convergence and statistical convergence. The notion of \mathcal{I} -convergence was further examined from the standpoint of sequence spaces and its connections to summability theory by Kostyrko et al. [25]. Subsequently, this concept has been extended in multiple directions. Das et al. [7] presented the idea of \mathcal{I} -convergence for double sequences in metric spaces and investigated various properties associated with this type of convergence.

Zadeh [45] is widely acknowledged as the pioneer who reshaped classical set theory by introducing fuzzy set theory. This seminal development has established itself as a fundamental component of modern mathematics, driving progress and fostering applications across diverse scientific and engineering disciplines, including chaos control and nonlinear dynamical systems. A significant breakthrough in the evolution of fuzzy set theory came with Atanassov's [3] formulation of intuitionistic fuzzy sets. By incorporating a non-membership function in addition to the standard membership function, this extension offers a more comprehensive and flexible framework. This pivotal advancement has led to the emergence of novel mathematical concepts, further broadening the scope of fuzzy set theory.

Expanding on this foundation, Smarandache [40] introduced neutrosophic sets, which generalize intuitionistic fuzzy sets by incorporating an additional function to account for indeterminacy. Within this framework, each element is represented by a triplet comprising truth-membership (T-M), indeterminacy-membership (I-M), and falsity-membership (F-M) functions.

This refined structure allows for a more detailed and nuanced representation of uncertainty, enabling precise characterization of elements based on their varying degrees of truth, indeterminacy, and falsity.

Building upon these developments, Kirişçi and Şimşek [23] proposed the concept of neutrosophic normed linear spaces (NFNS), opening new avenues for studying statistical convergence within this generalized setting. Their contributions have stimulated extensive research into different forms of sequence convergence in these spaces. For further insights into this rapidly evolving area, refer to [22]. Ideal convergence of sequences was investigated in NFNS by Kişi [24].

The notion of rough convergence was first formulated by Phu [35] within the setting of finite-dimensional normed spaces. In a subsequent study [36], Phu extended this concept by introducing rough continuity for linear mappings and proved that for any normed spaces X and Y with $\dim Y < \infty$ and any $r > 0$, every linear transformation $f : X \rightarrow Y$ is r -continuous at each point $x \in X$. Aytar [4] explored rough statistical convergence. More recently, Dündar and Çakan [10, 11] introduced rough \mathcal{I} -convergence and characterized the set of rough \mathcal{I} -limit points for sequences, further analyzing rough convergence properties and limit points in the framework of double sequences. This theory was later extended by Malik et al. [29], who established rough statistical convergence for double sequences based on double natural density on $\mathbb{N} \times \mathbb{N}$, following the generalization of Pringsheim-type double sequence convergence to statistical convergence. Furthermore, Dündar [9] explored rough \mathcal{I}_2 -convergence for double sequences. The concept of \mathcal{I} -statistical convergence for double sequences was introduced by Malik and Ghosh [28] as part of the broader framework of rough convergence theory. Since then, numerous studies have been conducted in this area up to the present (see, for example, [1, 2, 14, 15, 16, 17, 18, 19, 20, 21, 32]).

The concept of statistical convergence of order α for real number sequences was initially introduced by Çolak [6] in 2010. Later, Savaş and Das [8] expanded this idea by incorporating \mathcal{I} -statistical convergence of order α (where $\alpha \in (0, 1]$). Maity [27] explored several properties of rough statistical limit points within normed linear spaces and studied rough statistical convergence of order α . A set of necessary and sufficient conditions for a sequence (x_k) to be \mathcal{I} -statistically convergent of order α and \mathcal{I} -statistically bounded of order α was established in [5]. Furthermore, the authors defined \mathcal{I} -statistical rough convergence of order α in normed linear spaces.

1.1 Motivation

This work extends the results of Malik and Ghosh [28] to NFNS, providing a new perspective in the literature. The current study aims to define the concept of rough ideal statistical convergence of order α ($0 < \alpha < 1$) for double sequences in NFNS using a compact linear operator. Classical notions of convergence often fail to capture the complexities inherent in fuzzy and neutrosophic settings, particularly for double sequences. To address these challenges, this study introduces the notions of rough ideal statistical boundedness, rough ideal statistical cluster points, and rough ideal statistical limit points for double sequences. These new definitions expand the existing theories of convergence and provide a more robust approach for analyzing sequence behavior in NFNS. By accommodating uncertainties and vagueness in data, the study not only extends classical convergence theory but also offers potential applications in areas such as approximation theory, mathematical analysis, and real-world problems involving uncertainty.

2 Preliminary Concepts

This section will gather all pertinent results and methodologies that we will depend on to accomplish our primary objectives. To begin, let's establish some essential definitions.

Since Kostyrko et al. [25] first presented the concept of ideal convergence, there has been a significant increase in research focused on discovering new applications and advancing the study of summability within classical theories. This investigation has enhanced our comprehension of established mathematical structures and opened up new avenues for innovative applications in a variety of fields.

Let $\emptyset \neq Q$ be a set, and then a non empty class $\mathcal{I} \subseteq P(Q)$ is called an *ideal* on Q if and only if (i) \mathcal{I} is additive under union, (ii) for each $M \in \mathcal{I}$ and each $N \subseteq M$ we find $N \in \mathcal{I}$.

A non-empty family of sets \mathcal{F} is called *filter* on Q iff (i) $\emptyset \notin \mathcal{F}$, (ii) for each $M, N \in \mathcal{F}$ we get $M \cap N \in \mathcal{F}$, (iii) for every $M \in \mathcal{F}$ and each $N \supseteq M$, we obtain $N \in \mathcal{F}$.

Relationship between ideal and filter is given as follows:

$$\mathcal{F}(\mathcal{I}) = \{K \subset Q : K^c \in \mathcal{I}\},$$

where $K^c = Q - K$.

An ideal \mathcal{I} is called *non-trivial* if $\mathcal{I} \neq \emptyset$ and $Q \notin \mathcal{I}$.

A non-trivial ideal \mathcal{I} on Q is called *admissible* if and only if it includes all singleton sets.

Now, we present the concept of \mathcal{I}_2 -asymptotic density of \mathbb{N}^2 .

$U \subset \mathbb{N}^2$ is a subset with \mathcal{I}_2 -asymptotic density $d_{\mathcal{I}_2}(U)$ when

$$\delta_{\mathcal{I}_2}(U) = \mathcal{I}_2 - \lim_{u,v \rightarrow \infty} \frac{|U(u,v)|}{u.v},$$

where

$$U(u,v) = \{(u,v) \in \mathbb{N}^2 : u \leq t, v \leq q; (t,q) \in U\},$$

and $|U(u,v)|$ denotes number of elements of the set $U(u,v)$.

A nontrivial ideal \mathcal{I}_2 of $\mathbb{N} \times \mathbb{N}$ is called *strongly admissible* if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to \mathcal{I}_2 for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the paper we take \mathcal{I}_2 as a strongly admissible ideal in \mathbb{N}^2 .

Menger [30] proposed Triangular Norms (t-norms) as a generalization of probability distributions, incorporating the triangle inequality within the context of metric spaces. Triangular Conorms (t-conorms), recognized as dual operations to t-norms, play a crucial role in fuzzy operations, such as intersections and unions. Both t-norms and t-conorms are fundamental components for managing fuzzy operations within the framework of metric spaces.

Let Y be a linear space over the field \mathcal{V} and \circ and $*$ are t-norm and t-conorm, respectively. Let ρ, ς and τ be single valued fuzzy sets defined on $Y \times (0, \infty)$. We define the 6-tuple $(Y, \rho, \varsigma, \tau, \circ, *)$ as a NFNS if, for all $v, \gamma \in Y$ and $\tau, \kappa > 0$, the following conditions hold:

(A1) $\rho(v, \tau) + \varsigma(v, \tau) + \tau(v, \tau) \leq 3$,

(A2) $\rho(v, \tau) = 1, \varsigma(v, \tau) = 0$ and $\tau(v, \tau) = 0$ iff $v = 0$,

(A3) $\rho(\beta v, \tau) = \rho\left(v, \frac{\tau}{|\beta|}\right), \varsigma(\beta v, \tau) = \varsigma\left(v, \frac{\tau}{|\beta|}\right)$ and $\tau(\beta v, \tau) = \tau\left(v, \frac{\tau}{|\beta|}\right)$ for any $0 \neq \beta \in Y$,

(A4) $\rho(v+\gamma, \tau+\kappa) \geq \rho(v, \tau) \circ \rho(\gamma, \kappa), \varsigma(v+\gamma, \tau+\kappa) \leq \varsigma(v, \tau) * \varsigma(\gamma, \kappa)$ and $\tau(v+\gamma, \tau+\kappa) \leq \tau(v, \tau) * \tau(\gamma, \kappa)$,

(A5) $\rho(v, \cdot), \varsigma(v, \cdot)$ and $\tau(v, \cdot)$ are continuous on $(0, \infty)$,

(A6) $\lim_{\tau \rightarrow \infty} \rho(v, \tau) = 1, \lim_{\tau \rightarrow \infty} \varsigma(v, \tau) = 0$ and $\lim_{\mu \rightarrow \infty} \tau(v, \tau) = 0$,

(A7) $\lim_{\tau \rightarrow 0} \rho(v, \tau) = 0, \lim_{\mu \rightarrow 0} \varsigma(v, \tau) = 1$ and $\lim_{\mu \rightarrow 0} \tau(v, \tau) = 1$.

In this case, the 3-tuple (ρ, ς, τ) is referred to as a neutrosophic norm on Y .

The sequence $\check{z} = (\check{z}_u)$ is called \check{r} -convergent to \check{z}_* , denoted by $\check{z}_u \xrightarrow{\check{r}} \check{z}_*$ if

$$\forall \lambda > 0 \exists u_\lambda \in \mathbb{N} : u \geq u_\lambda \Rightarrow \|\check{z}_u - \check{z}_*\| < r + \lambda.$$

The set

$$\text{LIM}^{\check{r}} \check{z} := \{\check{z}_* \in \mathbb{R}^n : \check{z}_u \xrightarrow{\check{r}} \check{z}_*\}$$

is called the \check{r} -limit set of the sequence $\check{z} = (\check{z}_u)$.

A sequence $\check{z} = (\check{z}_u)$ is called \check{r} -convergent if $\text{LIM}^{\check{r}} \check{z} \neq \emptyset$. In this case, \check{r} is known as the degree of the sequence $\check{z} = (\check{z}_u)$. When $\check{r} = 0$, the sequence exhibits ordinary convergence. This concept has multiple applications (see [35]).

An operator \mathcal{T} defined by

$$\mathcal{T} : G \rightarrow H$$

is called compact linear operator (completely continuous linear operator) with G and H be two normed linear spaces if \mathcal{T} is linear and \mathcal{T} maps every bounded sequence $g = (g_k)$ in G onto a sequence $\mathcal{T}(g_k)$ in H which has a convergent subsequence. The set of all bounded linear operators $B(G, H)$ is normed linear space normed by

$$\|\mathcal{T}\| = \sup_{g \in G, \|g\|=1} \|\mathcal{T}g\|.$$

The set of all compact linear operator $C(G, H)$ is a closed subspace of $B(G, H)$ and $C(G, H)$ is a Banach space if H is a Banach (see details in [26]).

3 Main results

This section begins with defining the terms $St^{\check{r}}$ -convergence of order α ($\alpha \in (0, 1]$) and $\mathcal{I}_{St}^{\check{r}}$ of order α in NFNS using a compact linear operator. We then demonstrate several important findings. We shall use $\check{z} = \mathcal{T}(z_{uv}) = \check{z}_{uv}$ throughout the paper for simplicity in mathematical writing.

Definition 3.1. Let $(\check{Z}, \rho, \varsigma, \tau)$ be NFNS. A double sequence $\check{z} = \{\check{z}_{uv}\}$ in \check{Z} is said to be $St^{\check{r}}$ -convergent of order α ($\alpha \in (0, 1]$) to $\hat{w} \in \check{Z}$ with respect to norm (ρ, ς, τ) for some non-negative number \check{r} if for every $\lambda > 0$ and $\sigma \in (0, 1)$

$$\lim_{t,q \rightarrow \infty} \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma, \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma\}| = 0.$$

It is denoted by $\check{z}_{uv} \xrightarrow{St^{\check{r}, \alpha}} \hat{w}$ or $St^{\check{r}, \alpha} - \lim_{u,v \rightarrow \infty} \check{z}_{uv} = \hat{w}$.

Let $St\text{-LIM}_{\check{r}}^\alpha \check{z}_{uv}$ denotes the collection of $St^{\check{r}}$ -limit points of order α of the sequence $\check{z} = \{\check{z}_{uv}\}$.

Remark 3.2. For $\check{r} = 0$, the concept rough statistical convergence of order α corresponds to the usual statistical convergence of order α for the sequences using compact linear operator on NFNS. Moreover, when $\alpha = 1$, it aligns with rough statistical convergence in the presence of compact linear operators on NFNS.

Definition 3.3. Let $(\check{Z}, \rho, \varsigma, \tau)$ be NFNS. A sequence $\check{z} = \{\check{z}_{uv}\}$ in \check{Z} is said to be $\mathcal{I}_{st}^{\check{r}}$ -convergent of order α to $\hat{w} \in \check{Z}$ with respect to norm (ρ, ς, τ) for some non-negative number \check{r} if for every $\lambda, \psi > 0$ and $\sigma \in (0, 1)$

$$\left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma, \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma\}| \geq \psi \right\} \in \mathcal{I}_2,$$

or

$$\delta_{\mathcal{I}_2} \left(\{(u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma, \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma\} \right) = 0,$$

where $\delta_{\mathcal{I}_2}(A) = \mathcal{I}_2 - \lim_{t,q \rightarrow \infty} \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in A\}|$ if exists. Symbollically; it can be written as $\check{z}_{uv} \xrightarrow{\mathcal{I}_{St}^{\check{r}, \alpha}} \hat{w}$ or $\mathcal{I}_{St}^{\check{r}, \alpha} - \lim_{u,v \rightarrow \infty} \check{z}_{uv} = \hat{w}$.

Let $\mathcal{I}_2 - st\text{-LIM}_{\check{r}}^\alpha \check{z}_{uv}$ denotes the collection of all $\mathcal{I}_{St}^{\check{r}}$ -limits of order α of the sequence $\check{z} = \{\check{z}_{uv}\}$.

Remark 3.4. For $\check{r} = 0$, the notion rough ideal statistical convergence of order α agrees with the ideal statistical convergence of order α in NFNS.

From Definition 3.3, it is evident that the $\mathcal{I}_{st}^{\check{r}, \alpha}$ -limits of the sequence $\{\check{z}_{uv}\}$ is not unique. Hence, the set of all $\mathcal{I}_{st}^{\check{r}, \alpha}$ -limits of the sequence $\{\check{z}_{uv}\}$ is denoted and defined as

$$\mathcal{I}_2 - st - \text{LIM}_{\check{r}}^\alpha \check{z} = \left\{ \hat{w}^* \in \check{Z} : \check{z}_{uv} \xrightarrow{\mathcal{I}_{St}^{\check{r}, \alpha}} \hat{w}^* \right\}.$$

Furthermore, the sequence $\check{z} = \{\check{z}_{uv}\}$ is rough \mathcal{I}_2 -st-convergent if $\mathcal{I}_{St}^{\check{r}, \alpha} - \text{LIM}_{\check{r}}^\alpha \check{z} \neq \phi$ for some $\check{r} > 0$. It has been noted that for a double sequence $\check{z} = \{\check{z}_{uv}\}$ of real numbers, the set of rough limit points is given by

$$\text{LIM}_{\check{z}}^{\check{r}} = \{\limsup \check{z} - \check{r}, \liminf \check{z} + \check{r}\}.$$

In the similar way,

$$\mathcal{I}_2 - st - \text{LIM}_{\check{r}}^\alpha \check{z} = \{\mathcal{I}_2 - st - \limsup \check{z} - \check{r}, \mathcal{I}_2 - st - \liminf \check{z} - \check{r}\}.$$

Definition 3.5. A sequence $\check{z} = \{\check{z}_{uv}\}$ in NFNS $(\check{Z}, \rho, \varsigma, \tau)$ is \mathcal{I}_2 -st-bounded of order α ($\alpha \in (0, 1]$) if for $\psi > 0, \sigma \in (0, 1)$ and some $\check{r} > 0$, there exists $\check{U} > 0$ such that

$$\left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} \left| \{u \leq t, v \leq q : \rho(\check{z}_{uv}; \check{U}) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv}; \check{U}) \geq \sigma \right. \right. \\ \left. \left. \tau(\check{z}_{uv}; \check{U}) \geq \sigma \right\} \right| \geq \psi \} \in \mathcal{I}_2.$$

Theorem 3.6. Let $(\check{Z}, \rho, \varsigma, \tau)$ be NFNS with neutrosophic fuzzy norms (ρ, ς, τ) . A sequence $\check{z} = \{\check{z}_{uv}\}$ in \check{Z} is \mathcal{I}_2 -st-bounded of order α ($\alpha \in (0, 1]$) if and only if $\mathcal{I}_2 - st - LIM_\check{r}^\alpha \check{z} \neq \emptyset$ for some $\check{r} > 0$.

Proof. Necessary Part:- Firstly, let $\check{z} = \{\check{z}_{uv}\}$ which is \mathcal{I}_2 -st-bounded of order α on NFNS $(\check{Z}, \rho, \varsigma, \tau)$. Then, for any $\psi > 0, \sigma \in (0, 1)$ and some $\check{r} > 0, \exists \check{U} > 0$ such that

$$\left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} \left| \{u \leq t, v \leq q : \rho(\check{z}_{uv}; \check{U}) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv}; \check{U}) \geq \sigma, \right. \right. \\ \left. \left. \tau(\check{z}_{uv}; \check{U}) \geq \sigma \right\} \right| \geq \psi \} \in \mathcal{I}_2.$$

Since \mathcal{I}_2 is admissible, therefore $M = \mathbb{N}^2 \setminus Q$ is a non-empty set, where

$$Q = \left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} \left| \{u \leq t, v \leq q : \rho(\check{z}_{uv}; \check{U}) \leq 1 - \sigma \text{ or } \right. \right. \\ \left. \left. \varsigma(\check{z}_{uv}; \check{U}) \geq \sigma, \tau(\check{z}_{uv}; \check{U}) \geq \sigma \right\} \right| \geq \psi \}.$$

Choose $(u, v) \in M$, then

$$\frac{1}{(tq)^\alpha} \left| \{u \leq t, v \leq q : \rho(\check{z}_{uv}; \check{U}) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv}; \check{U}) \geq \sigma, \tau(\check{z}_{uv}; \check{U}) \geq \sigma \right| < \psi.$$

$$\implies \frac{1}{(tq)^\alpha} \left| \{u \leq t, v \leq q : \rho(\check{z}_{uv}; \check{U}) > 1 - \sigma \text{ and } \right. \\ \left. \varsigma(\check{z}_{uv}; \check{U}) < \sigma, \tau(\check{z}_{uv}; \check{U}) < \sigma \right\} \right| \geq 1 - \psi. \tag{3.1}$$

Let

$$B = \{u \leq t, v \leq q : \rho(\check{z}_{uv}; \check{U}) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv}; \check{U}) < \sigma, \tau(\check{z}_{uv}; \check{U}) < \sigma\}.$$

Also

$$\rho(\check{z}_{uv}; \check{r} + \check{U}) \geq \min \{\rho(0, \check{r}), \rho(\check{z}_{uv}, \check{U})\} = \min \{1, \rho(\check{z}_{uv}, \check{U})\} > 1 - \sigma, \\ \varsigma(\check{z}_{uv}; \check{r} + \check{U}) \leq \max \{\varsigma(0, \check{r}), \varsigma(\check{z}_{uv}, \check{U})\} = \max \{0, \varsigma(\check{z}_{uv}, \check{U})\} < \sigma, \\ \tau(\check{z}_{uv}; \check{r} + \check{U}) \leq \max \{\tau(0, \check{r}), \tau(\check{z}_{uv}, \check{U})\} = \max \{0, \tau(\check{z}_{uv}, \check{U})\} < \sigma.$$

Thus

$$B \subseteq \{u \leq t, v \leq q : \rho(\check{z}_{uv}; \check{r} + \check{U}) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv}; \check{r} + \check{U}) < \sigma, \tau(\check{z}_{uv}; \check{r} + \check{U}) < \sigma\}.$$

Using (3.1), it implies

$$1 - \psi \leq \frac{|B|}{(tq)^\alpha} \leq \frac{1}{(tq)^\alpha} \left| \{u \leq t, v \leq q : \rho(\check{z}_{uv}; \check{r} + \check{U}) > 1 - \sigma \text{ and } \right. \\ \left. \varsigma(\check{z}_{uv}; \check{r} + \check{U}) < \sigma, \tau(\check{z}_{uv}; \check{r} + \check{U}) < \sigma \right\} \right|.$$

Therefore,

$$\frac{1}{(tq)^\alpha} \left| \{u \leq t, v \leq q : \rho(\check{z}_{uv}; \check{r} + \check{U}) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv}; \check{r} + \check{U}) \geq \sigma, \right. \\ \left. \tau(\check{z}_{uv}; \check{r} + \check{U}) \geq \sigma \right\} \right| < 1 - (1 - \psi) = \psi.$$

Then, for $(t, q) \in \mathbb{N}^2$,

$$\frac{1}{(tq)^\alpha} \left| \{u \leq t, v \leq q : \rho(\check{z}_{uv}; \check{r} + \check{U}) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv}; \check{r} + \check{U}) \geq \sigma, \right. \\ \left. \tau(\check{z}_{uv}; \check{r} + \check{U}) \geq \sigma \right\} \right| \geq \psi \subset Q \in \mathcal{I}_2.$$

Hence, $0 \in \mathcal{I}_2 - st - LIM_\check{r}^\alpha \check{z}$. Therefore, $\mathcal{I}_2 - st - LIM_\check{r}^\alpha \check{z} \neq \emptyset$.

Sufficient Part:- Let $\mathcal{I}_2 - st - LIM_\check{r}^\alpha \check{z} \neq \emptyset$ for some $\check{r} > 0$. Then, there exists some $\hat{w} \in \check{Z}$ such

that $\hat{w} \in \mathcal{I}_2 - st\text{-}\lim_{\check{r}}^\alpha \check{z}$.

For every $\lambda, \psi > 0$ and $\sigma \in (0, 1)$, we have

$$\left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma, \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma\}| \geq \psi \right\} \in \mathcal{I}_2.$$

Consequently, almost all \check{z}_{uv} 's are enclosed in some ball with centre \hat{w} in NFNS, which indicates that $\check{z} = (\check{z}_{uv})$ is \mathcal{I}_2 -st-bounded of order α on NFNS ($\check{Z}, \rho, \varsigma, \tau$). \square

Next, we will show that the algebraic characterization also hold for (\check{z}_{uv}) sequences on NFNS.

Theorem 3.7. *Let $\check{z} = (\check{z}_{uv})$ and $\hat{h} = (\hat{h}_{uv})$ be two sequences in NFNS ($\check{Z}, \rho, \varsigma, \tau$), where \mathcal{I}_2 is an admissible ideal and $\alpha \in (0, 1]$. Then, the following properties hold,*

(i) *If $\mathcal{I}_{St}^{(\check{r}, \alpha)} - \lim_{u, v \rightarrow \infty} \check{z}_{uv} = \hat{w}$ and $\kappa \in \mathbb{R}$, then*

$$\mathcal{I}_{St}^{(\check{r}, \alpha)} - \lim_{u, v \rightarrow \infty} \kappa \check{z}_{uv} = \kappa \hat{w}.$$

(ii) *If $\mathcal{I}_{St}^{(\check{r}, \alpha)} - \lim_{u, v \rightarrow \infty} \check{z}_{uv} = \hat{w}_1$ and $\mathcal{I}_{St}^{(\check{r}, \alpha)} - \lim_{u, v \rightarrow \infty} \hat{h}_{uv} = \hat{w}_2$, then*

$$\mathcal{I}_{St}^{(\check{r}, \alpha)} - \lim_{u, v \rightarrow \infty} (\check{z}_{uv} + \hat{h}_{uv}) = \hat{w}_1 + \hat{w}_2.$$

Proof. If $\kappa = 0$, then the result holds trivially. Thus, we consider the case when $\kappa \neq 0$. Since $\mathcal{I}_{St}^{(\check{r}, \alpha)} - \lim_{u, v \rightarrow \infty} \check{z}_{uv} = \hat{w}$, then for given $\lambda, \psi > 0$ and $\sigma \in (0, 1)$

$$Q = \left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma, \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma\}| \geq \psi \right\} \in \mathcal{I}_2.$$

Since \mathcal{I}_2 is admissible, its complement $M = \mathbb{N}^2 \setminus Q$ must be non-empty. Selecting an arbitrary pair $(u, v) \in M$, we derive the following condition

$$\begin{aligned} & \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \leq 1 - \sigma \text{ or } \\ & \varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma, \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma\}| < \psi. \\ \Rightarrow & \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) > 1 - \sigma \text{ and } \\ & \varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) < \sigma, \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) < \sigma\}| \geq 1 - \psi. \\ \Rightarrow & \frac{1}{(tq)^\alpha} |B| \geq 1 - \psi, \end{aligned} \tag{3.2}$$

where

$$B = \{(u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) < \sigma, \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) < \sigma\}.$$

It is sufficient to prove that for all $\sigma \in (0, 1)$ and $\lambda > 0$

$$B \subseteq \{(u, v) \in \mathbb{N}^2 : \rho(\kappa \check{z}_{uv} - \kappa \hat{w}; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\kappa \check{z}_{uv} - \kappa \hat{w}; \check{r} + \lambda) < \sigma, \tau(\kappa \check{z}_{uv} - \kappa \hat{w}; \check{r} + \lambda) < \sigma\}.$$

Let $(u, v) \in B$, then $\rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) > 1 - \sigma$ and $\varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) < \sigma, \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) < \sigma$. So

$$\begin{aligned} \rho(\kappa \check{z}_{uv} - \kappa \hat{w}; \check{r} + \lambda) &= \rho\left(\check{z}_{uv} - \hat{w}; \frac{\check{r} + \lambda}{|\kappa|}\right) \\ &\geq \min\left\{\rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda), \rho\left(0; \frac{\check{r} + \lambda}{|\kappa|} - (\check{r} + \lambda)\right)\right\} \\ &\geq \min\{\rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda), 1\} \\ &= \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) > 1 - \sigma, \end{aligned}$$

$$\begin{aligned} \varsigma(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) &= \varsigma\left(\check{z}_{uv} - \hat{w}; \frac{\check{r} + \lambda}{|\kappa|}\right) \\ &\leq \max\left\{\rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda), \rho\left(0; \frac{\check{r} + \lambda}{|\kappa|} - (\check{r} + \lambda)\right)\right\} \\ &\leq \max\{\rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda), 0\} \\ &= \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) < \sigma \end{aligned}$$

and

$$\begin{aligned} \tau(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) &= \tau\left(\check{z}_{uv} - \hat{w}; \frac{\check{r} + \lambda}{|\kappa|}\right) \\ &\leq \max\left\{\tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda), \tau\left(0; \frac{\check{r} + \lambda}{|\kappa|} - (\check{r} + \lambda)\right)\right\} \\ &\leq \max\{\tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda), 0\} \\ &= \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) < \sigma. \end{aligned}$$

This implies

$$B \subseteq \{(u, v) \in \mathbb{N}^2 : \rho(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) < \sigma, \tau(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) < \sigma\}.$$

Using (3.2), we have

$$1 - \psi \leq \frac{B}{(tq)^\alpha} \leq \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) < \sigma, \tau(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) < \sigma\}|.$$

Therefore,

$$\frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) < \sigma, \tau(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) < \sigma\}| < 1 - (1 - \psi) = \psi.$$

Then,

$$\left\{(t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) \leq 1 - \sigma \text{ or } \varsigma(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) \geq \sigma, \tau(\kappa\check{z}_{uv} - \kappa\hat{w}; \check{r} + \lambda) \geq \sigma\}| \geq \psi\right\} \subset Q \in \mathcal{I}_2,$$

which shows that $\mathcal{I}_{St}^{(\check{r}, \alpha)} - \lim_{u, v \rightarrow \infty} \kappa\check{z}_{uv} = \kappa\hat{w}$.

(ii) In the similar manner, the proof for part (ii) can be established. Therefore, we omit the details of this proof. \square

In the following result, we will demonstrate that the set $\mathcal{I}_2 - st\text{-LIM}_r^\alpha \check{z}$ is closed.

Theorem 3.8. *Let $\check{z} = (\check{z}_{uv})$ be a double sequence in NFNS. Then, the set $\mathcal{I}_2 - st\text{-LIM}_r^\alpha \check{z}$ of a sequence (\check{z}_{uv}) in NFNS is a closed set.*

Proof. If $\mathcal{I}_2 - st\text{-LIM}_r^\alpha \check{z}$ is non-empty, the result is trivial since $\mathcal{I}_2 - st\text{-LIM}_r^\alpha \check{z}$ is either an empty set or a singleton set. Let $\mathcal{I}_2 - st\text{-LIM}_r^\alpha \check{z} \neq \emptyset$ for some $\check{r} > 0$. Take $\hat{h} = (\hat{h}_{uv})$ be a convergent sequence in $(\check{Z}, \rho, \varsigma, \tau)$ with respect to (ρ, ς, τ) , which converges to $\hat{h}_0 \in \check{Z}$. For $\lambda > 0$ and $\sigma \in (0, 1)$ there exist $u_0, v_0 \in \mathbb{N}$ such that

$$\rho\left(\hat{h}_{uv} - \hat{h}_0; \frac{\lambda}{2}\right) > 1 - \sigma \text{ and } \varsigma\left(\hat{h}_{uv} - \hat{h}_0; \frac{\lambda}{2}\right) < \sigma, \tau\left(\hat{h}_{uv} - \hat{h}_0; \frac{\lambda}{2}\right) < \sigma,$$

for $u \geq u_0$ and $v \geq v_0$.

Let $\hat{h}_{u_1 v_1} \in \mathcal{I}_2 - st\text{-LIM}_r^\alpha \check{z}$ which implies the existence of a set Q defined by

$$Q = \left\{(t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} \left| \left\{u \leq t, v \leq q : \rho\left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2}\right) \leq 1 - \sigma \text{ or } \varsigma\left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2}\right) \geq \sigma, \tau\left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2}\right) \geq \sigma\right\} \right| \geq \psi\right\} \in \mathcal{I}_2.$$

Since \mathcal{I}_2 is admissible, complement $G = \mathbb{N}^2 \setminus Q$ is non-empty. Choose $(t, q) \in G$, then

$$\begin{aligned} & \frac{1}{(tq)^\alpha} \left| \left\{ u \leq t, v \leq q : \rho \left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2} \right) \leq 1 - \sigma \text{ or} \right. \right. \\ & \left. \left. \varsigma \left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2} \right) \geq \sigma, \tau \left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2} \right) \geq \sigma \right\} \right| < \psi. \\ & \Rightarrow \frac{1}{(tq)^\alpha} \left| \left\{ u \leq t, v \leq q : \rho \left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2} \right) > 1 - \sigma \text{ and} \right. \right. \\ & \left. \left. \varsigma \left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2} \right) < \sigma, \tau \left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2} \right) < \sigma \right\} \right| \geq 1 - \psi. \end{aligned}$$

Let

$$\begin{aligned} B_{t,q} := & \left\{ u \leq t, v \leq q : \rho \left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2} \right) > 1 - \sigma \text{ and} \right. \\ & \left. \varsigma \left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2} \right) < \sigma, \tau \left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2} \right) < \sigma \right\}. \end{aligned}$$

Then, for $k, l \in B_{t,q}$, for $k \geq u_0$ and $l \geq v_0$, we have

$$\begin{aligned} \rho \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) & \geq \min \left\{ \rho \left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2} \right), \rho \left(\hat{h}_{u_1 v_1} - \hat{h}_0; \frac{\lambda}{2} \right) \right\} \\ & > 1 - \sigma, \\ \varsigma \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) & \leq \max \left\{ \varsigma \left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2} \right), \varsigma \left(\hat{h}_{u_1 v_1} - \hat{h}_0; \frac{\lambda}{2} \right) \right\} \\ & < \sigma, \end{aligned}$$

and

$$\tau \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) \leq \max \left\{ \tau \left(\check{z}_{uv} - \hat{h}_{u_1 v_1}; \check{r} + \frac{\lambda}{2} \right), \tau \left(\hat{h}_{u_1 v_1} - \hat{h}_0; \frac{\lambda}{2} \right) \right\} < \sigma.$$

Therefore,

$$(k, l) \in \left\{ (u, v) \in \mathbb{N}^2 : \rho \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) > 1 - \sigma \text{ and } \varsigma \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) < \sigma \right. \\ \left. \tau \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) < \sigma \right\}.$$

Hence,

$$B_{t,q} \subseteq \left\{ (u, v) \in \mathbb{N}^2 : \rho \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) > 1 - \sigma \text{ and } \varsigma \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) < \sigma \right. \\ \left. \tau \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) < \sigma \right\},$$

which implies that

$$1 - \psi \leq \frac{|B_{t,q}|}{(tq)^\alpha} \leq \frac{1}{(tq)^\alpha} \left| \left\{ u \leq t, v \leq q : \rho \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) > 1 - \sigma \text{ and} \right. \right. \\ \left. \left. \varsigma \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) < \sigma, \tau \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) < \sigma \right\} \right|.$$

Therefore,

$$\frac{1}{(tq)^\alpha} \left| \left\{ u \leq t, v \leq q : \rho \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) \leq 1 - \sigma \text{ or } \varsigma \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) \geq \sigma, \right. \right. \\ \left. \left. \tau \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) \geq \sigma \right\} \right| < 1 - (1 - \psi) = \psi.$$

Then,

$$\left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} \left| \left\{ u \leq t, v \leq q : \rho \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) \leq 1 - \sigma \text{ or} \right. \right. \right. \\ \left. \left. \left. \varsigma \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) \geq \sigma, \tau \left(\check{z}_{kl} - \hat{h}_0; \check{r} + \lambda \right) \geq \sigma \right\} \right| \geq \psi \right\} \subset Q \in \mathcal{I}_2,$$

which shows that $\hat{h}_0 \in \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$. □

The convexity of the set $\mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$ is established in the following result.

Theorem 3.9. *Let $\check{z} = (\check{z}_{uv})$ be a double sequence in NFNS. Then, $\mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$ is a convex set for some non-negative number \check{r} .*

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$. For convexity we have to show that $(1 - \hat{w})\varphi_1 + \hat{w}\varphi_2 \in \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$ for any real number $\hat{w} \in (0, 1)$. Since $\varphi_1, \varphi_2 \in \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}_{uv}$, then there exist $u, v \in \mathbb{N}$ for every $\lambda > 0$ and $\sigma \in (0, 1)$ such that

$$\mathbb{K}_0 = \left\{ (u, v) \in \mathbb{N}^2 : \rho \left(\check{z}_{uv} - \varphi_1; \frac{\check{r} + \lambda}{2(1 - \hat{w})} \right) \leq 1 - \sigma \text{ and } \varsigma \left(\check{z}_{uv} - \varphi_1; \frac{\check{r} + \lambda}{2(1 - \hat{w})} \right) \geq \sigma \right. \\ \left. \tau \left(\check{z}_{uv} - \varphi_1; \frac{\check{r} + \lambda}{2(1 - \hat{w})} \right) \geq \sigma \right\},$$

and

$$\mathbb{K}_1 = \left\{ (u, v) \in \mathbb{N}^2 : \rho \left(\check{z}_{uv} - \varphi_2; \frac{\check{r} + \lambda}{2\hat{w}} \right) \leq 1 - \sigma \text{ and } \varsigma \left(\check{z}_{uv} - \varphi_2; \frac{\check{r} + \lambda}{2\hat{w}} \right) \geq \sigma \right. \\ \left. \tau \left(\check{z}_{uv} - \varphi_2; \frac{\check{r} + \lambda}{2\hat{w}} \right) \geq \sigma \right\}.$$

For $\psi > 0$, we have

$$\left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_0 \cup \mathbb{K}_1\}| \geq \psi \right\} \in \mathcal{I}_2.$$

Now choose $0 < \psi_1 < 1$ such that $0 < 1 - \psi_1 < \psi$. Let

$$\mathbb{C} = \left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_0 \cup \mathbb{K}_1\}| \geq \psi_1 \right\} \in \mathcal{I}_2.$$

For $(t, q) \notin \mathbb{C}$

$$\frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_0 \cup \mathbb{K}_1\}| < 1 - \psi_1, \\ \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \notin \mathbb{K}_0 \cup \mathbb{K}_1\}| \geq 1 - (1 - \psi_1) = \psi_1.$$

This implies

$$\{u \leq t, v \leq q : (u, v) \notin \mathbb{K}_0 \cup \mathbb{K}_1\} \neq \emptyset.$$

Let $(u_0, v_0) \in (\mathbb{K}_0 \cup \mathbb{K}_1)^c = \mathbb{K}_0^c \cap \mathbb{K}_1^c$. Then,

$$\rho(\check{z}_{u_0 v_0} - [(1 - \hat{w})\varphi_1 + \hat{w}\varphi_2]; \check{r} + \lambda) \\ = \rho[(1 - \hat{w})(\check{z}_{u_0 v_0} - \varphi_1) + \hat{w}(\check{z}_{u_0 v_0} - \varphi_2); \check{r} + \lambda] \\ \geq \min \left\{ \rho \left((1 - \hat{w})(\check{z}_{u_0 v_0} - \varphi_1); \frac{\check{r} + \lambda}{2} \right), \rho \left(\hat{w}(\check{z}_{u_0 v_0} - \varphi_2); \frac{\check{r} + \lambda}{2} \right) \right\} \\ = \min \left\{ \rho \left(\check{z}_{u_0 v_0} - \varphi_1; \frac{\check{r} + \lambda}{2(1 - \hat{w})} \right), \rho \left(\check{z}_{u_0 v_0} - \varphi_2; \frac{\check{r} + \lambda}{2\hat{w}} \right) \right\} \\ > 1 - \sigma,$$

$$\varsigma(\check{z}_{u_0 v_0} - [(1 - \hat{w})\varphi_1 + \hat{w}\varphi_2]; \check{r} + \lambda) \\ = \varsigma[(1 - \hat{w})(\check{z}_{u_0 v_0} - \varphi_1) + \hat{w}(\check{z}_{u_0 v_0} - \varphi_2); \check{r} + \lambda] \\ \leq \max \left\{ \varsigma \left((1 - \hat{w})(\check{z}_{u_0 v_0} - \varphi_1); \frac{\check{r} + \lambda}{2} \right), \varsigma \left(\hat{w}(\check{z}_{u_0 v_0} - \varphi_2); \frac{\check{r} + \lambda}{2} \right) \right\} \\ = \max \left\{ \varsigma \left(\check{z}_{u_0 v_0} - \varphi_1; \frac{\check{r} + \lambda}{2(1 - \hat{w})} \right), \varsigma \left(\check{z}_{u_0 v_0} - \varphi_2; \frac{\check{r} + \lambda}{2\hat{w}} \right) \right\} \\ < \sigma,$$

and

$$\tau(\check{z}_{u_0 v_0} - [(1 - \hat{w})\varphi_1 + \hat{w}\varphi_2]; \check{r} + \lambda) \\ = \tau[(1 - \hat{w})(\check{z}_{u_0 v_0} - \varphi_1) + \hat{w}(\check{z}_{u_0 v_0} - \varphi_2); \check{r} + \lambda] \\ \leq \max \left\{ \tau \left((1 - \hat{w})(\check{z}_{u_0 v_0} - \varphi_1); \frac{\check{r} + \lambda}{2} \right), \tau \left(\hat{w}(\check{z}_{u_0 v_0} - \varphi_2); \frac{\check{r} + \lambda}{2} \right) \right\} \\ = \max \left\{ \tau \left(\check{z}_{u_0 v_0} - \varphi_1; \frac{\check{r} + \lambda}{2(1 - \hat{w})} \right), \tau \left(\check{z}_{u_0 v_0} - \varphi_2; \frac{\check{r} + \lambda}{2\hat{w}} \right) \right\} \\ < \sigma.$$

This implies $\mathbb{K}_0^c \cap \mathbb{K}_1^c \subset \mathbb{H}^c$, where

$$\mathbb{H} = \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{u_0 v_0} - [(1 - \hat{w})\varphi_1 + \hat{w}\varphi_2]; \check{r} + \lambda) \leq 1 - \sigma \text{ or } \right. \\ \left. \varsigma(\check{z}_{u_0 v_0} - [(1 - \hat{w})\varphi_1 + \hat{w}\varphi_2]; \check{r} + \lambda) \geq \sigma \right. \\ \left. \tau(\check{z}_{u_0 v_0} - [(1 - \hat{w})\varphi_1 + \hat{w}\varphi_2]; \check{r} + \lambda) \geq \sigma \right\}.$$

So for $(t, q) \notin \mathbb{C}$,

$$\begin{aligned} \psi_1 &\leq \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \notin \mathbb{K}_0 \cup \mathbb{K}_1\}| \\ &\leq \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \notin \mathbb{H}\}|, \end{aligned}$$

or

$$\frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{H}\}| < 1 - \psi_1 < \psi.$$

Thus

$$\mathbb{C}^c \subset \left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{H}\}| < \psi \right\}.$$

Since $\mathbb{C}^c \in \mathcal{F}(\mathcal{I}_2)$, So,

$$\left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{H}\}| < \psi \right\} \in \mathcal{F}(\mathcal{I}_2),$$

which implies

$$\left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{H}\}| \geq \psi \right\} \in \mathcal{I}_2.$$

This implies $\mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}_{uv}$ is a convex set. □

Theorem 3.10. *A sequence $\check{z} = \{\check{z}_{uv}\}$ in NFNS $(\check{Z}, \rho, \varsigma, \tau)$ is $\mathcal{I}_{St}^{(\check{r}, \alpha)}$ -convergent of order α to $\hat{w} \in \check{Z}$ with respect to norm (ρ, ς, τ) for some $\check{r} > 0$ if there exists a sequence $\hat{h} = (\hat{h}_{uv})$ in \check{Z} such that $\mathcal{I}_{St}^\alpha - \lim_{u, v \rightarrow \infty} \hat{h}_{uv} = \hat{w}$ in \check{Z} and for every $\sigma \in (0, 1)$ have $\rho(\check{z}_{uv} - \hat{h}_{uv}; \check{r} + \lambda) > 1 - \sigma$ and $\varsigma(\check{z}_{uv} - \hat{h}_{uv}; \check{r} + \lambda) < \sigma, \tau(\check{z}_{uv} - \hat{h}_{uv}; \check{r} + \lambda) < \sigma$ for all $u, v \in \mathbb{N}$.*

Proof. Let $\mathcal{I}_{St}^\alpha - \lim_{u, v \rightarrow \infty} \hat{h}_{uv} = \hat{w}$ in \check{Z} and $\rho(\check{z}_{uv} - \hat{h}_{uv}; \check{r} + \lambda) > 1 - \sigma$ and $\varsigma(\check{z}_{uv} - \hat{h}_{uv}; \check{r} + \lambda) < \sigma, \tau(\check{z}_{uv} - \hat{h}_{uv}; \check{r} + \lambda) > 1 - \sigma$. Then, by definition, for any $\lambda, \psi > 0$ and $\sigma \in (0, 1)$, the set

$$\begin{aligned} M = \left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} \left| \left\{ u \leq t, v \leq q : \rho(\hat{h}_{uv} - \hat{w}; \lambda) \leq 1 - \sigma \text{ or} \right. \right. \right. \\ \left. \left. \left. \varsigma(\hat{h}_{uv} - \hat{w}; \lambda) \geq \sigma, \tau(\hat{h}_{uv} - \hat{w}; \lambda) \geq \sigma \right\} \right| \geq \psi \right\} \in \mathcal{I}_2. \end{aligned}$$

Define

$$\mathbb{K}_1 = \left\{ (u, v) \in \mathbb{N}^2 : \rho(\hat{h}_{uv} - \hat{w}; \lambda) \leq 1 - \sigma \text{ and } \varsigma(\hat{h}_{uv} - \hat{w}; \lambda) \geq \sigma \right. \\ \left. \tau(\hat{h}_{uv} - \hat{w}; \lambda) \geq \sigma \right\},$$

and

$$\mathbb{K}_2 = \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \hat{h}_{uv}; \check{r}) \leq 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \hat{h}_{uv}; \check{r}) \geq \sigma \right. \\ \left. \tau(\check{z}_{uv} - \hat{h}_{uv}; \check{r}) \geq \sigma \right\}.$$

For $\psi > 0$, we have

$$\left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_1 \cup \mathbb{K}_2\}| \geq \psi \right\} \in \mathcal{I}_2.$$

Now choose $0 < \psi_1 < 1$ such that $0 < 1 - \psi_1 < \psi$ and let

$$\mathbb{K} = \left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_1 \cup \mathbb{K}_2\}| \geq \psi_1 \right\} \in \mathcal{I}_2.$$

For $(t, q) \notin \mathbb{K}$

$$\begin{aligned} \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_1 \cup \mathbb{K}_2\}| &< 1 - \psi_1, \\ \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \notin \mathbb{K}_1 \cup \mathbb{K}_2\}| &\geq 1 - (1 - \psi_1) = \psi_1. \end{aligned}$$

This implies

$$\{u \leq t, v \leq q : (u, v) \notin \mathbb{K}_1 \cup \mathbb{K}_2\} \neq \emptyset.$$

Let $(u, v) \in (\mathbb{K}_1 \cup \mathbb{K}_2)^c = \mathbb{K}_1^c \cap \mathbb{K}_2^c$. Then,

$$\begin{aligned} \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) &\geq \min \left\{ \rho(\check{z}_{uv} - \hat{h}_{uv}; \check{r}), \rho(\hat{h}_{uv} - \hat{w}; \lambda) \right\} \\ &> 1 - \sigma, \end{aligned}$$

$$\begin{aligned} \varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) &\leq \max \left\{ \varsigma(\check{z}_{uv} - \hat{h}_{uv}; \check{r}), \varsigma(\hat{h}_{uv} - \hat{w}; \lambda) \right\} \\ &< \sigma, \end{aligned}$$

and

$$\begin{aligned} \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) &\leq \max \left\{ \tau(\check{z}_{uv} - \hat{h}_{uv}; \check{r}), \tau(\hat{h}_{uv} - \hat{w}; \lambda) \right\} \\ &< \sigma, \end{aligned}$$

which gives $\mathbb{K}_1^c \cap \mathbb{K}_2^c \subset \mathbb{C}^c$, where

$$\mathbb{C} = \{(u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \leq 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma \\ \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \geq \sigma\}.$$

So for $(u, v) \notin \mathbb{K}$,

$$\begin{aligned} \psi_1 &\leq \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \notin \mathbb{K}_1 \cup \mathbb{K}_2\}| \\ &\leq \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \notin \mathbb{C}\}| \end{aligned}$$

or

$$\frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{C}\}| < 1 - \psi_1 < \psi.$$

Thus

$$\mathbb{K}^c \subset \left\{ (u, v) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{C}\}| < \psi \right\}.$$

Since $\mathbb{K}^c \in \mathcal{F}(\mathcal{I}_2)$, So,

$$\left\{ (u, v) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{C}\}| < \psi \right\} \in \mathcal{F}(\mathcal{I}_2),$$

which implies

$$\left\{ (u, v) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{C}\}| \geq \psi \right\} \in \mathcal{I}_2.$$

Hence, $\mathcal{I}_{St}^{(\check{r}, \alpha)} - \lim_{u, v \rightarrow \infty} \check{z}_{uv} = \hat{w}$. □

Theorem 3.11. *Let $\check{z} = \{\check{z}_{uv}\}$ be a sequence in NFNS $(\check{Z}, \rho, \varsigma, \tau)$ then there does not exist two elements $\alpha_1, \alpha_2 \in \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$ for $\check{r} > 0$ and $\sigma \in (0, 1)$ such that $\rho(\alpha_1 - \alpha_2; t\check{r}) \leq 1 - \sigma$ or $\varsigma(\alpha_1 - \alpha_2; t\check{r}) \geq \sigma$, $\tau(\alpha_1 - \alpha_2; t\check{r}) \geq \sigma$ for $t > 2$.*

Proof. We will employ a contradiction argument to justify our conclusion. Suppose that there are two elements $\alpha_1, \alpha_2 \in \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$ such that

$$\rho(\alpha_1 - \alpha_2; t\check{r}) \leq 1 - \sigma \text{ or } \varsigma(\alpha_1 - \alpha_2; t\check{r}) \geq \sigma, \tau(\alpha_1 - \alpha_2; t\check{r}) \geq \sigma \text{ for } t > 2. \tag{3.3}$$

Since $\alpha_1, \alpha_2 \in \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$ then for every $\lambda > 0$ and $\sigma \in (0, 1)$. Establish

$$\mathbb{K}_1 = \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \alpha_1; \check{r} + \frac{\lambda}{2}) \leq 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \alpha_1; \check{r} + \frac{\lambda}{2}) \geq \sigma \\ \tau(\check{z}_{uv} - \alpha_1; \check{r} + \frac{\lambda}{2}) \geq \sigma \right\},$$

and

$$\mathbb{K}_1 = \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \alpha_2; \check{r} + \frac{\lambda}{2}) \leq 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \alpha_2; \check{r} + \frac{\lambda}{2}) \geq \sigma \\ \tau(\check{z}_{uv} - \alpha_2; \check{r} + \frac{\lambda}{2}) \geq \sigma \right\}.$$

Then,

$$\frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_1 \cup \mathbb{K}_2\}| \leq \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_1\}| + \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_2\}|.$$

So, by the property of \mathcal{I}_2 -convergence, we have

$$\begin{aligned} & \mathcal{I}_2 - \lim_{t,q \rightarrow \infty} \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_1 \cup \mathbb{K}_2\}| \\ & \leq \mathcal{I}_2 - \lim_{t,q \rightarrow \infty} \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_1\}| \\ & + \mathcal{I}_2 - \lim_{t,q \rightarrow \infty} \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_2\}| = 0. \end{aligned}$$

Thus

$$\left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_1 \cup \mathbb{K}_2\}| \geq \psi \right\} \in \mathcal{I}_2,$$

for all $\psi > 0$.

Now choose $0 < \psi_1 = 1/2 < 1$ such that $0 < 1 - \psi_1 < \psi$. Let

$$\mathbb{A} = \left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_1 \cup \mathbb{K}_2\}| \geq \psi_1 \right\} \in \mathcal{I}_2.$$

Now for $(t, q) \notin \mathbb{A}$

$$\begin{aligned} & \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in \mathbb{K}_1 \cup \mathbb{K}_2\}| < 1 - \psi_1 = 1/2, \\ & \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \notin \mathbb{K}_1 \cup \mathbb{K}_2\}| \geq 1 - (1 - \psi_1) = 1/2. \end{aligned}$$

This implies

$$\{u \leq t, v \leq q : (u, v) \notin \mathbb{K}_1 \cup \mathbb{K}_2\} \neq \emptyset.$$

Then, for $(u, v) \in \mathbb{K}_1^c \cap \mathbb{K}_2^c$ we have

$$\begin{aligned} \rho(\alpha_1 - \alpha_2; 2\check{r} + \lambda) & \geq \min \left\{ \rho(\check{z}_{uv} - \alpha_2; \check{r} + \frac{\lambda}{2}), \rho(\check{z}_{uv} - \alpha_1; \check{r} + \frac{\lambda}{2}) \right\} > 1 - \sigma, \\ \varsigma(\alpha_1 - \alpha_2; 2\check{r} + \lambda) & \leq \max \left\{ \varsigma(\check{z}_{uv} - \alpha_2; \check{r} + \frac{\lambda}{2}), \varsigma(\check{z}_{uv} - \alpha_1; \check{r} + \frac{\lambda}{2}) \right\} < \sigma, \\ \tau(\alpha_1 - \alpha_2; 2\check{r} + \lambda) & \leq \max \left\{ \tau(\check{z}_{uv} - \alpha_2; \check{r} + \frac{\lambda}{2}), \tau(\check{z}_{uv} - \alpha_1; \check{r} + \frac{\lambda}{2}) \right\} < \sigma. \end{aligned}$$

Hence,

$$\rho(\alpha_1 - \alpha_2; 2\check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\alpha_1 - \alpha_2; 2\check{r} + \lambda) < \sigma, \tau(\alpha_1 - \alpha_2; 2\check{r} + \lambda) < \sigma. \tag{3.4}$$

It follows from (3.4) that $\rho(\alpha_1 - \alpha_2; t\check{r}) > 1 - \sigma$ and $\varsigma(\alpha_1 - \alpha_2; t\check{r}) < \sigma, \tau(\alpha_1 - \alpha_2; t\check{r}) < \sigma$ for $t > 2$. This contradicts (3.3). Consequently, there does not exist two elements such that $\rho(\alpha_1 - \alpha_2; t\check{r}) \leq 1 - \sigma$ or $\varsigma(\alpha_1 - \alpha_2; t\check{r}) \geq \sigma, \tau(\alpha_1 - \alpha_2; t\check{r}) \geq \sigma$ for $t > 2$. \square

Definition 3.12. Let $(\check{Z}, \rho, \varsigma, \tau)$ be NFNS. Then, $\gamma \in \check{Z}$ is called rough \mathcal{I}_2 -statistical cluster point of order α of the sequence $\check{z} = \{\check{z}_{uv}\}$ in \check{Z} with respect to norm (ρ, ς, τ) for some $\check{r} > 0$ if, for every $\lambda > 0$ and $\sigma \in (0, 1)$

$$\delta_{\mathcal{I}_2} \left(\left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \gamma; \check{r} + \lambda) > 1 - \sigma \text{ or } \varsigma(\check{z}_{uv} - \gamma; \check{r} + \lambda) < \sigma \right. \right. \\ \left. \left. \tau(\check{z}_{uv} - \gamma; \check{r} + \lambda) < \sigma \right\} \right) \neq 0,$$

where

$$\delta_{\mathcal{I}_2}(A) = \mathcal{I}_2 - \lim_{t,q \rightarrow \infty} \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : (u, v) \in A\}|$$

if exists. In this case, γ is known as \check{r} - \mathcal{I}_2 -statistical cluster point of order α of a sequence $\check{z} = (\check{z}_{uv})$.

Let $\Gamma_{st(\rho,\varsigma,\tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)$ denotes the set of all \check{r} - \mathcal{I}_2 -statistical cluster points of order α with respect to norm (ρ, ς, τ) of a sequence $\check{z} = (\check{z}_{uv})$ in NFNS $(\check{Z}, \rho, \varsigma, \tau)$.

Theorem 3.13. *Let $(\check{Z}, \rho, \varsigma, \tau)$ be NFNS. Then, the set $\Gamma_{st(\rho,\varsigma,\tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)$ of any sequence $\check{z} = (\check{z}_{uv})$ is closed for some $\check{r} > 0$.*

Proof. If $\Gamma_{st(\rho,\varsigma,\tau)}^{\check{r}(\alpha)}(\mathcal{I}_2) = \emptyset$, there is nothing to prove. Thus, assume that $\Gamma_{st(\rho,\varsigma,\tau)}^{\check{r}(\alpha)}(\mathcal{I}_2) \neq \emptyset$. Consider a sequence $\hat{h} = (\hat{h}_{uv}) \subseteq \Gamma_{st(\rho,\varsigma,\tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)$ such that $\hat{h}_{uv} \xrightarrow{(\rho,\varsigma,\tau)} \hat{h}_0$. It is sufficient to prove that $\hat{h}_0 \in \Gamma_{st(\rho,\varsigma,\tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)$. Since $\hat{h}_{uv} \xrightarrow{(\rho,\varsigma,\tau)} \hat{h}_0$, then for any $\lambda > 0$ and $\sigma \in (0, 1)$, there exist $u_\lambda, v_\lambda \in \mathbb{N}$ such that $\rho(\hat{h}_{uv} - \hat{h}_0; \frac{\lambda}{2}) > 1 - \sigma$ and $\varsigma(\hat{h}_{uv} - \hat{h}_0; \frac{\lambda}{2}) < \sigma, \tau(\hat{h}_{uv} - \hat{h}_0; \frac{\lambda}{2}) < \sigma$ for $u \geq u_\lambda$ and $v \geq v_\lambda$. Now, choose $u_0, v_0 \in \mathbb{N}$ satisfying $u_0 \geq u_\lambda$ and $v_0 \geq v_\lambda$. Then, we have $\rho(\hat{h}_{u_0v_0} - \hat{h}_0; \frac{\lambda}{2}) > 1 - \sigma$ and $\varsigma(\hat{h}_{u_0v_0} - \hat{h}_0; \frac{\lambda}{2}) < \sigma, \tau(\hat{h}_{u_0v_0} - \hat{h}_0; \frac{\lambda}{2}) < \sigma$. Again as $\hat{h} = (\hat{h}_{uv}) \subseteq \Gamma_{st(\rho,\varsigma,\tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)$, we have $\hat{h}_{u_0v_0} \in \Gamma_{st(\rho,\varsigma,\tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)$

$$\delta_{\mathcal{I}_2} \left(\left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}) < \sigma \right. \right. \\ \left. \left. \tau(\check{z}_{uv} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}) < \sigma \right\} \right) \neq 0. \tag{3.5}$$

Define the set

$$\mathbb{H} = \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}) < \sigma \right. \\ \left. \tau(\check{z}_{uv} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}) < \sigma \right\}.$$

Choose $k, l \in \mathbb{H}$, then we have

$$\rho(\check{z}_{kl} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}) > 1 - \sigma \text{ and } \varsigma(\check{z}_{kl} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}) < \sigma, \tau(\check{z}_{kl} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}) < \sigma.$$

Now,

$$\rho(\check{z}_{kl} - \hat{h}_0; \check{r} + \frac{\lambda}{2}) \geq \min \left\{ \rho(\check{z}_{kl} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}), \rho(\hat{h}_{u_0v_0} - \hat{h}_0; \check{r} + \frac{\lambda}{2}) \right\} \\ > 1 - \sigma,$$

$$\varsigma(\check{z}_{kl} - \hat{h}_0; \check{r} + \frac{\lambda}{2}) \leq \max \left\{ \varsigma(\check{z}_{kl} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}), \varsigma(\hat{h}_{u_0v_0} - \hat{h}_0; \check{r} + \frac{\lambda}{2}) \right\} \\ < \sigma,$$

and

$$\tau(\check{z}_{kl} - \hat{h}_0; \check{r} + \frac{\lambda}{2}) \leq \max \left\{ \tau(\check{z}_{kl} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}), \tau(\hat{h}_{u_0v_0} - \hat{h}_0; \check{r} + \frac{\lambda}{2}) \right\} \\ < \sigma.$$

Thus

$$(k, l) \in \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \hat{h}_0; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \hat{h}_0; \check{r} + \lambda) < \sigma \right. \\ \left. \tau(\check{z}_{uv} - \hat{h}_0; \check{r} + \lambda) < \sigma \right\}.$$

Hence,

$$\left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}) < \sigma \right. \\ \left. \tau(\check{z}_{uv} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2}) < \sigma \right\} \\ \subseteq \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \hat{h}_0; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \hat{h}_0; \check{r} + \lambda) < \sigma \right. \\ \left. \tau(\check{z}_{uv} - \hat{h}_0; \check{r} + \lambda) < \sigma \right\}.$$

So, we have

$$\begin{aligned} & \delta_{\mathcal{I}_2} \left(\left\{ (u, v) \in \mathbb{N}^2 : \rho \left(\check{z}_{uv} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2} \right) > 1 - \sigma \text{ and } \varsigma \left(\check{z}_{uv} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2} \right) < \sigma \right. \right. \\ & \left. \left. \tau \left(\check{z}_{uv} - \hat{h}_{u_0v_0}; \check{r} + \frac{\lambda}{2} \right) < \sigma \right\} \right) \\ & \leq \delta_{\mathcal{I}_2} \left(\left\{ (u, v) \in \mathbb{N}^2 : \rho \left(\check{z}_{uv} - \hat{h}_0; \check{r} + \lambda \right) > 1 - \sigma \text{ and } \varsigma \left(\check{z}_{uv} - \hat{h}_0; \check{r} + \lambda \right) < \sigma \right. \right. \\ & \left. \left. \tau \left(\check{z}_{uv} - \hat{h}_0; \check{r} + \lambda \right) < \sigma \right\} \right). \end{aligned} \tag{3.6}$$

From (3.5), the set on left side hand of (3.6) has natural density more than zero, which implies that

$$\delta_{\mathcal{I}_2} \left(\left\{ (u, v) \in \mathbb{N}^2 : \rho \left(\check{z}_{uv} - \hat{h}_0; \check{r} + \lambda \right) > 1 - \sigma \text{ and } \varsigma \left(\check{z}_{uv} - \hat{h}_0; \check{r} + \lambda \right) < \sigma \right. \right. \\ \left. \left. \tau \left(\check{z}_{uv} - \hat{h}_0; \check{r} + \lambda \right) < \sigma \right\} \right) \neq 0.$$

Therefore, $\hat{h}_0 \in \Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)$ Thus, the desired result follows. □

Theorem 3.14. *Let $\Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)$ be the set of all \mathcal{I}_2 -statistical cluster points of order α of the sequence $\check{z} = (\check{z}_{uv})$ in NFNS $(\check{Z}, \rho, \varsigma, \tau)$. Then, for any arbitrary $\nu \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)$, $\check{r} > 0$ and $\sigma \in (0, 1)$, we have $\rho(\sigma - \nu; \check{r}) > 1 - \sigma$ and $\varsigma(\sigma - \nu; \check{r}) < \sigma$, $\tau(\sigma - \nu; \check{r}) < \sigma$ for all $\sigma \in \Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)$.*

Proof. Since $\nu \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)$ then for $\lambda > 0$ and $\sigma \in (0, 1)$, we have

$$\delta_{\mathcal{I}_2} \left(\left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \nu; \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \nu; \lambda) < \sigma \right. \right. \\ \left. \left. \tau(\check{z}_{uv} - \nu; \lambda) < \sigma \right\} \right) \neq 0. \tag{3.7}$$

Now, it is enough to demonstrate that if any $\sigma \in \check{Z}$ satisfies $\rho(\sigma - \nu; \lambda) > 1 - \sigma$ and $\varsigma(\sigma - \nu; \lambda) < \sigma$, $\tau(\sigma - \nu; \lambda) < \sigma$ then $\sigma \in \Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)$.

Suppose

$$(k, l) \in \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \nu; \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \nu; \lambda) < \sigma \right. \\ \left. \tau(\check{z}_{uv} - \nu; \lambda) < \sigma \right\}.$$

Now,

$$\rho(\check{z}_{kl} - \sigma; \check{r} + \lambda) \geq \min \{ \rho(\check{z}_{kl} - \nu; \lambda), \rho(\sigma - \nu; \check{r}) \} \\ > 1 - \sigma,$$

$$\varsigma(\check{z}_{kl} - \sigma; \check{r} + \lambda) \leq \max \{ \varsigma(\check{z}_{kl} - \nu; \lambda), \varsigma(\sigma - \nu; \check{r}) \} \\ < \sigma,$$

and

$$\tau(\check{z}_{kl} - \sigma; \check{r} + \lambda) \leq \max \{ \tau(\check{z}_{kl} - \nu; \lambda), \tau(\sigma - \nu; \check{r}) \} \\ < \sigma.$$

So, we have $\rho(\check{z}_{kl} - \sigma; \check{r} + \lambda) > 1 - \sigma$, $\varsigma(\check{z}_{kl} - \sigma; \check{r} + \lambda) < \sigma$ and $\tau(\check{z}_{kl} - \sigma; \check{r} + \lambda) < \sigma$. Thus

$$(k, l) \in \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \sigma; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \sigma; \check{r} + \lambda) < \sigma, \tau(\check{z}_{uv} - \sigma; \check{r} + \lambda) < \sigma \right\}$$

which gives the inclusion

$$\begin{aligned} & \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \nu; \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \nu; \lambda) < \sigma \right. \\ & \left. \tau(\check{z}_{uv} - \nu; \lambda) < \sigma \right\} \\ & \subseteq \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \sigma; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \sigma; \check{r} + \lambda) < \sigma \right. \\ & \left. \tau(\check{z}_{uv} - \sigma; \check{r} + \lambda) < \sigma \right\}. \end{aligned}$$

Then,

$$\begin{aligned} &\delta_{\mathcal{I}_2} (\{(u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \nu; \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \nu; \lambda) < \sigma \\ &\tau(\check{z}_{uv} - \nu; \lambda) < \sigma\}) \\ &\delta_{\mathcal{I}_2} (\{(u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \sigma; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \sigma; \check{r} + \lambda) < \sigma \\ &\tau(\check{z}_{uv} - \sigma; \check{r} + \lambda) < \sigma\}). \end{aligned}$$

Therefore, from (3.7),

$$\delta_{\mathcal{I}_2} (\{(u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \nu; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \nu; \check{r} + \lambda) < \sigma \\ \tau(\check{z}_{uv} - \nu; \check{r} + \lambda) < \sigma\}) \neq 0.$$

Hence, $\sigma \in \Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)$. □

Theorem 3.15. Let $\check{z} = (\check{z}_{uv})$ be a sequence in NFNS $(\check{Z}, \rho, \varsigma, \tau)$ and

$$\overline{B(\varkappa, \sigma, \check{r})} = \{z \in \check{Z} : \rho(z - \varkappa; \check{r}) \geq 1 - \sigma, \varsigma(z - \varkappa; \check{r}) < \sigma \text{ and } \tau(z - \varkappa; \check{r}) < \sigma\},$$

is a closed ball for some $\check{r} > 0$, $\alpha \in (0, 1]$ and $\sigma \in (0, 1)$ and fixed $\varkappa \in \check{Z}$, then

$$\Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_2) = \bigcup_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^{\alpha}(\mathcal{I}_2)} \overline{B(\varkappa, \sigma, \check{r})}.$$

Proof. Let $\sigma \in \bigcup_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^{\alpha}(\mathcal{I}_2)} \overline{B(\varkappa, \sigma, \check{r})}$ then there exists some $\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^{\alpha}(\mathcal{I}_2)$ for $\check{r} > 0$, $\sigma \in (0, 1)$ and $\alpha \in (0, 1]$ such that $\rho(\varkappa - \sigma; \check{r}) > 1 - \sigma$ and $\varsigma(\varkappa - \sigma; \check{r}) < \sigma$, $\tau(\varkappa - \sigma; \check{r}) < \sigma$. As $\sigma \in \Gamma_{st(\rho, \varsigma, \tau)}^{\alpha}(\mathcal{I}_2)$ (\check{z}) then there exists a set

$$\mathbb{Q} = \{(u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \varkappa; \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \varkappa; \lambda) < \sigma \\ \tau(\check{z}_{uv} - \varkappa; \lambda) < \sigma\},$$

with $\delta_{\mathcal{I}_2}(\mathbb{Q}) \neq 0$. For $(u, v) \in \mathbb{Q}$,

$$\rho(\check{z}_{uv} - \sigma; \check{r} + \lambda) \geq \min\{\rho(\check{z}_{uv} - \varkappa; \lambda), \rho(\varkappa - \sigma; \check{r})\} > 1 - \sigma,$$

$$\varsigma(\check{z}_{uv} - \sigma; \check{r} + \lambda) \leq \max\{\varsigma(\check{z}_{uv} - \varkappa; \lambda), \varsigma(\varkappa - \sigma; \check{r})\} < \sigma,$$

and

$$\tau(\check{z}_{uv} - \sigma; \check{r} + \lambda) \leq \max\{\tau(\check{z}_{uv} - \varkappa; \lambda), \tau(\varkappa - \sigma; \check{r})\} < \sigma.$$

This implies

$$\delta_{\mathcal{I}_2} (\{(u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \sigma; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \sigma; \check{r} + \lambda) < \sigma \\ \tau(\check{z}_{uv} - \sigma; \check{r} + \lambda) < \sigma\}) \neq 0.$$

Hence, $\sigma \in \Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)$. So,

$$\bigcup_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^{\alpha}(\mathcal{I}_d)} \overline{B(\varkappa, \sigma, \check{r})} \subseteq \Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_d).$$

Conversely, take $\sigma \in \Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)$ and we want to show that $\sigma \in \bigcup_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^{\alpha}(\mathcal{I}_d)} \overline{B(\varkappa, \sigma, \check{r})}$.

We will prove the result by contadiction, if possible let $\sigma \notin \bigcup_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^{\alpha}(\mathcal{I}_2)} \overline{B(\varkappa, \sigma, \check{r})}$ i.e. $\sigma \notin \overline{B(\varkappa, \sigma, \check{r})}$ for all $\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^{\alpha}(\mathcal{I}_2)$. Then, $\rho(\varkappa - \sigma; \check{r}) \leq 1 - \sigma$ or $\varsigma(\varkappa - \sigma; \check{r}) \geq \sigma$, $\tau(\varkappa - \sigma; \check{r}) \geq \sigma$ for every $\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^{\alpha}(\mathcal{I}_2)$. But by Theorem 3.14, we have $\rho(\varkappa - \sigma; \check{r}) > 1 - \sigma$ and $\varsigma(\varkappa - \sigma; \check{r}) < \sigma$, $\tau(\varkappa - \sigma; \check{r}) < \sigma$ for every $\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^{\alpha}(\mathcal{I}_2)$, which is contadiction to our supposition. Hence, $\sigma \in \bigcup_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^{\alpha}(\mathcal{I}_2)} \overline{B(\varkappa, \sigma, \check{r})}$. This completes the proof. □

Theorem 3.16. Let $\check{z} = (\check{z}_{uv})$ be a sequence in NFNS $(\dot{Z}, \rho, \varsigma, \tau)$. Then, for $\sigma \in (0, 1)$ and $\alpha \in (0, 1]$,

- (i) If $\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)$, then $\mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z} \subseteq \overline{B(\varkappa, \sigma, \check{r})}$.
- (ii)

$$\mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z} = \bigcap_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)} \overline{B(\varkappa, \sigma, \check{r})} = \left\{ \hat{w} \in \dot{Z} : \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2) \subseteq \overline{B(\hat{w}, \sigma, \check{r})} \right\}.$$

Proof. Let $\hat{w} \in \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha$ and $\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)$. For $\lambda > 0$, $\sigma \in (0, 1)$ and $\alpha \in (0, 1]$, consider

$$\mathbb{A} = \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) < \sigma \right. \\ \left. \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) < \sigma \right\},$$

and

$$\mathbb{B} = \left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - \varkappa; \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \varkappa; \lambda) < \sigma \right. \\ \left. \tau(\check{z}_{uv} - \varkappa; \lambda) < \sigma \right\},$$

with $\delta_{\mathcal{I}_2}(\mathbb{A}^c) = 0$ and $\delta_{\mathcal{I}_2}(\mathbb{B}) \neq 0$ respectively.

Now for $(u, v) \in \mathbb{A} \cap \mathbb{B}$,

$$\rho(\hat{w} - \varkappa; \check{r}) \geq \min \{ \rho(\check{z}_{uv} - \varkappa; \lambda), \rho(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \} \\ > 1 - \sigma,$$

$$\varsigma(\hat{w} - \varkappa; \check{r}) \leq \max \{ \varsigma(\check{z}_{uv} - \varkappa; \lambda), \varsigma(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \} \\ < \sigma,$$

and

$$\tau(\hat{w} - \varkappa; \check{r}) \leq \max \{ \tau(\check{z}_{uv} - \varkappa; \lambda), \tau(\check{z}_{uv} - \hat{w}; \check{r} + \lambda) \} \\ < \sigma,$$

which gives $\varkappa \in \overline{B(\varkappa, \sigma, \check{r})}$.

(ii) From (i) part, one can easily see that

$$\mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z} \subseteq \bigcap_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)} \overline{B(\varkappa, \sigma, \check{r})}.$$

Take $z \in \bigcap_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)} \overline{B(\varkappa, \sigma, \check{r})}$. Then, $\rho(z - \varkappa; \check{r}) \geq 1 - \sigma$ and $\varsigma(z - \varkappa; \check{r}) \leq \sigma, \tau(z - \varkappa; \check{r}) \leq \sigma$ for all $\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)$. This implies $\Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)(z) \subseteq \overline{B(z, \sigma, \check{r})}$ i.e.

$$\bigcap_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)} \overline{B(\varkappa, \sigma, \check{r})} \subseteq \left\{ \hat{w} \in \dot{Z} : \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2) \subseteq \overline{B(\hat{w}, \sigma, \check{r})} \right\}.$$

Now suppose $z \notin \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$, then

$$\delta_{\mathcal{I}_2} \left(\left\{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{kl} - z; \check{r} + \lambda) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{kl} - z; \check{r} + \lambda) \geq \sigma \right. \right. \\ \left. \left. \tau(\check{z}_{kl} - z; \check{r} + \lambda) \geq \sigma \right\} \right) \neq 0,$$

for all $\lambda > 0, \sigma \in (0, 1]$. It means there exists some \mathcal{I}_2 -statistical cluster point \varkappa for the sequence $\check{z} = (\check{z}_{uv})$ with

$$\rho(\check{z} - \varkappa; \check{r} + \lambda) \leq 1 - \sigma \text{ or } \varsigma(\check{z} - \varkappa; \check{r} + \lambda) \geq \sigma, \tau(\check{z} - \varkappa; \check{r} + \lambda) \geq \sigma.$$

Thus,

$$\Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2) \not\subseteq \overline{B(z, \sigma, \check{r})}$$

and

$$z \notin \left\{ \hat{w} \in \dot{Z} : \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2) \subseteq \overline{B(\hat{w}, \sigma, \check{r})} \right\}.$$

Hence,

$$\left\{ \hat{w} \in \dot{Z} : \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2) \subseteq \overline{B(\hat{w}, \sigma, \check{r})} \right\} \subseteq \mathcal{I}_2 - st - LIM_{r_s}^\alpha,$$

and

$$\bigcap_{\rho \in \Gamma_{st(\kappa, \Omega)}^\alpha(\mathcal{I}_2)} \overline{B(\rho, \zeta, r_s)} \subseteq \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha.$$

So,

$$\mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z} = \bigcap_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)} \overline{B(\varkappa, \sigma, \check{r})} = \left\{ \hat{w} \in \dot{Z} : \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2) \subseteq \overline{B(\hat{w}, \sigma, \check{r})} \right\}$$

□

Theorem 3.17. *Let $\check{z} = (\check{z}_{uv})$ be a sequence in NFNS $(\dot{Z}, \rho, \varsigma, \tau)$, which is ideal statistically convergent of order α to \varkappa then $\overline{B(\varkappa, \sigma, \check{r})} = \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha$.*

Proof. Since (\check{z}_{uv}) is ideal statistical convergent of order α to \varkappa and \mathcal{I}_2 is admissible so $G = \mathbb{N}^2 \setminus \mathbb{A}$ is a non-empty set, then for $(u, v) \in G^c$,

$$\frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\check{z}_{uv} - \varkappa; \lambda) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv} - \varkappa; \lambda) \geq \sigma, \tau(\check{z}_{uv} - \varkappa; \lambda) \geq \sigma\}| < \psi.$$

$$\Rightarrow \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\check{z}_{uv} - \varkappa; \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \varkappa; \lambda) < \sigma, \tau(\check{z}_{uv} - \varkappa; \lambda) < \sigma\}| \geq 1 - \psi.$$

Put

$$\mathbb{B}_{t,q} = \{u \leq t, v \leq q : \rho(\check{z}_{uv} - \varkappa; \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - \varkappa; \lambda) < \sigma, \tau(\check{z}_{uv} - \varkappa; \lambda) < \sigma\},$$

for all $k \geq u$ and $l \geq v$.

Now for $(k, l) \in \mathbb{B}_{t,q}$, we have $\rho(\check{z}_{kl} - \varkappa; \lambda) > 1 - \sigma$ and $\varsigma(\check{z}_{kl} - \varkappa; \lambda) < \sigma, \tau(\check{z}_{kl} - \varkappa; \lambda) < \sigma$.

Let $z \in \overline{B(\varkappa, \sigma, \check{r})}$. We will prove $z \in \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha z$.

$$\rho(\check{z}_{kl} - z; \check{r} + \lambda) \geq \min \{ \rho(\check{z}_{kl} - \varkappa; \lambda), \rho(z - \varkappa; \check{r}) \} > 1 - \sigma,$$

$$\varsigma(\check{z}_{kl} - z; \check{r} + \lambda) \leq \max \{ \varsigma(\check{z}_{kl} - \varkappa; \lambda), \varsigma(z - \varkappa; \check{r}) \} < \sigma,$$

and

$$\tau(\check{z}_{kl} - z; \check{r} + \lambda) \leq \max \{ \tau(\check{z}_{kl} - \varkappa; \lambda), \tau(z - \varkappa; \check{r}) \} < \sigma.$$

Hence,

$$\mathbb{B}_{t,q} \subseteq \{ (u, v) \in \mathbb{N}^2 : \rho(\check{z}_{uv} - z; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - z; \check{r} + \lambda) < \sigma, \tau(\check{z}_{uv} - z; \check{r} + \lambda) < \sigma \},$$

which implies that

$$1 - \psi \leq \frac{|\mathbb{B}_{t,q}|}{(tq)^\alpha} \leq \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\check{z}_{uv} - z; \check{r} + \lambda) > 1 - \sigma \text{ and } \varsigma(\check{z}_{uv} - z; \check{r} + \lambda) < \sigma, \tau(\check{z}_{uv} - z; \check{r} + \lambda) < \sigma\}|.$$

Therefore,

$$\frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\check{z}_{uv} - z; \check{r} + \lambda) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv} - z; \check{r} + \lambda) \geq \sigma, \tau(\check{z}_{uv} - z; \check{r} + \lambda) \geq \sigma\}| < 1 - (1 - \psi) = \psi.$$

Then,

$$\left\{ (t, q) \in \mathbb{N}^2 : \frac{1}{(tq)^\alpha} |\{u \leq t, v \leq q : \rho(\check{z}_{uv} - z; \check{r} + \lambda) \leq 1 - \sigma \text{ or } \varsigma(\check{z}_{uv} - z; \check{r} + \lambda) \geq \sigma, \tau(\check{z}_{uv} - z; \check{r} + \lambda) \geq \sigma\}| \geq \psi \right\} \subset \mathbb{A} \in \mathcal{I}_2,$$

which gives that $z \in \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$ in $(\check{Z}, \rho, \varsigma, \tau)$. Hence,

$$\overline{B(\varkappa, \sigma, \check{r})} \subseteq \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha.$$

Also,

$$\mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \subseteq \overline{B(\varkappa, \sigma, \check{r})}.$$

Therefore,

$$\overline{B(\varkappa, \sigma, \check{r})} = \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha.$$

□

Theorem 3.18. *Let $\check{z} = (\check{z}_{uv})$ be a sequence in NFNS $(\check{Z}, \rho, \varsigma, \tau)$, which is ideal statistically convergent of order α to \hat{w} if and only if $\Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_2) = \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha$.*

Proof. Firstly, suppose that $\check{z}_{uv} \xrightarrow{\mathcal{I}_2 - st(\rho, \varsigma, \tau)} \hat{w}$, which gives $\Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_2)(\check{z}) = \{\hat{w}\}$. Then, for any $\lambda > 0, \sigma \in (0, 1)$ and $\alpha \in (0, 1]$ by Theorem 3.15, we have $\Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_2) = \overline{B(\hat{w}, \sigma, \check{r})}$. Also from Theorem 3.17, $\overline{B(\hat{w}, \sigma, \check{r})} = \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$. Hence, $\Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_2) = \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$.

Conversely, assume $\Gamma_{st(\rho, \varsigma, \tau)}^{\check{r}(\alpha)}(\mathcal{I}_2) = \mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z}$, then by Theorem 3.15 and 3.16(ii),

$$\bigcap_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)} \overline{B(\varkappa, \sigma, \check{r})} = \bigcup_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)} \overline{B(\varkappa, \sigma, \check{r})}.$$

This is possible only if either $\Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2) = \emptyset$ or $\Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)$ is a singleton set. Then,

$$\mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z} = \bigcap_{\varkappa \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)} \overline{B(\varkappa, \sigma, \check{r})} = \overline{B(\hat{w}, \sigma, \check{r})}$$

for some $\hat{w} \in \Gamma_{st(\rho, \varsigma, \tau)}^\alpha(\mathcal{I}_2)$. Also, by Theorem 3.15, $\mathcal{I}_2 - st - LIM_{\check{r}}^\alpha \check{z} = \hat{w}$. □

4 Conclusion and future developments

In this study, we introduced the concept of rough ideal statistical convergence of order α ($0 < \alpha < 1$) for double sequences in NFNS using a compact linear operator. We defined and examined several related notions, including rough \mathcal{I}_2 -statistical boundedness, rough \mathcal{I}_2 -statistical cluster points, and rough \mathcal{I}_2 -statistical limit points. Various properties and relationships among these concepts were established, offering new insights into convergence behavior in NFNS. Research on sequence convergence in NFNS is still at an early stage, with limited progress so far. Building on the insights from this study, future work could extend these concepts to include compact operators and further investigate their connections with double sequences in neutrosophic n -normed linear spaces. Moreover, these ideas can be applied to convergence-related problems in various fields of science and engineering, offering valuable insights and potential solutions. Further studies may also explore the implications of these results in different types of fuzzy spaces and examine broader applications in areas such as mathematical analysis and approximation theory.

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