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A characterization of Poisson structures compatible with the canonical metric on $\mathbb{R}^4$

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Abstract *The object of the present paper is to construct a characterization of a four-dimensional Poisson manifold with canonical metric.*

1 Introduction

Poisson manifolds are nowadays an important field of research in expansion, in differential geometry. Poisson manifolds were introduce by A. Lichnerowhich, in [4], but the notion of Poisson bracket were first introduced by S. D. Poisson in [5]. They have their application in mechanics in her Hamiltonnian formulation and in quantization because they are used to quantify physical system. This motived several study of such manifold (see [7]).

A Poisson structure on a smooth manifold M is the data of a Lie bracket $\{\cdot,\cdot\}$ satisfying the Jacoby identity. It's also equivalent to the date of a bivector π satisfying some condition. This bivector is called Poison tensor.

The notion of Riemann Poisson manifold were first introduced by I. Vaisman ([6]) and M. Boucetta characterised the compatibility between Poisson and Riemannian structures ([1, 2, 3]). Hence, in [2], the characterization of Poisson tensor which are compatible with the canonical metric of a three-dimensional manifold were done.

In this paper, we characterize the Poisson structures compatible with the canonical metric of $\mathbb{R}^4$, this compatibility (condition) will be defined in section 2. The aim of this work is to continue the work of M. Boucetta [2].

The present paper is organized as follows:
In section 2 we recall some preliminary results. Section 3 contains a characterization of three and four-dimensional Poisson manifolds.

2 Background on Riemann-Poisson manifolds

In this section, we recall some basics notions on Poisson manifolds taken from the book [6]. We begin with some Poisson structure on a differential manifold. Let M be a differentiable manifold with a pseudo-Riemannian metric g and a bivector field π. To the field of bivectors π is associated, in a natural way, a field of endomorphisms $\sharp_\pi : T^*M \to TM$ by $\beta(\sharp_\pi(\alpha)) = \pi(\alpha,\beta)$ and on the space of differential 1-forms $\Omega^1(M)$, the Poisson tensor induces a Lie bracket defined by (see [6]):

$$[\alpha,\beta]_\pi = \mathcal{L}_{\sharp(\alpha)}\beta - \mathcal{L}_{\sharp(\beta)}\alpha - d\Big(\pi(\alpha,\beta)\Big). \tag{2.1}$$

And the Shouten-Nihenjuis's bracket $[\pi,\pi]$ defined by

$$[\pi,\pi](\alpha,\beta,\gamma) = \gamma\Big(\sharp_\pi([\alpha,\beta]_\pi) - [\sharp_\pi\alpha,\sharp_\pi\beta]_\pi\Big). \tag{2.2}$$

The bivector field π defines a Poisson structure on M if and only if $[\pi,\pi]=0$ (see [6, Proposition 1.4]). In this case,the equation (2.2) implies that $\sharp_\pi$ is an isometry. We denote by g^* the cometric defined on T^*M. The metric contravariant connection associated to (π, g^*) is the unique contravariant connection $\mathcal{D}$ with respect to π such that, for any $\alpha,\beta,\gamma\in\Omega^1(M)$:

1) *the metric g^* is parallel with respect to $\mathcal{D}$ or $\mathcal{D}$ is compatible with g^*, i.e.*

$$\sharp_\pi(\alpha)g^*(\beta,\gamma)=g^*(\mathcal{D}_\alpha\beta,\gamma)+g^*(\beta,\mathcal{D}_\alpha\gamma),$$

2) *$\mathcal{D}$ is torsion-free, i.e $\mathcal{D}_\alpha\beta-\mathcal{D}_\beta\alpha=[\alpha,\beta]$.*

With above assertions, the triple (M,π,g^*) is called a Riemann–Poisson manifold if $\mathcal{D}_\pi=0$.

3 Main results

Let (M,g) be a 4-dimension Riemannian manifold endowed with her canonical metric g:

$$g=dx^2+dy^2+dz^2+dt^2. \tag{3.1}$$

Let π be a Poisson tensor given in local cordinate (x,y,z,t) by:

$$\pi=\pi_{12}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\pi_{13}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}+\pi_{14}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}+\pi_{23}\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}+\pi_{24}\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}+\pi_{34}\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}.$$

where

$$\begin{aligned}&\pi_{12}=\pi\left(dx,dy\right),\ \ \pi_{13}=\pi\left(dx,dz\right),\ \ \pi_{14}=\pi\left(dx,dt\right),\\&\pi_{23}=\pi\left(dy,dz\right),\ \ \pi_{24}=\pi\left(dy,dt\right),\ \ \pi_{34}=\pi\left(dz,dt\right)\end{aligned}$$

are differentiable functions on $\mathbb{R}^4$. Therefore we have

$$\begin{cases}\sharp_\pi\left(dx\right)&=\pi_{12}\frac{\partial}{\partial y}+\pi_{13}\frac{\partial}{\partial z}+\pi_{14}\frac{\partial}{\partial t},\ \ \sharp_\pi\left(dy\right)=-\pi_{12}\frac{\partial}{\partial x}+\pi_{23}\frac{\partial}{\partial z}+\pi_{24}\frac{\partial}{\partial t}\\ \sharp_\pi\left(dz\right)&=-\pi_{13}\frac{\partial}{\partial x}-\pi_{23}\frac{\partial}{\partial y}+\pi_{34}\frac{\partial}{\partial t},\ \ \sharp_\pi\left(dt\right)=-\pi_{14}\frac{\partial}{\partial x}-\pi_{24}\frac{\partial}{\partial y}-\pi_{34}\frac{\partial}{\partial z}\end{cases}$$

and the bracket $[.,.]_\pi$ is defined by

$$\begin{cases}[dx,dy]_\pi&=d\left(\pi\left(dx,dy\right)\right)=d\pi_{12}=\frac{\partial\pi_{12}}{\partial x}dx+\frac{\partial\pi_{12}}{\partial y}dy+\frac{\partial\pi_{12}}{\partial z}dz+\frac{\partial\pi_{12}}{\partial t}dt\\ [dx,dz]_\pi&=d\left(\pi\left(dx,dz\right)\right)=d\pi_{13}=\frac{\partial\pi_{13}}{\partial x}dx+\frac{\partial\pi_{13}}{\partial y}dy+\frac{\partial\pi_{13}}{\partial z}dz+\frac{\partial\pi_{13}}{\partial t}dt\\ [dx,dt]_\pi&=d\left(\pi\left(dx,dt\right)\right)=d\pi_{14}=\frac{\partial\pi_{14}}{\partial x}dx+\frac{\partial\pi_{14}}{\partial y}dy+\frac{\partial\pi_{14}}{\partial z}dz+\frac{\partial\pi_{14}}{\partial t}dt\\ [dy,dz]_\pi&=d\left(\pi\left(dy,dz\right)\right)=d\pi_{23}=\frac{\partial\pi_{23}}{\partial x}dx+\frac{\partial\pi_{23}}{\partial y}dy+\frac{\partial\pi_{23}}{\partial z}dz+\frac{\partial\pi_{23}}{\partial t}dt\\ [dy,dt]_\pi&=d\left(\pi\left(dy,dt\right)\right)=d\pi_{24}=\frac{\partial\pi_{24}}{\partial x}dx+\frac{\partial\pi_{24}}{\partial y}dy+\frac{\partial\pi_{24}}{\partial z}dz+\frac{\partial\pi_{24}}{\partial t}dt\\ [dz,dt]_\pi&=d\left(\pi\left(dz,dt\right)\right)=d\pi_{34}=\frac{\partial\pi_{34}}{\partial x}dx+\frac{\partial\pi_{34}}{\partial y}dy+\frac{\partial\pi_{34}}{\partial z}dz+\frac{\partial\pi_{34}}{\partial t}dt.\end{cases} \tag{3.2}$$

Let g^* be the metric on the cotangent bundle defined by $g^*(\alpha,\beta)=g(\sharp_\pi\alpha,\sharp_\pi\beta)$. Let $\mathcal{D}$ be the contravariante Levi-Civita connection associated to g^*. It's caracterised by the Koszul type formula:

$$\begin{aligned}2g^*(\mathcal{D}_\alpha\beta,\gamma)=&\sharp_\pi(\alpha)g^*(\beta,\gamma)+\sharp_\pi(\beta)g^*(\alpha,\gamma)-\sharp_\pi(\gamma)g^*(\alpha,\beta)\\&+g^*([\alpha,\beta]_\pi,\gamma)+g^*([\gamma,\alpha]_\pi,\beta)+g^*([\gamma,\beta]_\pi,\alpha).\end{aligned} \tag{3.3}$$

The non vanishing Christoffel's coefficiants of $\mathcal{D}$ are given by:

$$\begin{gathered}
\Gamma_{12}^{1}=-\Gamma_{11}^{2}=\frac{\partial\pi_{12}}{\partial x},\quad \Gamma_{13}^{1}=-\Gamma_{11}^{3}=\frac{\partial\pi_{13}}{\partial x},\quad \Gamma_{14}^{1}=-\Gamma_{11}^{4}=\frac{\partial\pi_{14}}{\partial x}\\
\Gamma_{22}^{1}=-\Gamma_{21}^{2}=\frac{\partial\pi_{12}}{\partial y},\quad \Gamma_{23}^{2}=-\Gamma_{22}^{3}=\frac{\partial\pi_{23}}{\partial y},\quad \Gamma_{24}^{2}=-\Gamma_{22}^{4}=-\frac{\partial\pi_{24}}{\partial y}\\
\Gamma_{33}^{1}=-\Gamma_{31}^{3}=\frac{\partial\pi_{13}}{\partial z},\quad \Gamma_{44}^{1}=-\Gamma_{41}^{4}=\frac{\partial\pi_{14}}{\partial t},\quad \Gamma_{33}^{2}=-\Gamma_{32}^{3}=\frac{\partial\pi_{23}}{\partial z}\\
\Gamma_{44}^{2}=-\Gamma_{42}^{4}=\frac{\partial\pi_{24}}{\partial t},\quad \Gamma_{34}^{3}=-\Gamma_{33}^{4}=\frac{\partial\pi_{34}}{\partial z},\quad \Gamma_{44}^{3}=-\Gamma_{43}^{4}=\frac{\partial\pi_{34}}{\partial t},
\end{gathered}$$

$$\begin{aligned}
\Gamma_{23}^{1}&=-\Gamma_{21}^{3}=\tfrac{1}{2}\left(\tfrac{\partial\pi_{23}}{\partial x}+\tfrac{\partial\pi_{13}}{\partial y}+\tfrac{\partial\pi_{12}}{\partial z}\right),\quad \Gamma_{24}^{1}=-\Gamma_{21}^{4}=\tfrac{1}{2}\left(\tfrac{\partial\pi_{24}}{\partial x}+\tfrac{\partial\pi_{14}}{\partial y}+\tfrac{\partial\pi_{12}}{\partial t}\right)\\
\Gamma_{32}^{2}&=-\Gamma_{32}^{1}=\tfrac{1}{2}\left(\tfrac{\partial\pi_{23}}{\partial x}-\tfrac{\partial\pi_{13}}{\partial y}-\tfrac{\partial\pi_{12}}{\partial z}\right),\quad \Gamma_{34}^{1}=-\Gamma_{31}^{4}=\tfrac{1}{2}\left(\tfrac{\partial\pi_{34}}{\partial x}+\tfrac{\partial\pi_{14}}{\partial z}+\tfrac{\partial\pi_{13}}{\partial t}\right)\\
\Gamma_{41}^{2}&=-\Gamma_{42}^{1}=\tfrac{1}{2}\left(\tfrac{\partial\pi_{24}}{\partial x}-\tfrac{\partial\pi_{14}}{\partial y}-\tfrac{\partial\pi_{12}}{\partial t}\right),\quad \Gamma_{41}^{3}=-\Gamma_{43}^{1}=\tfrac{1}{2}\left(\tfrac{\partial\pi_{34}}{\partial x}-\tfrac{\partial\pi_{14}}{\partial z}-\tfrac{\partial\pi_{13}}{\partial t}\right)\\
\Gamma_{12}^{3}&=-\Gamma_{13}^{2}=\tfrac{1}{2}\left(-\tfrac{\partial\pi_{23}}{\partial x}-\tfrac{\partial\pi_{13}}{\partial y}+\tfrac{\partial\pi_{12}}{\partial z}\right),\quad \Gamma_{14}^{2}=-\Gamma_{12}^{4}=\tfrac{1}{2}\left(\tfrac{\partial\pi_{24}}{\partial x}+\tfrac{\partial\pi_{14}}{\partial y}-\tfrac{\partial\pi_{12}}{\partial t}\right)\\
\Gamma_{34}^{2}&=-\Gamma_{32}^{4}=\tfrac{1}{2}\left(\tfrac{\partial\pi_{34}}{\partial y}+\tfrac{\partial\pi_{24}}{\partial z}+\tfrac{\partial\pi_{23}}{\partial t}\right),\quad \Gamma_{42}^{3}=-\Gamma_{43}^{2}=\tfrac{1}{2}\left(\tfrac{\partial\pi_{34}}{\partial y}-\tfrac{\partial\pi_{24}}{\partial z}-\tfrac{\partial\pi_{23}}{\partial t}\right)\\
\Gamma_{14}^{3}&=-\Gamma_{13}^{4}=\tfrac{1}{2}\left(\tfrac{\partial\pi_{34}}{\partial x}+\tfrac{\partial\pi_{14}}{\partial z}-\tfrac{\partial\pi_{13}}{\partial t}\right),\quad \Gamma_{24}^{3}=-\Gamma_{23}^{4}=\tfrac{1}{2}\left(\tfrac{\partial\pi_{34}}{\partial y}+\tfrac{\partial\pi_{24}}{\partial z}-\tfrac{\partial\pi_{23}}{\partial t}\right).
\end{aligned}$$

To introduce the conditions of compatibility for triplet $\left(\mathbb{R}^4, g^*, \pi\right)$ we need the following lemma which gives us a necessary condition so that $\left(\mathbb{R}^3, g^*, \pi\right)$ is a Riemann-Poisson manifold using divergence. This is a verification test of our work out method.

Lemma 3.1. *If the bivector fieldr π is compatible with g^* then there exists a differential function μ on $\mathbb{R}^3$ such that*

$$\pi=-\left(\frac{\partial\mu}{\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\left(\frac{\partial\mu}{\partial y}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}-\left(\frac{\partial\mu}{\partial x}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}. \tag{3.4}$$

Proof. According to the work of M.Boucetta [2], the compatibility condition leads to the application being $C^\infty\left(\mathbb{R}^3\right)$-bilinear

$$(\alpha,\beta)\longmapsto \mathcal{L}_{H_h}g^*(\alpha,\beta)=g^*(\mathcal{D}_\alpha dh,\beta)+g^*(\alpha,\mathcal{D}_\beta dh)$$

is anti-symmetric for any differentiable function h on $\mathbb{R}^3$, consequently it has zero trace; but its trace in the canonical base is equal to

$$\phi(h):=g^*(\mathcal{D}_{dx}dh,dx)+g^*(\mathcal{D}_{dy}dh,dy)+g^*(\mathcal{D}_{dz}dh,dz)\ \ \forall\ h\in C^\infty\left(\mathbb{R}^3\right). \tag{3.5}$$

We calculate $\phi(h)$, as

$$\begin{aligned}
g^*(\mathcal{D}_{dx}dh,dx)&=\frac{\partial h}{\partial x}g^*(\mathcal{D}_{dx}dx,dx)+\frac{\partial h}{\partial y}g^*(\mathcal{D}_{dx}dy,dx)+\frac{\partial h}{\partial z}g^*(\mathcal{D}_{dx}dz,dx)+\sharp_\pi(dx)\cdot\left(\frac{\partial h}{\partial x}\right)\\
&=\frac{\partial h}{\partial x}\cdot\Gamma_1^{11}+\frac{\partial h}{\partial y}\cdot\Gamma_1^{12}+\frac{\partial h}{\partial z}\cdot\Gamma_1^{13}+\left(\frac{\partial\pi_{12}}{\partial y}+\frac{\partial\pi_{13}}{\partial z}\right)\left(\frac{\partial h}{\partial x}\right)\\
&=\frac{\partial h}{\partial y}\cdot\frac{\partial\pi_{12}}{\partial x}+\frac{\partial h}{\partial z}\cdot\frac{\partial\pi_{13}}{\partial x}+\pi_{12}\frac{\partial^2 h}{\partial y\partial x}+\pi_{13}\frac{\partial^2 h}{\partial z\partial x}.
\end{aligned}$$

So

$$\begin{aligned}
g^*(\mathcal{D}_{dy}dh,dy)&=\frac{\partial h}{\partial x}\cdot\Gamma_2^{21}+\frac{\partial h}{\partial y}\cdot\Gamma_2^{22}+\frac{\partial h}{\partial z}\cdot\Gamma_2^{23}+\left(-\frac{\partial\pi_{12}}{\partial x}+\frac{\partial\pi_{23}}{\partial z}\right)\left(\frac{\partial h}{\partial y}\right)\\
&=-\frac{\partial h}{\partial x}\cdot\frac{\partial\pi_{12}}{\partial y}+\frac{\partial h}{\partial z}\cdot\frac{\partial\pi_{23}}{\partial y}-\pi_{12}\frac{\partial^2 h}{\partial x\partial y}+\pi_{23}\frac{\partial^2 h}{\partial z\partial y}
\end{aligned}$$

and

$$g^*\left(\mathcal{D}_{dz}dh, dz\right) = \frac{\partial h}{\partial x}\cdot\Gamma_3^{31} + \frac{\partial h}{\partial y}\cdot\Gamma_3^{32} + \frac{\partial h}{\partial z}\cdot\Gamma_3^{33} + \left(-\frac{\partial \pi_{13}}{\partial x} - \frac{\partial \pi_{23}}{\partial y}\right)\left(\frac{\partial h}{\partial z}\right)$$
$$= -\frac{\partial h}{\partial x}\cdot\frac{\partial \pi_{13}}{\partial z} - \frac{\partial h}{\partial y}\cdot\frac{\partial \pi_{23}}{\partial z} - \pi_{13}\frac{\partial^2 h}{\partial x\partial z} - \pi_{23}\frac{\partial^2 h}{\partial y\partial z}.$$

Therefore

$$\phi(h) = \frac{\partial h}{\partial y}\cdot\frac{\partial \pi_{12}}{\partial x} + \frac{\partial h}{\partial z}\cdot\frac{\partial \pi_{13}}{\partial x} - \frac{\partial h}{\partial x}\cdot\frac{\partial \pi_{12}}{\partial y} + \frac{\partial h}{\partial z}\cdot\frac{\partial \pi_{23}}{\partial y} - \frac{\partial h}{\partial x}\cdot\frac{\partial \pi_{13}}{\partial z}$$
$$= \left(-\frac{\partial \pi_{13}}{\partial z} - \frac{\partial \pi_{12}}{\partial y}\right)\frac{\partial h}{\partial x} + \left(\frac{\partial \pi_{12}}{\partial x} - \frac{\partial \pi_{23}}{\partial z}\right)\frac{\partial h}{\partial y} + \left(\frac{\partial \pi_{23}}{\partial y} + \frac{\partial \pi_{13}}{\partial x}\right)\frac{\partial h}{\partial z}$$
$$= \left\{\left(-\frac{\partial \pi_{13}}{\partial z} - \frac{\partial \pi_{12}}{\partial y}\right)\frac{\partial}{\partial x} + \left(\frac{\partial \pi_{12}}{\partial x} - \frac{\partial \pi_{23}}{\partial z}\right)\frac{\partial}{\partial y} + \left(\frac{\partial \pi_{23}}{\partial y} + \frac{\partial \pi_{13}}{\partial x}\right)\frac{\partial}{\partial z}\right\}\cdot h;$$

since $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$ is a basis of $C^\infty\left(\mathbb{R}^3\right)$-free module of $\mathcal{A}\left(\mathbb{R}^3\right)$. Then $\phi(h) = 0\ \forall\ h \in C^\infty\left(\mathbb{R}^3\right)$ is equivalent to the following system:

$$\left\{\begin{array}{rcl} -\frac{\partial \pi_{13}}{\partial z} - \frac{\partial \pi_{12}}{\partial y} & = & 0 \\ \frac{\partial \pi_{12}}{\partial x} - \frac{\partial \pi_{23}}{\partial z} & = & 0 \\ \frac{\partial \pi_{23}}{\partial y} + \frac{\partial \pi_{13}}{\partial x} & = & 0, \end{array}\right.$$

which is equivalent to:

$$\frac{\partial}{\partial x}\left(\pi_{12} + \pi_{13}\right) + \frac{\partial}{\partial y}\left(-\pi_{12} + \pi_{23}\right) + \frac{\partial}{\partial z}\left(-\pi_{13} - \pi_{23}\right) = 0.$$

By putting

$$a = -\pi_{13} - \pi_{23},\quad b = -\pi_{12} + \pi_{23},\quad c = \pi_{12} + \pi_{13},$$

$$\frac{\partial}{\partial x}c + \frac{\partial}{\partial y}b + \frac{\partial}{\partial z}a = 0 \quad \Leftrightarrow div\cdot v = 0 \text{ où } v\left(c, b, a\right),$$

the 2-form $\alpha = adx\wedge dy + bdx\wedge dz + cdy\wedge dz$ on $\mathbb{R}^3$ is closed. Since $\mathbb{R}^3$ is simply connected, then α is exact, there are three differentiable functions η, μ, ν and a 1-form $\beta = \eta dx + \mu dy + \nu dz$ on $\mathbb{R}^3$ such that $\alpha = d\beta$

$$d\beta = \left(\frac{\partial \mu}{\partial x} - \frac{\partial \eta}{\partial y}\right)dx\wedge dy + \left(\frac{\partial \nu}{\partial x} - \frac{\partial \eta}{\partial z}\right)dx\wedge dz + \left(\frac{\partial \eta}{\partial y} - \frac{\partial \mu}{\partial z}\right)dy\wedge dz \Rightarrow$$

$$\left\{\begin{array}{ll} a & = -\pi_{13} - \pi_{23} = \frac{\partial \mu}{\partial x} - \frac{\partial \eta}{\partial y} \\ b & = -\pi_{12} + \pi_{23} = \frac{\partial \eta}{\partial z} - \frac{\partial \nu}{\partial x} \\ c & = \pi_{12} + \pi_{13} = \frac{\partial \nu}{\partial y} - \frac{\partial \mu}{\partial z} \end{array}\right. \Rightarrow \quad a + b + c = 0 \tag{3.6}$$

which is equivalent to

$$\frac{\partial}{\partial x}\left(\mu - \nu\right) + \frac{\partial}{\partial y}\left(\nu - \eta\right) + \frac{\partial}{\partial z}\left(\eta - \mu\right) = 0,$$

which is also equivalent to

$$\begin{cases} \eta & = \nu \\ \eta & = \mu \\ \mu & = \nu \end{cases} \Leftrightarrow \quad \eta = \mu = \nu.$$

Therefore the system (3.6) becomes

$$\begin{cases} a = -\pi_{13} - \pi_{23} & = & \dfrac{\partial \mu}{\partial x} - \dfrac{\partial \mu}{\partial y} \\ b = -\pi_{12} + \pi_{23} & = & \dfrac{\partial \mu}{\partial z} - \dfrac{\partial \mu}{\partial x} \\ c = \pi_{12} + \pi_{13} & = & \dfrac{\partial \mu}{\partial y} - \dfrac{\partial \mu}{\partial z}. \end{cases}$$

By identification we obtain

$$\pi_{12} = -\frac{\partial \mu}{\partial z}, \quad \pi_{13} = \frac{\partial \mu}{\partial y}, \quad \pi_{23} = -\frac{\partial \mu}{\partial x}.$$

Which gives the desired result. □

Remark 3.2. By setting $f = -\mu$ we find the results of M. Boucetta [2].

The following Proposition 3.3 follows from the lemma 3.1, give us a necessary condition of the compatibility between the Poisson structure π and the canonical Riemannian metric g^*.

Proposition 3.3. *If the bivector field π is compatible with g^* then there exist six differential functions $f_1, f_2, f_3, f_4, f_5, f_6$ on $\mathbb{R}^4$ satisfying the following system*

$$\begin{cases} f_6 & = f_5 - f_4 \\ f_6 & = f_3 - f_2 \\ f_1 & = f_3 - f_5 \\ f_1 & = f_2 - f_4 \end{cases} \tag{3.7}$$

such as

$$\pi = \left(-\frac{\partial f_2}{\partial z} + \frac{\partial f_2}{\partial t}\right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_2}{\partial y}\right) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t}, \tag{3.8}$$

or else

$$\pi = \left(-\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial t} + \left(\frac{\partial f_1}{\partial x} - \frac{\partial f_1}{\partial t}\right) \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}, \tag{3.9}$$

or else

$$\pi = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_3}{\partial t}\right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \left(-\frac{\partial f_3}{\partial x} + \frac{\partial f_3}{\partial z}\right) \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial t}. \tag{3.10}$$

In particular if the bivector field π is compatible with g^* then there exist six differential functions $f_1, f_2, f_3, f_4, f_5, f_6$ on $\mathbb{R}^4$ satisfying the following qualities

$$\begin{aligned} \pi_{12} &= \frac{\partial f_6}{\partial x} = \frac{\partial f_6}{\partial y}, \quad \pi_{13} = -\frac{\partial f_5}{\partial x} = -\frac{\partial f_5}{\partial z}, \quad \pi_{14} = \frac{\partial f_4}{\partial x} = \frac{\partial f_4}{\partial t}, \\ \pi_{23} &= \frac{\partial f_3}{\partial y} = \frac{\partial f_3}{\partial z}, \quad \pi_{24} = -\frac{\partial f_2}{\partial y} = -\frac{\partial f_2}{\partial t}, \quad \pi_{34} = \frac{\partial f_1}{\partial z} = \frac{\partial f_1}{\partial t}, \end{aligned} \tag{3.11}$$

such that the bivector field π can be written in the following form:

$$\begin{aligned} \pi = &\left(\frac{\partial f_6}{\partial y}\right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - \left(\frac{\partial f_5}{\partial z}\right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \left(\frac{\partial f_4}{\partial t}\right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial t} \\ &+ \left(\frac{\partial f_3}{\partial z}\right) \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} - \left(\frac{\partial f_2}{\partial t}\right) \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial t} + \left(\frac{\partial f_1}{\partial t}\right) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t}. \end{aligned} \tag{3.12}$$

Proof. The compatiblity of the Riemann metric g^* and the Poisson structure π on M implies the vanishing of the modular vector field which held to:

$$\phi_{g^*}(f) = \sum_{i=1}^{n} g^*(\mathcal{D}_{\alpha_i} df, \alpha_i) = 0,$$

where $\{\alpha_1, \ldots, \alpha_n\}$ is a local basis of the space of 1-forms, and $f \in \mathcal{C}^\infty(M)$.

In our case we have :

$$\phi_{g^*}(f) = 0 \;\Leftrightarrow\; g^*(\mathcal{D}_{dx} df, dx) + g^*(\mathcal{D}_{dy} df, dy) + g^*(\mathcal{D}_{dz} df, dz) + g^*(\mathcal{D}_{dt} df, dt) = 0.$$

Since,

$$\begin{aligned}
\mathcal{D}_{dx} df \;=\;& \Big(\Gamma^1_{11}\frac{\partial f}{\partial x} + \Gamma^1_{12}\frac{\partial f}{\partial y} + \Gamma^1_{13}\frac{\partial f}{\partial z} + \Gamma^1_{14}\frac{\partial f}{\partial t} + \sharp(dx)(\frac{\partial f}{\partial x})\Big) dx \\
&+\Big(\Gamma^2_{11}\frac{\partial f}{\partial x} + \Gamma^2_{12}\frac{\partial f}{\partial y} + \Gamma^2_{13}\frac{\partial f}{\partial z} + \Gamma^2_{14}\frac{\partial f}{\partial t} + \sharp(dx)(\frac{\partial f}{\partial y})\Big) dy \\
&+\Big(\Gamma^3_{11}\frac{\partial f}{\partial x} + \Gamma^3_{12}\frac{\partial f}{\partial y} + \Gamma^3_{13}\frac{\partial f}{\partial z} + \Gamma^3_{14}\frac{\partial f}{\partial t} + \sharp(dx)(\frac{\partial f}{\partial z})\Big) dz \\
&+\Big(\Gamma^4_{11}\frac{\partial f}{\partial x} + \Gamma^4_{12}\frac{\partial f}{\partial y} + \Gamma^4_{13}\frac{\partial f}{\partial z} + \Gamma^4_{14}\frac{\partial f}{\partial t} + \sharp(dx)(\frac{\partial f}{\partial t})\Big) dt,
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_{dy} df \;=\;& \Big(\Gamma^1_{21}\frac{\partial f}{\partial x} + \Gamma^1_{22}\frac{\partial f}{\partial y} + \Gamma^1_{23}\frac{\partial f}{\partial z} + \Gamma^1_{24}\frac{\partial f}{\partial t} + \sharp(dy)(\frac{\partial f}{\partial x})\Big) dx \\
&+\Big(\Gamma^2_{21}\frac{\partial f}{\partial x} + \Gamma^2_{22}\frac{\partial f}{\partial y} + \Gamma^2_{23}\frac{\partial f}{\partial z} + \Gamma^2_{24}\frac{\partial f}{\partial t} + \sharp(dy)(\frac{\partial f}{\partial y})\Big) dy \\
&+\Big(\Gamma^3_{21}\frac{\partial f}{\partial x} + \Gamma^3_{22}\frac{\partial f}{\partial y} + \Gamma^3_{23}\frac{\partial f}{\partial z} + \Gamma^3_{24}\frac{\partial f}{\partial t} + \sharp(dy)(\frac{\partial f}{\partial z})\Big) dz \\
&+\Big(\Gamma^4_{21}\frac{\partial f}{\partial x} + \Gamma^4_{22}\frac{\partial f}{\partial y} + \Gamma^4_{23}\frac{\partial f}{\partial z} + \Gamma^4_{24}\frac{\partial f}{\partial t} + \sharp(dy)(\frac{\partial f}{\partial t})\Big) dt,
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_{dz} df \;=\;& \Big(\Gamma^1_{31}\frac{\partial f}{\partial x} + \Gamma^1_{32}\frac{\partial f}{\partial y} + \Gamma^1_{33}\frac{\partial f}{\partial z} + \Gamma^1_{34}\frac{\partial f}{\partial t} + \sharp(dz)(\frac{\partial f}{\partial x})\Big) dx \\
&+\Big(\Gamma^2_{31}\frac{\partial f}{\partial x} + \Gamma^2_{32}\frac{\partial f}{\partial y} + \Gamma^2_{33}\frac{\partial f}{\partial z} + \Gamma^2_{34}\frac{\partial f}{\partial t} + \sharp(dz)(\frac{\partial f}{\partial y})\Big) dy \\
&+\Big(\Gamma^3_{31}\frac{\partial f}{\partial x} + \Gamma^3_{32}\frac{\partial f}{\partial y} + \Gamma^3_{33}\frac{\partial f}{\partial z} + \Gamma^3_{34}\frac{\partial f}{\partial t} + \sharp(dz)(\frac{\partial f}{\partial z})\Big) dz \\
&+\Big(\Gamma^4_{31}\frac{\partial f}{\partial x}\Gamma^4_{32}\frac{\partial f}{\partial y} + \Gamma^4_{33}\frac{\partial f}{\partial y} + \Gamma^4_{34}\frac{\partial f}{\partial t} + \sharp(dz)(\frac{\partial f}{\partial t})\Big) dt,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}_{dt} df \;=\;& \Big(\Gamma^1_{41}\frac{\partial f}{\partial x} + \Gamma^1_{42}\frac{\partial f}{\partial y} + \Gamma^1_{43}\frac{\partial f}{\partial z} + \Gamma^1_{44}\frac{\partial f}{\partial t} + \sharp(dt)(\frac{\partial f}{\partial x})\Big) dx \\
&+\Big(\Gamma^2_{41}\frac{\partial f}{\partial x} + \Gamma^2_{42}\frac{\partial f}{\partial y} + \Gamma^2_{43}\frac{\partial f}{\partial z} + \Gamma^2_{44}\frac{\partial f}{\partial t} + \sharp(dt)(\frac{\partial f}{\partial y})\Big) dy \\
&+\Big(\Gamma^3_{41}\frac{\partial f}{\partial x} + \Gamma^3_{42}\frac{\partial f}{\partial y} + \Gamma^3_{43}\frac{\partial f}{\partial z} + \Gamma^3_{44}\frac{\partial f}{\partial t} + \sharp(dt)(\frac{\partial f}{\partial z})\Big) dz \\
&+\Big(\Gamma^4_{41}\frac{\partial f}{\partial x} + \Gamma^4_{42}\frac{\partial f}{\partial y} + \Gamma^4_{43}\frac{\partial f}{\partial y} + \Gamma^4_{44}\frac{\partial f}{\partial t} + \sharp(dt)(\frac{\partial f}{\partial t})\Big) dt.
\end{aligned}$$

Then

$$\begin{aligned}\phi_{g^*}(f) &= \Gamma^1_{11}\frac{\partial f}{\partial x} + \Gamma^1_{12}\frac{\partial f}{\partial y} + \Gamma^1_{13}\frac{\partial f}{\partial z} + \Gamma^1_{14}\frac{\partial f}{\partial t} + \sharp(dx)(\frac{\partial f}{\partial x})\\
&+\Gamma^2_{21}\frac{\partial f}{\partial x} + \Gamma^2_{22}\frac{\partial f}{\partial y} + \Gamma^2_{23}\frac{\partial f}{\partial z} + \Gamma^2_{24}\frac{\partial f}{\partial t} + \sharp(dy)(\frac{\partial f}{\partial y})\\
&+\Gamma^3_{31}\frac{\partial f}{\partial x} + \Gamma^3_{32}\frac{\partial f}{\partial y} + \Gamma^3_{33}\frac{\partial f}{\partial z} + \Gamma^3_{34}\frac{\partial f}{\partial t} + \sharp(dz)(\frac{\partial f}{\partial z})\\
&+\Gamma^4_{41}\frac{\partial f}{\partial x} + \Gamma^4_{42}\frac{\partial f}{\partial y} + \Gamma^4_{43}\frac{\partial f}{\partial y} + \Gamma^4_{44}\frac{\partial f}{\partial t} + \sharp(dt)(\frac{\partial f}{\partial t})\\
&= (\Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31} + \Gamma^4_{41})\frac{\partial f}{\partial x} + (\Gamma^1_{12} + \Gamma^2_{22} + \Gamma^3_{32} + \Gamma^4_{42})\frac{\partial f}{\partial y}\\
&+(\Gamma^1_{13} + \Gamma^2_{23} + \Gamma^3_{33} + \Gamma^4_{43})\frac{\partial f}{\partial z} + (\Gamma^1_{14} + \Gamma^2_{24} + \Gamma^3_{34} + \Gamma^4_{44})\frac{\partial f}{\partial t}.\end{aligned}$$

On can cheich that the vanishing of $\phi_{<,>}(f)$ is equivalente to the following system:

$$\begin{cases}\Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31} + \Gamma^4_{41} = 0\\
\Gamma^1_{12} + \Gamma^2_{22} + \Gamma^3_{32} + \Gamma^4_{42} = 0\\
\Gamma^1_{13} + \Gamma^2_{23} + \Gamma^3_{33} + \Gamma^4_{43} = 0\\
\Gamma^1_{14} + \Gamma^2_{24} + \Gamma^3_{34} + \Gamma^4_{44} = 0\end{cases}$$

which is equivalent to

$$\begin{cases}-\frac{\partial \pi_{12}}{\partial y} - \frac{\partial \pi_{13}}{\partial z} - \frac{\partial \pi_{14}}{\partial t} = 0\\
\frac{\partial \pi_{12}}{\partial x} - \frac{\partial \pi_{23}}{\partial z} - \frac{\partial \pi_{14}}{\partial t} = 0\\
\frac{\partial \pi_{13}}{\partial x} + \frac{\partial \pi_{23}}{\partial y} - \frac{\partial \pi_{34}}{\partial t} = 0,\\
\frac{\partial \pi_{14}}{\partial x} + \frac{\partial \pi_{24}}{\partial y} + \frac{\partial \pi_{34}}{\partial z} = 0\end{cases}$$

which also is equivalent to

$$\begin{aligned}&\frac{\partial}{\partial x}(\pi_{12} + \pi_{13} + \pi_{14}) + \frac{\partial}{\partial y}(-\pi_{12} + \pi_{23} + \pi_{24}) +\\
&\frac{\partial}{\partial z}(-\pi_{13} - \pi_{23} + \pi_{34}) + \frac{\partial}{\partial t}(-\pi_{14} - \pi_{24} - \pi_{34}) = 0.\end{aligned}$$

Thus the 3-form

$$\begin{aligned}\alpha &= -(\pi_{14} + \pi_{24} + \pi_{34})\, dx\wedge dy\wedge dz + (-\pi_{13} - \pi_{23} + \pi_{34})\, dx\wedge dy\wedge dt\\
&\quad -(-\pi_{12} + \pi_{23} + \pi_{24})\, dx\wedge dz\wedge dt + (\pi_{12} + \pi_{13} + \pi_{14})\, dy\wedge dz\wedge dt\end{aligned}$$

on $\mathbb{R}^4$ is closed. as $\mathbb{R}^4$ is simply connected, this form is exact therefore there are six differential functions f_1, f_2, f_3, f_4, f_5, f_6 on $\mathbb{R}^4$ and a 2-form

$$\beta = f_1 dx\wedge dy + f_2 dx\wedge dz + f_3 dx\wedge dt + f_4 dy\wedge dz + f_5 dy\wedge dt + f_6 dz\wedge dt$$

on $\mathbb{R}^4$ such that $\alpha = d\beta \Longleftrightarrow$

$$\begin{cases}A = -\pi_{14} - \pi_{24} - \pi_{34} = -\frac{\partial f_1}{\partial z} + \frac{\partial f_2}{\partial y} - \frac{\partial f_4}{\partial x}\\
B = -\pi_{13} - \pi_{23} + \pi_{34} = \frac{\partial f_1}{\partial t} - \frac{\partial f_3}{\partial y} + \frac{\partial f_5}{\partial x}\\
C = -\pi_{12} + \pi_{23} + \pi_{24} = -\frac{\partial f_2}{\partial t} + \frac{\partial f_3}{\partial z} - \frac{\partial f_6}{\partial x}\\
D = \pi_{12} + \pi_{13} + \pi_{14} = \frac{\partial f_4}{\partial t} - \frac{\partial f_5}{\partial z} + \frac{\partial f_6}{\partial y}\end{cases} \quad \text{or} \quad A + B + C + D = 0 \tag{3.13}$$

$$\Longleftrightarrow \frac{\partial}{\partial x}(f_5-f_6-f_4)+\frac{\partial}{\partial y}(f_6-f_3+f_2)+\frac{\partial}{\partial z}(f_3-f_5-f_1)+\frac{\partial}{\partial t}(f_1-f_2+f_4)=0$$

which is equivalent to

$$\begin{cases} f_6 &= f_5-f_4 \\ f_6 &= f_3-f_2 \\ f_1 &= f_3-f_5 \\ f_1 &= f_2-f_4 \end{cases} \Longleftrightarrow \begin{cases} f_3 &= f_2+f_6 \\ f_3 &= f_1+f_5 \\ f_4 &= f_5-f_6 \\ f_4 &= f_2-f_1 \end{cases} \Longleftrightarrow \begin{cases} f_5 &= f_4+f_6 \\ f_5 &= f_3-f_1 \\ f_2 &= f_3-f_6 \\ f_2 &= f_1+f_4 \end{cases}.$$

So the system (3.11) becomes

$$\begin{cases} A &= -\pi_{14}-\pi_{24}-\pi_{34} = \frac{\partial}{\partial z}(f_4-f_2)+\frac{\partial f_2}{\partial y}-\frac{\partial f_4}{\partial x} \\ B &= -\pi_{13}-\pi_{23}+\pi_{34} = \frac{\partial}{\partial t}(f_3-f_5)-\frac{\partial f_3}{\partial y}+\frac{\partial f_5}{\partial x} \\ C &= -\pi_{12}+\pi_{23}+\pi_{24} = -\frac{\partial f_2}{\partial t}+\frac{\partial f_3}{\partial z}+\frac{\partial}{\partial x}(f_2-f_3) \\ D &= \pi_{12}+\pi_{13}+\pi_{14} = \frac{\partial f_4}{\partial t}-\frac{\partial f_5}{\partial z}+\frac{\partial}{\partial y}(f_5-f_4) \end{cases}$$

$$A+B=-(C+D) \quad \text{and} \quad A+C=-(B+D) \quad \text{and} \quad A+D=-(B+C).$$

So we have

$$\begin{cases} \pi_{13}+\pi_{14}+\pi_{23}+\pi_{24} &= -\frac{\partial f_6}{\partial x}+\frac{\partial f_6}{\partial y}+\frac{\partial f_1}{\partial z}-\frac{\partial f_1}{\partial t} \\ \pi_{12}+\pi_{13}-\pi_{24}-\pi_{34} &= -\frac{\partial f_4}{\partial x}+\frac{\partial f_3}{\partial y}-\frac{\partial f_3}{\partial z}+\frac{\partial f_4}{\partial t} \\ \pi_{12}+\pi_{14}-\pi_{23}+\pi_{34} &= \frac{\partial f_5}{\partial x}-\frac{\partial f_2}{\partial y}-\frac{\partial f_5}{\partial z}+\frac{\partial f_2}{\partial t} \end{cases} \tag{3.14}$$

If $f_1=f_6=0 \Rightarrow f_2=f_3=f_4=f_5$, then the previous system (3.14) becomes

$$\begin{cases} \pi_{13}+\pi_{14}+\pi_{23}+\pi_{24} &= 0 \\ \pi_{12}+\pi_{13}-\pi_{24}-\pi_{34} &= -\frac{\partial f_2}{\partial x}+\frac{\partial f_2}{\partial y}-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t} \\ \pi_{12}+\pi_{14}-\pi_{23}+\pi_{34} &= \frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}. \end{cases}$$

Through identification we have:

$$\pi_{12}=-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}, \quad \pi_{34}=\frac{\partial f_4}{\partial x}-\frac{\partial f_3}{\partial y} \quad \text{and} \quad \pi_{13}=\pi_{14}=\pi_{23}=\pi_{24}=0 \Rightarrow$$
$$\pi=\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}.$$

If $f_3=f_4=0 \Rightarrow \quad f_1=f_2=-f_5=-f_6$ then the system (3.14) becomes:

$$\begin{cases} \pi_{13}+\pi_{14}+\pi_{23}+\pi_{24} &= \frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}-\frac{\partial f_1}{\partial t} \\ \pi_{12}+\pi_{13}-\pi_{24}-\pi_{34} &= 0 \\ \pi_{12}+\pi_{14}-\pi_{23}+\pi_{34} &= -\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial y}+\frac{\partial f_5}{\partial z}+\frac{\partial f_1}{\partial t}. \end{cases}$$

By identification we get

$$\pi_{14}=-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}, \quad \pi_{23}=\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t} \quad \text{and} \quad \pi_{12}=\pi_{13}=\pi_{14}=\pi_{34}=0 \Rightarrow$$
$$\pi=\left(-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}+\left(\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.$$

If $f_2 = f_5 = 0 \Rightarrow \quad f_1 = f_3 = f_6 = -f_4$ then the system (3.14) becomes:

$$\begin{cases} \pi_{13} + \pi_{14} + \pi_{23} + \pi_{24} & = -\dfrac{\partial f_3}{\partial x} + \dfrac{\partial f_3}{\partial y} + \dfrac{\partial f_3}{\partial z} - \dfrac{\partial f_3}{\partial t} \\ \pi_{12} + \pi_{13} - \pi_{24} - \pi_{34} & = \dfrac{\partial f_3}{\partial x} + \dfrac{\partial f_3}{\partial y} - \dfrac{\partial f_3}{\partial z} - \dfrac{\partial f_3}{\partial t} \\ \pi_{12} + \pi_{14} - \pi_{23} + \pi_{34} & = 0. \end{cases}$$

By identification, we have

$$\pi_{13} = \frac{\partial f_3}{\partial y} - \frac{\partial f_3}{\partial t}, \quad \pi_{24} = -\frac{\partial f_3}{\partial x} + \frac{\partial f_3}{\partial z} \quad \text{and} \quad \pi_{12} = \pi_{14} = \pi_{23} = \pi_{34} = 0 \Rightarrow$$
$$\pi = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_3}{\partial t}\right)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \left(-\frac{\partial f_3}{\partial x} + \frac{\partial f_3}{\partial z}\right)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial t}.$$

Which gives the desired result.
In particular if the functions f_1, f_2, f_3, f_4, f_5 and f_6 verify the following equalities

$$\frac{\partial f_6}{\partial x} = \frac{\partial f_6}{\partial y}, \quad -\frac{\partial f_5}{\partial x} = -\frac{\partial f_5}{\partial z}, \quad \frac{\partial f_4}{\partial x} = \frac{\partial f_4}{\partial t},$$
$$\frac{\partial f_3}{\partial y} = \frac{\partial f_3}{\partial z}, \quad -\frac{\partial f_2}{\partial y} = -\frac{\partial f_2}{\partial t}, \quad \frac{\partial f_1}{\partial z} = \frac{\partial f_1}{\partial t},$$

then, by identification of the system (3.12) we have:

$$\pi_{12} = \frac{\partial f_6}{\partial x} = \frac{\partial f_6}{\partial y}, \quad \pi_{13} = -\frac{\partial f_5}{\partial x} = -\frac{\partial f_5}{\partial z}, \quad \pi_{14} = \frac{\partial f_4}{\partial x} = \frac{\partial f_4}{\partial t},$$
$$\pi_{23} = \frac{\partial f_3}{\partial y} = \frac{\partial f_3}{\partial z}, \quad \pi_{24} = -\frac{\partial f_2}{\partial y} = -\frac{\partial f_2}{\partial t}, \quad \pi_{34} = \frac{\partial f_1}{\partial z} = \frac{\partial f_1}{\partial t}.$$

We can write

$$\pi = \left(\frac{\partial f_6}{\partial y}\right)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} - \left(\frac{\partial f_5}{\partial z}\right)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} + \left(\frac{\partial f_4}{\partial t}\right)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial t} +$$
$$+ \left(\frac{\partial f_3}{\partial z}\right)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} - \left(\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial t} + \left(\frac{\partial f_1}{\partial t}\right)\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t}.$$

□

Corollary 3.4. *By Proposition 3.3 the contravariant derivative $\mathcal{D}$ will have the following Christoffel's symbols:*

1) If $\pi = \left(-\frac{\partial f_2}{\partial z} + \frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}$ *one has*

$$\pi_{12} = -\frac{\partial f_2}{\partial z} + \frac{\partial f_2}{\partial t}, \quad \pi_{34} = \frac{\partial f_2}{\partial x} - \frac{\partial f_2}{\partial y} \quad \textit{and} \quad \pi_{13} = \pi_{14} = \pi_{23} = \pi_{24} = 0$$

$$\Gamma^2_{11} = -\Gamma^1_{12} = \frac{\partial^2 f_2}{\partial x\partial z} - \frac{\partial^2 f_2}{\partial x\partial t}; \quad \Gamma^1_{22} = -\Gamma^2_{21} = -\frac{\partial^2 f_2}{\partial y\partial z} + \frac{\partial^2 f_2}{\partial y\partial t};$$

$$\Gamma^3_{34} = -\Gamma^4_{33} = -\frac{\partial^2 f_2}{\partial z\partial x} + \frac{\partial^2 f_2}{\partial z\partial y}; \quad \Gamma^3_{44} = -\Gamma^4_{43} = -\frac{\partial^2 f_2}{\partial t\partial x} + \frac{\partial^2 f_2}{\partial t\partial y};$$

$$\Gamma^1_{23} = -\Gamma^3_{21} = \frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial z^2} + \frac{\partial^2 f_2}{\partial z\partial t}\right); \quad \Gamma^1_{24} = -\Gamma^4_{21} = \frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial t\partial z} + \frac{\partial^2 f_2}{\partial t^2}\right);$$

$$\Gamma^2_{31} = -\Gamma^1_{32} = \frac{1}{2}\left(\frac{\partial^2 f_2}{\partial z^2} - \frac{\partial^2 f_2}{\partial z\partial t}\right); \quad \Gamma^1_{34} = -\Gamma^4_{31} = \frac{1}{2}\left(\frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial x\partial y}\right);$$

$$\Gamma^2_{41} = -\Gamma^1_{42} = \frac{1}{2}\left(\frac{\partial^2 f_2}{\partial t\partial z} - \frac{\partial^2 f_2}{\partial t^2}\right); \quad \Gamma^3_{41} = -\Gamma^1_{43} = \frac{1}{2}\left(\frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial x\partial y}\right);$$

$$\Gamma^3_{12} = -\Gamma^2_{13} = \frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial z^2} + \frac{\partial^2 f_2}{\partial z\partial t}\right); \quad \Gamma^2_{14} = -\Gamma^4_{12} = \frac{1}{2}\left(\frac{\partial^2 f_2}{\partial t\partial z} - \frac{\partial^2 f_2}{\partial t^2}\right);$$

$$\Gamma^2_{34} = -\Gamma^4_{32} = \frac{1}{2}\left(\frac{\partial^2 f_2}{\partial y\partial x} - \frac{\partial^2 f_2}{\partial y^2}\right); \quad \Gamma^3_{42} = -\Gamma^2_{43} = \frac{1}{2}\left(\frac{\partial^2 f_2}{\partial y\partial x} - \frac{\partial^2 f_2}{\partial y^2}\right);$$

$$\Gamma^3_{14} = -\Gamma^4_{13} = \frac{1}{2}\left(\frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_2}{\partial x\partial y}\right); \quad \Gamma^3_{24} = -\Gamma^4_{23} = \frac{1}{2}\left(\frac{\partial^2 f_2}{\partial y\partial x} - \frac{\partial^2 f_2}{\partial y^2}\right).$$

2) If $\pi = \left(-\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t} + \left(\frac{\partial f_1}{\partial x} - \frac{\partial f_1}{\partial z}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$ *we have*

$$\pi_{14} = -\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}, \quad \pi_{23} = \frac{\partial f_1}{\partial x} - \frac{\partial f_1}{\partial z} \quad \textit{et} \quad \pi_{12} = \pi_{13} = \pi_{24} = \pi_{34} = 0$$

$$\Gamma^1_{14} = -\Gamma^4_{11} = -\frac{\partial^2 f_1}{\partial x\partial y} + \frac{\partial^2 f_1}{\partial x\partial z}; \quad \Gamma^2_{23} = -\Gamma^3_{22} = \frac{\partial^2 f_1}{\partial y\partial x} - \frac{\partial^2 f_1}{\partial y\partial t};$$

$$\Gamma^1_{44} = -\Gamma^4_{41} = -\frac{\partial^2 f_1}{\partial t\partial y} + \frac{\partial^2 f_1}{\partial t\partial z}; \quad \Gamma^2_{33} = -\Gamma^3_{32} = \frac{\partial^2 f_1}{\partial z\partial x} - \frac{\partial^2 f_1}{\partial z\partial t};$$

$$\Gamma^1_{23} = -\Gamma^3_{21} = \frac{1}{2}\left(\frac{\partial^2 f_1}{\partial x^2} - \frac{\partial^2 f_1}{\partial x\partial t}\right); \quad \Gamma^1_{24} = -\Gamma^4_{21} = \frac{1}{2}\left(-\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial y\partial z}\right);$$

$$\Gamma^2_{31} = -\Gamma^1_{32} = \frac{1}{2}\left(\frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial y\partial z}\right); \quad \Gamma^1_{34} = -\Gamma^4_{31} = \frac{1}{2}\left(-\frac{\partial^2 f_1}{\partial z\partial y} + \frac{\partial^2 f_1}{\partial z^2}\right);$$

$$\Gamma^2_{41} = -\Gamma^1_{42} = \frac{1}{2}\left(\frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial y\partial z}\right); \quad \Gamma^3_{41} = -\Gamma^1_{43} = \frac{1}{2}\left(\frac{\partial^2 f_1}{\partial y\partial z} - \frac{\partial^2 f_1}{\partial z^2}\right);$$

$$\Gamma^3_{12} = -\Gamma^2_{13} = \frac{1}{2}\left(-\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial x\partial t}\right); \quad \Gamma^2_{14} = -\Gamma^4_{12} = \frac{1}{2}\left(-\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial y\partial z}\right);$$

$$\Gamma^2_{34} = -\Gamma^4_{32} = \frac{1}{2}\left(\frac{\partial^2 f_1}{\partial t\partial x} - \frac{\partial^2 f_1}{\partial t^2}\right); \quad \Gamma^3_{42} = -\Gamma^2_{43} = \frac{1}{2}\left(-\frac{\partial^2 f_1}{\partial t\partial x} + \frac{\partial^2 f_1}{\partial t^2}\right);$$

$$\Gamma^3_{14} = -\Gamma^4_{13} = \frac{1}{2}\left(-\frac{\partial^2 f_1}{\partial z\partial y} + \frac{\partial^2 f_1}{\partial z^2}\right); \quad \Gamma^3_{24} = -\Gamma^4_{23} = \frac{1}{2}\left(-\frac{\partial^2 f_1}{\partial t\partial x} + \frac{\partial^2 f_1}{\partial t^2}\right).$$

3) If $\pi = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_3}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} + \left(-\frac{\partial f_3}{\partial x} + \frac{\partial f_3}{\partial z}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}$ *one has*

$$\pi_{13} = \frac{\partial f_3}{\partial y} - \frac{\partial f_3}{\partial t}, \quad \pi_{24} = -\frac{\partial f_3}{\partial x} + \frac{\partial f_3}{\partial z} \quad \text{and} \quad \pi_{12} = \pi_{14} = \pi_{23} = \pi_{34} = 0$$

$$\Gamma^1_{13} = -\Gamma^3_{11} = \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_3}{\partial x\partial t}; \quad \Gamma^2_{24} = -\Gamma^4_{22} = -\frac{\partial^2 f_3}{\partial y\partial x} + \frac{\partial^2 f_3}{\partial y\partial z};$$

$$\Gamma^1_{33} = -\Gamma^3_{31} = \frac{\partial^2 f_3}{\partial y\partial z} - \frac{\partial^2 f_3}{\partial t\partial z}; \quad \Gamma^2_{44} = -\Gamma^4_{42} = -\frac{\partial^2 f_3}{\partial t\partial x} + \frac{\partial^2 f_3}{\partial t\partial z};$$

$$\Gamma^1_{23} = -\Gamma^3_{21} = \frac{1}{2}\left(\frac{\partial^2 f_3}{\partial y^2} - \frac{\partial^2 f_3}{\partial y\partial t}\right); \quad \Gamma^1_{24} = -\Gamma^4_{21} = \frac{1}{2}\left(-\frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial x\partial z}\right);$$

$$\Gamma^2_{31} = -\Gamma^1_{32} = \frac{1}{2}\left(-\frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3}{\partial y\partial t}\right); \quad \Gamma^4_{31} = -\Gamma^1_{34} = \frac{1}{2}\left(\frac{\partial^2 f_3}{\partial t\partial y} - \frac{\partial^2 f_3}{\partial t^2}\right);$$

$$\Gamma^2_{41} = -\Gamma^1_{42} = \frac{1}{2}\left(-\frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial x\partial z}\right); \quad \Gamma^3_{41} = -\Gamma^1_{43} = \frac{1}{2}\left(-\frac{\partial^2 f_3}{\partial t\partial y} + \frac{\partial^2 f_3}{\partial t^2}\right);$$

$$\Gamma^3_{12} = -\Gamma^2_{13} = \frac{1}{2}\left(-\frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3}{\partial t\partial y}\right); \quad \Gamma^2_{14} = -\Gamma^4_{12} = \frac{1}{2}\left(-\frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial x\partial z}\right);$$

$$\Gamma^2_{34} = -\Gamma^4_{32} = \frac{1}{2}\left(-\frac{\partial^2 f_3}{\partial z\partial x} + \frac{\partial^2 f_3}{\partial z^2}\right); \quad \Gamma^3_{42} = -\Gamma^2_{43} = \frac{1}{2}\left(\frac{\partial^2 f_3}{\partial z\partial x} - \frac{\partial^2 f_3}{\partial z^2}\right);$$

$$\Gamma^3_{14} = -\Gamma^4_{13} = \frac{1}{2}\left(-\frac{\partial^2 f_3}{\partial t\partial y} + \frac{\partial^2 f_3}{\partial t^2}\right); \quad \Gamma^3_{24} = -\Gamma^4_{23} = \frac{1}{2}\left(-\frac{\partial^2 f_3}{\partial z\partial x} + \frac{\partial^2 f_3}{\partial z^2}\right).$$

For the condition of Proposition 3.3 to become sufficient, an additional constraint is required on the functions f_1, f_2 *and* f_3 *hence the following theorem:*

Theorem 3.5. *The bivector field* π *is compatible with* g^* *if and only if there exist differentiable functions* f_1, f_2 *and* f_3 *on* $\mathbb{R}^4$ *such that:*

$$\pi = \left(-\frac{\partial f_2}{\partial z} + \frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}, \tag{3.15}$$

or else

$$\pi = \left(-\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t} + \left(\frac{\partial f_1}{\partial x} - \frac{\partial f_1}{\partial t}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}, \tag{3.16}$$

or else

$$\pi = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_3}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} + \left(-\frac{\partial f_3}{\partial x} + \frac{\partial f_3}{\partial z}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}, \tag{3.17}$$

and for all $k = \overline{1,3}$*, the functions* f_k *satisfy the following equations:*

$$\nabla\cdot f_k = 2\psi_k(x,z) \quad \text{where} \quad \psi_k(x,z) = \frac{\partial f_k}{\partial x} + \frac{\partial f_k}{\partial z}. \tag{3.18}$$

Proof. For $(\mathbb{R}^4, g^*, \pi)$ to be a Riemann-Poisson manifold it is necessary and sufficient that:

$$\mathcal{D}\pi : \Omega^1(\mathbb{R}^4) \longrightarrow \mathcal{X}(\mathbb{R}^4)$$

is identically zero: that's to say

$$\mathcal{D}_{dx}\pi = \mathcal{D}_{dy}\pi = \mathcal{D}_{dz}\pi = \mathcal{D}_{dt}\pi = 0.$$

If

$$\pi = \left(-\frac{\partial f_2}{\partial z} + \frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t},$$

one has:

$$\begin{aligned}\mathcal{D}_{dx}\pi &= \mathcal{D}_{dx}\left\{\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right\}\\ &= \left[\sharp_\pi(dx)\cdot\left(-\frac{\partial f_2}{\partial z}+\frac{\partial h}{\partial t}\right)\right]\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\left[\sharp_\pi(dx)\cdot\left(-\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\right]\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\\ &\quad+\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\mathcal{D}_{dx}\left(\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right)+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\mathcal{D}_{dx}\left(\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right).\end{aligned}$$

Firstly we have

$$\sharp_\pi(dx)\cdot\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)=\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial y}\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)$$

$$\sharp_\pi(dx)\cdot\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)=\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)$$

and on the other hand we have:

$$\begin{aligned}\mathcal{D}_{dx}\left(\frac{\partial}{\partial x}\right) &= -\left\{\frac{\partial}{\partial x}(\mathcal{D}_{dx}dx)\right\}\frac{\partial}{\partial x}-\left\{\frac{\partial}{\partial x}(\mathcal{D}_{dx}dy)\right\}\frac{\partial}{\partial y}-\left\{\frac{\partial}{\partial x}(\mathcal{D}_{dx}dz)\right\}\frac{\partial}{\partial z}\\ &\quad-\left\{\frac{\partial}{\partial x}(\mathcal{D}_{dx}dt)\right\}\frac{\partial}{\partial t}\\ &= -\Gamma^1_{11}\frac{\partial}{\partial x}-\Gamma^1_{12}\frac{\partial}{\partial y}-\Gamma^1_{13}\frac{\partial}{\partial z}-\Gamma^1_{14}\frac{\partial}{\partial t}=\frac{\partial^2 f_2}{\partial x\partial z}\frac{\partial}{\partial y}-\frac{\partial^2 f_2}{\partial x\partial t}\frac{\partial}{\partial y}.\end{aligned}$$

Likewise

$$\begin{aligned}\mathcal{D}_{dx}\left(\frac{\partial}{\partial y}\right) &= -\Gamma^2_{11}\frac{\partial}{\partial x}-\Gamma^2_{12}\frac{\partial}{\partial y}-\Gamma^2_{13}\frac{\partial}{\partial z}-\Gamma^2_{14}\frac{\partial}{\partial t}\\ &= \left(\frac{\partial^2 f_2}{\partial x\partial z}-\frac{\partial^2 f_2}{\partial z\partial y}\right)\frac{\partial}{\partial x}+\frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial z^2}+\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial z}-\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial t\partial z}-\frac{\partial^2 f_2}{\partial t^2}\right)\frac{\partial}{\partial t}.\end{aligned}$$

Next

$$\begin{aligned}\mathcal{D}_{dx}\left(\frac{\partial}{\partial z}\right) &= -\Gamma^3_{11}\frac{\partial}{\partial x}-\Gamma^3_{12}\frac{\partial}{\partial y}-\Gamma^3_{13}\frac{\partial}{\partial z}-\Gamma^3_{14}\frac{\partial}{\partial t}\\ &= -\frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial z^2}+\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial y}-\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial x^2}-\frac{\partial^2 f_2}{\partial x\partial y}\right)\frac{\partial}{\partial t}.\end{aligned}$$

Finally

$$\begin{aligned}\mathcal{D}_{dx}\left(\frac{\partial}{\partial t}\right) &= -\Gamma^4_{11}\frac{\partial}{\partial x}-\Gamma^4_{12}\frac{\partial}{\partial y}-\Gamma^4_{13}\frac{\partial}{\partial z}-\Gamma^4_{14}\frac{\partial}{\partial t}\\ &= \frac{1}{2}\left(\frac{\partial^2 f_2}{\partial t\partial z}-\frac{\partial^2 f_2}{\partial t^2}\right)\frac{\partial}{\partial y}+\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial x^2}-\frac{\partial^2 f_2}{\partial x\partial y}\right)\frac{\partial}{\partial z}.\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{D}_{dx}\left(\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right) &= \mathcal{D}_{dx}\left(\frac{\partial}{\partial x}\right)\wedge\frac{\partial}{\partial y}+\frac{\partial}{\partial x}\wedge\mathcal{D}_{dx}\left(\frac{\partial}{\partial y}\right)\\ &= \frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial z^2}+\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}-\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial t\partial z}-\frac{\partial^2 f_2}{\partial t^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_{dx}\left(\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right) &= \mathcal{D}_{dx}\left(\frac{\partial}{\partial z}\right)\wedge\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\wedge\mathcal{D}_{dx}\left(\frac{\partial}{\partial t}\right)\\ &= -\frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial z^2}+\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}-\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial t\partial z}-\frac{\partial^2 f_2}{\partial t^2}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.\end{aligned}$$

Therefore

$$\begin{aligned}\mathcal{D}_{dx}\pi = \Bigg\{ & \left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\left[\frac{\partial}{\partial y}\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right]\\ & +\frac{1}{2}\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\left[\left(-\frac{\partial^2 f_2}{\partial z^2}+\frac{\partial f_2}{\partial z\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}-\left(\frac{\partial^2 f_2}{\partial t\partial z}-\frac{\partial^2 f_2}{\partial t^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}\right]\\ & -\frac{1}{2}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\left[\left(-\frac{\partial f_2}{\partial z^2}+\frac{\partial f_2}{\partial z\partial t}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}+\left(\frac{\partial f_2}{\partial t\partial z}-\frac{\partial f_2}{\partial t^2}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right]\Bigg\}.\end{aligned}$$

In the same way we have:

$$\begin{aligned}\mathcal{D}_{dy}\pi &= \mathcal{D}_{dy}\left\{\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right\}\\ &= \left[\sharp_\pi(dy)\cdot\left(-\frac{\partial f_2}{\partial z}+\frac{\partial h}{\partial t}\right)\right]\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\left[\sharp_\pi(dy)\cdot\left(-\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\right]\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\\ &\quad+\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\mathcal{D}_{dy}\left(\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right)+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\mathcal{D}_{dy}\left(\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right).\end{aligned}$$

On the one hand we have

$$\begin{aligned}\sharp_\pi(dy)\cdot\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right) &= -\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\\ \sharp_\pi(dy)\cdot\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right) &= -\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\end{aligned}$$

and on the other hand we have:

$$\begin{aligned}\mathcal{D}_{dy}\left(\frac{\partial}{\partial x}\right) &= -\left\{\frac{\partial}{\partial x}\left(\mathcal{D}_{dy}dx\right)\right\}\frac{\partial}{\partial x}-\left\{\frac{\partial}{\partial x}\left(\mathcal{D}^\pi_{dy}dy\right)\right\}\frac{\partial}{\partial y}-\left\{\frac{\partial}{\partial x}\left(\mathcal{D}^\pi_{dy}dz\right)\right\}\frac{\partial}{\partial z}\\ &\qquad-\left\{\frac{\partial}{\partial x}\left(\mathcal{D}_{dy}dt\right)\right\}\frac{\partial}{\partial t}\\ &= -\Gamma^1_{21}\frac{\partial}{\partial x}-\Gamma^1_{22}\frac{\partial}{\partial y}-\Gamma^1_{23}\frac{\partial}{\partial z}-\Gamma^1_{24}\frac{\partial}{\partial t}\\ &= \left(\frac{\partial^2 f_2}{\partial y\partial z}-\frac{\partial^2 f_2}{\partial y\partial t}\right)\frac{\partial}{\partial y}-\frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial z^2}+\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial z}-\frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial t\partial z}+\frac{\partial^2 f_2}{\partial t^2}\right)\frac{\partial}{\partial t}.\end{aligned}$$

Likewise

$$\mathcal{D}_{dy}\left(\frac{\partial}{\partial y}\right) = -\Gamma^2_{21}\frac{\partial}{\partial x}-\Gamma^2_{22}\frac{\partial}{\partial y}-\Gamma^2_{23}\frac{\partial}{\partial z}-\Gamma^2_{24}\frac{\partial}{\partial t} = \left(\frac{\partial^2 f_2}{\partial z\partial z}-\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial x}.$$

Next

$$\begin{aligned}\mathcal{D}_{dz}\left(\frac{\partial}{\partial z}\right) &= -\Gamma^3_{21}\frac{\partial}{\partial x}-\Gamma^3_{22}\frac{\partial}{\partial y}-\Gamma^3_{23}\frac{\partial}{\partial z}-\Gamma^3_{24}\frac{\partial}{\partial t}\\ &= \frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial z^2}+\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial x}-\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial y\partial x}-\frac{\partial^2 f_2}{\partial y^2}\right)\frac{\partial}{\partial t}\end{aligned}$$

at the end

$$\begin{aligned}\mathcal{D}_{dt}\left(\frac{\partial}{\partial t}\right) &= -\Gamma^4_{21}\frac{\partial}{\partial x}-\Gamma^4_{22}\frac{\partial}{\partial y}-\Gamma^4_{23}\frac{\partial}{\partial z}-\Gamma^4_{24}\frac{\partial}{\partial t}\\ &= \frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial t\partial z}+\frac{\partial^2 f_2}{\partial t^2}\right)\frac{\partial}{\partial x}+\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial y\partial x}-\frac{\partial^2 f_2}{\partial y^2}\right)\frac{\partial}{\partial z}.\end{aligned}$$

Thus

$$\mathcal{D}_{dy}\left(\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right)=\mathcal{D}_{dy}\left(\frac{\partial}{\partial x}\right)\wedge\frac{\partial}{\partial y}+\frac{\partial}{\partial x}\wedge\mathcal{D}_{dy}\left(\frac{\partial}{\partial y}\right)$$
$$=\frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial z^2}+\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}+\frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial t\partial z}+\frac{\partial^2 f_2}{\partial t^2}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}$$

and

$$\mathcal{D}_{dy}\left(\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right)=\mathcal{D}_{dx}\left(\frac{\partial}{\partial z}\right)\wedge\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\wedge\mathcal{D}_{dx}\left(\frac{\partial}{\partial t}\right)$$
$$=\frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial z^2}+\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}-\frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial t\partial z}+\frac{\partial^2 f_2}{\partial t^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}.$$

So

$$\mathcal{D}_{dy}\pi=\left\{-\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\left[\frac{\partial}{\partial x}\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right]\right.$$
$$+\frac{1}{2}\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\left[\left(-\frac{\partial^2 f_2}{\partial z^2}+\frac{\partial f_2}{\partial z\partial t}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}+\left(-\frac{\partial^2 f_2}{\partial t\partial z}+\frac{\partial^2 f_2}{\partial t^2}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}\right]$$
$$\left.+\frac{1}{2}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\left[\left(-\frac{\partial f_2}{\partial z^2}+\frac{\partial f_2}{\partial z\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}-\left(-\frac{\partial f_2}{\partial t\partial z}+\frac{\partial f_2}{\partial t^2}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}\right]\right\}$$

$$\mathcal{D}_{dz}\pi=\mathcal{D}_{dz}\left\{\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right\}$$
$$=\left[\sharp_\pi(dz)\cdot\left(-\frac{\partial f_2}{\partial z}+\frac{\partial h}{\partial t}\right)\right]\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\left[\sharp_\pi(dz)\cdot\left(-\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\right]\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}$$
$$+\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\mathcal{D}_{dz}\left(\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right)+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\mathcal{D}_{dz}\left(\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right).$$

On the one hand we have:

$$\sharp_\pi(dz)\cdot\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)=\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial t}\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)$$
$$\sharp_\pi(dz)\cdot\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)=\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial t}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)$$

and on the other hand one has:

$$\mathcal{D}_{dz}\left(\frac{\partial}{\partial x}\right)=-\left\{\frac{\partial}{\partial x}(\mathcal{D}_{dz}dx)\right\}\frac{\partial}{\partial x}-\left\{\frac{\partial}{\partial x}(\mathcal{D}_{dz}dy)\right\}\frac{\partial}{\partial y}-\left\{\frac{\partial}{\partial x}(\mathcal{D}_{dz}dz)\right\}\frac{\partial}{\partial z}$$
$$-\left\{\frac{\partial}{\partial x}(\mathcal{D}_{dz}dt)\right\}\frac{\partial}{\partial t}$$
$$=-\Gamma^1_{31}\frac{\partial}{\partial x}-\Gamma^1_{32}\frac{\partial}{\partial y}-\Gamma^1_{33}\frac{\partial}{\partial z}-\Gamma^1_{34}\frac{\partial}{\partial t}$$
$$=\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial z^2}-\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial y}-\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial x^2}-\frac{\partial^2 f_2}{\partial x\partial y}\right)\frac{\partial}{\partial t}.$$

Likewise

$$\mathcal{D}_{dz}\left(\frac{\partial}{\partial y}\right)=-\Gamma^2_{31}\frac{\partial}{\partial x}-\Gamma^2_{32}\frac{\partial}{\partial y}-\Gamma^2_{33}\frac{\partial}{\partial z}-\Gamma^2_{34}\frac{\partial}{\partial t}$$
$$=-\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial z^2}-\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial x}-\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial y\partial x}-\frac{\partial^2 f_2}{\partial y^2}\right)\frac{\partial}{\partial t}$$

continued

$$\mathcal{D}_{dz}\left(\frac{\partial}{\partial z}\right)=-\Gamma_{31}^{3}\frac{\partial}{\partial x}-\Gamma_{32}^{3}\frac{\partial}{\partial y}-\Gamma_{33}^{3}\frac{\partial}{\partial z}-\Gamma_{34}^{3}\frac{\partial}{\partial t}=\left(\frac{\partial^2 f_2}{\partial z\partial x}-\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial t}.$$

Finally

$$\begin{aligned}\mathcal{D}_{dz}\left(\frac{\partial}{\partial t}\right)&=-\Gamma_{31}^{4}\frac{\partial}{\partial x}-\Gamma_{32}^{4}\frac{\partial}{\partial y}-\Gamma_{23}^{4}\frac{\partial}{\partial z}-\Gamma_{34}^{4}\frac{\partial}{\partial t}\\&=\frac{1}{2}\left(-\frac{\partial^2 f_2}{\partial x^2}-\frac{\partial^2 f_2}{\partial x\partial y}\right)\frac{\partial}{\partial x}+\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial y\partial x}-\frac{\partial^2 f_2}{\partial y^2}\right)\frac{\partial}{\partial y}+\left(-\frac{\partial^2 f_2}{\partial z^2}+\frac{\partial^2 f_2}{\partial z\partial t}\right)\frac{\partial}{\partial z}.\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{D}_{dz}\left(\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right)&=\mathcal{D}_{dz}\left(\frac{\partial}{\partial x}\right)\wedge\frac{\partial}{\partial y}+\frac{\partial}{\partial x}\wedge\mathcal{D}_{dz}\left(\frac{\partial}{\partial y}\right)\\&=-\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial x^2}-\frac{\partial^2 f_2}{\partial x\partial y}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}-\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial y\partial x}-\frac{\partial^2 f_2}{\partial y^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_{dx}\left(\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right)&=\mathcal{D}_{dx}\left(\frac{\partial}{\partial z}\right)\wedge\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\wedge\mathcal{D}_{dx}\left(\frac{\partial}{\partial t}\right)\\&=-\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial x^2}-\frac{\partial^2 f_2}{\partial x\partial y}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}-\frac{1}{2}\left(\frac{\partial^2 f_2}{\partial y\partial x}-\frac{\partial^2 f_2}{\partial y^2}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}.\end{aligned}$$

So

$$\begin{aligned}\mathcal{D}_{dz}\pi=\Bigg\{&\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\left[\frac{\partial}{\partial t}\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right]\\&+\frac{1}{2}\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\left[\left(\frac{\partial^2 f_2}{\partial x^2}-\frac{\partial f_2}{\partial x\partial y}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}-\left(\frac{\partial^2 f_2}{\partial y\partial x}-\frac{\partial^2 f_2}{\partial y^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}\right]\\&-\frac{1}{2}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\left[\left(\frac{\partial f_2}{\partial x^2}-\frac{\partial f_2}{\partial x\partial y}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}+\left(\frac{\partial f_2}{\partial y\partial x}-\frac{\partial f_2}{\partial y^2}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right]\Bigg\}.\end{aligned}$$

Likewise

$$\begin{aligned}\mathcal{D}_{dt}\pi=\Bigg\{&-\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\left[\frac{\partial}{\partial z}\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right]\\&-\frac{1}{2}\left(-\frac{\partial f_2}{\partial z}+\frac{\partial f_2}{\partial t}\right)\left[\left(\frac{\partial^2 f_2}{\partial x^2}-\frac{\partial f_2}{\partial x\partial y}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}-\left(\frac{\partial^2 f_2}{\partial y\partial x}-\frac{\partial^2 f_2}{\partial y^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}\right]\\&-\frac{1}{2}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_2}{\partial y}\right)\left[\left(\frac{\partial f_2}{\partial x^2}-\frac{\partial f_2}{\partial x\partial y}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}+\left(\frac{\partial f_2}{\partial y\partial x}-\frac{\partial f_2}{\partial y^2}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}\right]\Bigg\}.\end{aligned}$$

Finally we have:

$$\mathcal{D}_{dx}\pi=\mathcal{D}_{dy}\pi=\mathcal{D}_{dz}\pi=\mathcal{D}_{dt}\pi=0.$$

One has:

$$\left\{\begin{array}{ccc}\dfrac{\partial f_2}{\partial z}&=&\dfrac{\partial f_2}{\partial t}\\&\text{et}&\\\dfrac{\partial f_2}{\partial x}&=&\dfrac{\partial f_2}{\partial y}\end{array}\right.\iff\quad\nabla\cdot f_2=2\psi_2\left(x,z\right)\quad\text{where}\quad\psi_2\left(x,z\right)=\frac{\partial f_2}{\partial x}+\frac{\partial f_2}{\partial z}.$$

If $\pi=\left(-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}+\left(\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z},$ one has:

$$\begin{aligned}\mathcal{D}_{dx}\pi &= \left\{\left(-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\right)\left[\frac{\partial}{\partial t}\left(-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}+\frac{\partial}{\partial t}\left(\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right]\right.\\
&+\tfrac{1}{2}\left(-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\right)\left[\left(-\frac{\partial^2 f_1}{\partial y^2}+\frac{\partial f_1}{\partial y\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\left(-\frac{\partial^2 f_1}{\partial z\partial y}+\frac{\partial^2 f_1}{\partial z^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}\right]\\
&\left.+\tfrac{1}{2}\left(\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t}\right)\left[\left(-\frac{\partial^2 f_1}{\partial y^2}+\frac{\partial^2 f_1}{\partial y\partial z}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}-\left(-\frac{\partial^2 f_1}{\partial z\partial y}+\frac{\partial^2 f_1}{\partial z^2}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}\right]\right\};\end{aligned}$$

$$\begin{aligned}\mathcal{D}_{dy}\pi &= \left\{\left(\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t}\right)\left[\frac{\partial}{\partial z}\left(-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right]\right.\\
&-\tfrac{1}{2}\left(-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\right)\left[\left(\frac{\partial^2 f_1}{\partial x^2}-\frac{\partial f_1}{\partial x\partial t}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}-\left(-\frac{\partial^2 f_1}{\partial t\partial x}+\frac{\partial^2 f_1}{\partial t^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}\right]\\
&\left.-\tfrac{1}{2}\left(\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t}\right)\left[\left(\frac{\partial^2 f_1}{\partial x^2}-\frac{\partial^2 f_1}{\partial x\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}-\left(-\frac{\partial^2 f_1}{\partial t\partial x}+\frac{\partial^2 f_1}{\partial t^2}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}\right]\right\};\end{aligned}$$

$$\begin{aligned}\mathcal{D}_{dz}\pi &= \left\{-\left(\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t}\right)\left[\frac{\partial}{\partial y}\left(-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}+\frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right]\right.\\
&+\tfrac{1}{2}\left(-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\right)\left[\left(\frac{\partial^2 f_1}{\partial y^2}-\frac{\partial f_1}{\partial y\partial z}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}+\left(\frac{\partial^2 f_1}{\partial t\partial x}-\frac{\partial^2 f_1}{\partial t^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right]\\
&\left.-\tfrac{1}{2}\left(\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t}\right)\left[\left(\frac{\partial^2 f_1}{\partial y^2}-\frac{\partial^2 f_1}{\partial y\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}-\left(\frac{\partial^2 f_1}{\partial t\partial x}-\frac{\partial^2 f_1}{\partial t^2}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right]\right\};\end{aligned}$$

$$\begin{aligned}\mathcal{D}_{dt}\pi &= \left\{-\left(\frac{\partial f_1}{\partial y}-\frac{\partial f_1}{\partial z}\right)\left[\frac{\partial}{\partial x}\left(-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right]\right.\\
&+\tfrac{1}{2}\left(-\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial z}\right)\left[\left(\frac{\partial^2 f_1}{\partial y^2}-\frac{\partial f_1}{\partial y\partial z}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}+\left(\frac{\partial^2 f_1}{\partial z\partial y}-\frac{\partial^2 f_1}{\partial z^2}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right]\\
&\left.-\tfrac{1}{2}\left(\frac{\partial f_1}{\partial x}-\frac{\partial f_1}{\partial t}\right)\left[\left(\frac{\partial^2 f_1}{\partial y^2}-\frac{\partial^2 f_1}{\partial y\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}-\left(\frac{\partial^2 f_1}{\partial z\partial y}-\frac{\partial^2 f_1}{\partial z^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right]\right\}.\end{aligned}$$

Finally one has:

$$\mathcal{D}_{dx}\pi=\mathcal{D}_{dy}\pi=\mathcal{D}_{dz}\pi=\mathcal{D}_{dt}\pi=0.$$

Which is equivalent to saying that

$$\left\{\begin{array}{ccc}\dfrac{\partial f_1}{\partial x} & = & \dfrac{\partial f_1}{\partial t}\\ & \text{and} & \\ \dfrac{\partial f_1}{\partial y} & = & \dfrac{\partial f_1}{\partial z}\end{array}\right. \iff \nabla\cdot f_1 = 2\psi_1(x,z) \quad \text{where} \quad \psi_1(x,z)=\frac{\partial f_1}{\partial x}+\frac{\partial f_1}{\partial z}.$$

If $\pi = \left(\frac{\partial f_3}{\partial y}-\frac{\partial f_3}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}+\left(-\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial z}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t},$ one has:

$$\begin{aligned}\mathcal{D}_{dx}\pi &= \left\{\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_3}{\partial t}\right)\left[\frac{\partial}{\partial z}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_3}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}+\frac{\partial}{\partial z}\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_3}{\partial z}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}\right]\right.\\
&-\tfrac{1}{2}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_3}{\partial t}\right)\left[\left(-\frac{\partial^2 f_3}{\partial y^2}+\frac{\partial f_3}{\partial t\partial y}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}+\left(-\frac{\partial^2 f_3}{\partial t\partial y}+\frac{\partial^2 f_3}{\partial t^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}\right]\\
&\left.+\tfrac{1}{2}\left(-\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial z}\right)\left[\left(-\frac{\partial^2 f_3}{\partial y^2}+\frac{\partial^2 f_3}{\partial t\partial y}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}+\left(-\frac{\partial^2 f_3}{\partial t\partial y}+\frac{\partial^2 f_3}{\partial t^2}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right]\right\};\end{aligned}$$

$$\begin{aligned}\mathcal{D}_{dy}\pi \quad &= \Bigg\{ \left(-\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial z}\right)\left[\frac{\partial}{\partial t}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_3}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\left(-\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial z}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}\right]\\ &+\tfrac{1}{2}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_3}{\partial t}\right)\left[\left(\frac{\partial^2 f_3}{\partial x^2}-\frac{\partial f_3}{\partial x\partial z}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}-\left(-\frac{\partial^2 f_3}{\partial z\partial x}+\frac{\partial^2 f_3}{\partial z^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}\right]\\ &-\tfrac{1}{2}\left(-\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial z}\right)\left[\left(-\frac{\partial^2 f_3}{\partial x^2}+\frac{\partial^2 f_3}{\partial x\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}-\left(-\frac{\partial^2 f_3}{\partial z\partial x}+\frac{\partial^2 f_3}{\partial z^2}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}\right]\Bigg\};\end{aligned}$$

$$\begin{aligned}\mathcal{D}_{dz}\pi \quad &= \Bigg\{ -\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_3}{\partial t}\right)\left[\frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_3}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}+\frac{\partial}{\partial x}\left(-\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial z}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}\right]\\ &+\tfrac{1}{2}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_3}{\partial t}\right)\left[\left(-\frac{\partial^2 f_3}{\partial y^2}+\frac{\partial f_3}{\partial y\partial t}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}+\left(\frac{\partial^2 f_3}{\partial t\partial y}-\frac{\partial^2 f_3}{\partial t^2}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right]\\ &-\tfrac{1}{2}\left(-\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial z}\right)\left[\left(-\frac{\partial^2 f_3}{\partial y^2}+\frac{\partial^2 f_3}{\partial y\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}+\left(\frac{\partial^2 f_3}{\partial t\partial y}-\frac{\partial^2 f_3}{\partial t^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right]\Bigg\};\end{aligned}$$

$$\begin{aligned}\mathcal{D}_{dt}\pi \quad &= \Bigg\{ -\left(-\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial z}\right)\left[\frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_3}{\partial t}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}+\frac{\partial}{\partial y}\left(-\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial z}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}\right]\\ &+\tfrac{1}{2}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_3}{\partial t}\right)\left[\left(-\frac{\partial^2 f_3}{\partial x^2}+\frac{\partial f_3}{\partial x\partial z}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}-\left(\frac{\partial^2 f_3}{\partial z\partial x}-\frac{\partial^2 f_3}{\partial z^2}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\right]\\ &+\tfrac{1}{2}\left(-\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial z}\right)\left[\left(-\frac{\partial^2 f_3}{\partial x^2}+\frac{\partial^2 f_3}{\partial x\partial z}\right)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t}+\left(\frac{\partial^2 f_3}{\partial z\partial x}-\frac{\partial^2 f_3}{\partial z^2}\right)\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial t}\right]\Bigg\}.\end{aligned}$$

Finally we have:

$$\mathcal{D}_{dx}\pi=\mathcal{D}_{dy}\pi=\mathcal{D}_{dz}\pi=\mathcal{D}_{dt}\pi=0.$$

Which is equivalent to saying that

$$\left\{\begin{array}{ccc}\dfrac{\partial f_3}{\partial y} & = & \dfrac{\partial f_3}{\partial t}\\ & \text{and} & \\ \dfrac{\partial f_3}{\partial x} & = & \dfrac{\partial f_3}{\partial z}\end{array}\right. \iff \quad \nabla\cdot f_3 = 2\psi_3(x,z) \quad \text{where} \quad \psi_3(x,z)=\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial z}.$$

Which gives the desired result.

□

4 Conclusion remarks

This paper aims is to obtain a characterization of 4*-dimensional Poisson manifold endowed of canonical metric. Also, some characterization have been discussed. Therefore, the results of this work are variant, significant and so it is interesting and capable to develop its study in the future.*

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